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When does Zeilberger's algorithm succeed?

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Abstract

A terminating condition of the well-known Zeilberger's algorithm for a given hypergeometric term $T(n,k)$ is presented. It is shown that the only information on $T(n,k)$ that one needs in order to determine in advance whether this algorithm will succeed is the rational function $T(n,k+1)/T(n,k)$.

1. Introduction

Let K be a field of characteristic 0. A hypergeometric term (or simply a term) $T(k)$ in k over K satisfies a linear recurrence of the form

$$f(k)T(k+1) + g(k)T(k) = 0, \tag{1}$$

$f, g \in K[k] \setminus \{0\}$, the variable k is integer-valued. The *certificate* $\mathcal{C}_k(T)$ of the term $T(k)$ is the rational function $T(k+1)/T(k) = -g(k)/f(k)$. A term $T(n,k)$ in two integer-valued variables over K satisfies the recurrences

$$f_1(n,k)T(n+1,k) + g_1(n,k)T(n,k) = 0, \tag{2}$$

$$f_2(n,k)T(n,k+1) + g_2(n,k)T(n,k) = 0, \tag{3}$$

$f_1, g_1, f_2, g_2 \in K[n,k] \setminus \{0\}$. $T(n,k)$ has the n-certificate $\mathcal{C}_n(T) = T(n+1,k)/T(n,k)$ and the k-certificate $\mathcal{C}_k(T) = T(n,k+1)/T(n,k)$ which are rational functions of n and k.

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By using a standard notation E_n, E_k for the shift operators w.r.t. n and k, respectively, we can write $\mathcal{C}_n(T) = E_nT/T$ and $\mathcal{C}_k(T) = E_kT/T$.

Throughout the paper until Section 5.3, the field K is mostly $\mathbb{C}$ for the case of terms in two variables n, k, and K is $\mathbb{C}(n)$ for the case of terms in one variable k (and we will note this explicitly). In Section 5.3 we discuss more practical suppositions on the field K. The usage of a field of rational functions of $u, v, x, \ldots$ allows us to consider terms depending on the parameters u, v, x, et cetera.

Zeilberger's algorithm, named hereafter as $\mathcal{Z}$, is a useful tool for proving combinatorial identities that involve definite sums of hypergeometric terms [10,13,18]. Given a term $T(n,k)$, $\mathcal{Z}$ tries to construct for $T(n,k)$ a *Z-duplex* (L, F) which consists of a linear difference operator L with coefficients which are polynomials in n over $\mathbb{C}$

$$L = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n + a_0(n), \tag{4}$$

i.e., $L \in \mathbb{C}[n, E_n]$, and a rational function $F(n,k) \in \mathbb{C}(n,k)$ such that

$$LT(n,k) = G(n,k+1) - G(n,k) \tag{5}$$

where

$$G(n,k) = F(n,k)T(n,k). \tag{6}$$

Obviously $G(n,k)$ is a term.

It is not true that a Z-duplex exists for every term $T(n,k)$. Even if a Z-duplex exists for $T(n,k)$, it is not uniquely defined. In this case, $\mathcal{Z}$ terminates with one of the Z-duplexes and the operator L in the returned Z-duplex is of minimal order [18] (though the induced recurrence, e.g., for the definite sum $s(n) = \sum_{k=0}^{n} T(n,k)$ can have the order that is not minimal). The algorithm uses an item-by-item examination on the order ρ of L. It starts with the value of 0 for ρ and increases ρ until it finds a Z-duplex (L, F) for T. In this case, $\mathcal{Z}$ is said to be *applicable* to $T(n,k)$. If a Z-duplex does not exist for $T(n,k)$, then $\mathcal{Z}$ does not terminate, and it is said to be *not applicable* to $T(n,k)$. So in the context of this paper "$\mathcal{Z}$ is applicable to $T(n,k)$" means "$\mathcal{Z}$ succeeds on $T(n,k)$."

Algorithmically speaking, $\mathcal{Z}$ works with the certificates of T in order to find the coefficients $a_0(n), \ldots, a_\rho(n)$ in (4) and the rational function $F(n,k)$ in (6). From this standpoint, the basis of $\mathcal{Z}$ can be formulated as in the following proposition.

Proposition 1. *If $\mathcal{Z}$ is applicable to a term $T(n,k)$, then $\mathcal{Z}$ is applicable to any term $T'(n,k)$ that has the same certificates.*

The question to which terms $\mathcal{Z}$ is applicable was not conclusively answered although a sufficient condition has been known for quite a long time. The "fundamental theorem" [10,13,16,17] states that a Z-duplex exists if $T(n,k)$ is a *proper term* (or, in short, a *p-term*), i.e., it can be written in the form

$$P(n,k)\frac{\prod_{i=1}^{l}\Gamma(a_in + b_ik + c_i)}{\prod_{i=1}^{m}\Gamma(a_i'n + b_i'k + c_i')}u^n v^k, \tag{7}$$

where $P(n,k)\in\mathbb{C}[n,k]$, $a_i, b_i, a_i', b_i' \in \mathbb{Z}$, $l, m \in \mathbb{N}$, $c_i, c_i', u, v \in \mathbb{C}$. A polynomial $p \in \mathbb{C}[n,k]$ is defined to be *integer-linear* if it has the form $an + bk + c$, where $a, b \in \mathbb{Z}$, $c \in \mathbb{C}$ (note that any constant $c \in \mathbb{C}$ is an integer-linear polynomial with $a = b = 0$). Equivalently, any p-term can be written in the form

$$P(n,k)\frac{\prod_{i=1}^{l}\Gamma(\alpha_i(n,k))}{\prod_{i=1}^{m}\Gamma(\beta_i(n,k))}u^n v^k, \tag{8}$$

where $\alpha_i(n,k), \beta_i(n,k)$ are integer-linear polynomials, while $P(n,k)$, u, v, l, m are as in (7). If a p-term in (8) has $P = 1$, then we call this term a *factorial term*. If T can be written as RT' where R is a rational function and T' is a factorial term, then we call T an *r-term* (the prefix r refers to rational functions; in the same manner, p refers to polynomials, and also to the word “proper”). Each p-term is evidently an r-term.

It is possible, however, to give examples showing that the condition “T is a p-term” is not a necessary condition for the existence of a Z-duplex for T. The main contribution of this paper is a criterion for the applicability of $\mathcal{Z}$ to a given term, i.e., a necessary and sufficient condition for the applicability of $\mathcal{Z}$. (This criterion can be formulated in different forms.) Additionally, an algorithm to recognize the applicability of $\mathcal{Z}$ to a given term is presented.

Before embarking upon further discussion, we would like to stress one more time that the following three statements are equivalent:

(a) $\mathcal{Z}$ is applicable to T, i.e., it terminates in finite time for the given certificates of T as input;
(b) there exists a Z-duplex for T;
(c) $\mathcal{Z}$ constructs a Z-duplex for T in finite time.

Traditionally, programs that implement $\mathcal{Z}$ are organized in such a manner that each of them *tries* to construct a Z-duplex (L, G) for a given term T such that the order of L does not exceed a fixed bound B, e.g., $B = 6$. Consequently, the lack of a criterion prevents the use of $\mathcal{Z}$ to its full capacity.

In [4] a criterion for the applicability of $\mathcal{Z}$ to a given rational function is presented (the rational functions are a particular case of terms). This criterion can be described as follows. Consider a given rational function $R(n,k)$ as a rational function in k over $\mathbb{C}(n)$. It is then possible to apply an algorithm to solve the *additive decomposition* problem (or synonymously, the decomposition problem of indefinite sum) [1,2,14] to R to represent this rational function as

$$(E_k - 1)U + V, \tag{9}$$

where $U, V \in \mathbb{C}(n)(k)$ are such that the denominator of V has the minimal degree w.r.t. k. (We will refer to this representation as an additive decomposition of R with *summable component* U and *non-summable component* V.) By the criterion, $\mathcal{Z}$ is applicable to $R(n,k)$ iff V, represented as a ratio of two relatively prime polynomials from $\mathbb{C}[n,k]$, has the denominator that is a product of integer-linear polynomials.

Note that additive decomposition (9) of a rational function R is not unique in general. But if $R = (E_k - 1)U' + V'$ is another additive decomposition, then the denominator of V factors into integer-linear factors iff the denominator of V' does.

It is self-evident that the set of rational functions is a proper subset of the set of all terms, and we shall present in this paper a conclusive answer to the question of specifying the class of terms $T(n,k)$ to which $\mathcal{Z}$ is applicable.

As aforementioned, $\mathcal{Z}$ works with the certificates $\mathcal{C}_n(T)$ and $\mathcal{C}_k(T)$ instead of with T. Our algorithm which determines the applicability of $\mathcal{Z}$ follows the same concept. The criterion and the algorithm that will be presented are based on the additive decomposition (of terms in one variable over a field K). In this sense this result is a generalization of [4].

The algorithm that we present needs only the rational function $\mathcal{C}_k(T)$ as input.

A preliminary version of this paper has appeared as [3].

2. Preliminaries

In addition to the "fundamental theorem," we shall use recent results on a special type of a term in two variables (r-terms) [6], on the additive decomposition of terms in one variable (the construction of this decomposition uses a special form of representation of rational functions in one variable) [5]. We shall also use a tool to determine whether a polynomial from $\mathbb{C}[n,k]$ factors into a product of integer-linear polynomials [4]. In this section, we give a summary of these results.

Throughout the paper, we consider rational functions of k over $\mathbb{C}(n)$, i.e., elements of the field $\mathbb{C}(n)(k)$, as the ratios of relatively prime polynomials from $\mathbb{C}[n,k]$, and irreducibles from $\mathbb{C}(n)[k]$ in the form of irreducibles from $\mathbb{C}[n,k]$. This allows us to identify the irreducibles of $\mathbb{C}(n)[k]$, $\mathbb{C}[n][k]$ and $\mathbb{C}[n,k]$.

2.1. A structure theorem for terms in two variables

Two rational functions $S_1(n,k)$ and $S_2(n,k)$ are *compatible* if

$$S_1(n,k)S_2(n+1,k) = S_1(n,k+1)S_2(n,k). \tag{10}$$

Theorem 1 [6]. *Let the non-zero rational functions S_1, S_2 be compatible. Then there exists an r-term $T(n,k)$ such that $\mathcal{C}_n(T) = S_1$, $\mathcal{C}_k(T) = S_2$.*

This theorem is a "conservative version" of the well-known Ore–Sato theorem [11,15]. This "conservatism" is motivated by examples such as the following. Let $T(n,k) = |n-k|$. Notice that T satisfies the equations of the form (2) and (3), namely:

$$(n-k)T(n+1,k) + (k-n-1)T(n,k) = 0,$$

$$(n-k)T(n,k+1) + (k-n+1)T(n,k) = 0,$$

although $|n-k|$ is not an r-term [8]. However, the same equations hold for $n-k$ which is an r-term. So, though it is not true that any term is an r-term, it is always possible to construct an r-term which has the same certificates as the given term.

Theorem 1 plays a key role in the verification of the criterion to be proposed in this paper (it is due to Proposition 1).

2.2. Rational normal forms

We write $p \perp q$ to indicate that the polynomials p, q are relatively prime. Let Λ be a field of characteristic 0 and $f, f_1, f_2 \in \Lambda[k]$. If $f_1 \perp E_k^m f_2$ for all $m \in \mathbb{Z}$ then the rational function $F = f_1/f_2$ is *shift-reduced*. If $f \perp E_k^m f$ for all $m \in \mathbb{Z} \setminus \{0\}$ then the polynomial f is *shift-free*.

Define a normal form for rational functions which reveals the shift structure of their factors. For a given non-zero rational function $R \in K(k)$, let $F, V \in \Lambda(k)$ be such that

$$R = F\frac{E_k V}{V}, \tag{11}$$

where F is shift-reduced, then the right-hand side of (11) is a *rational normal form* (*RNF*) of R.

If (11) is an RNF of R,

$$F = \frac{f_1}{f_2}, \quad f_1 \perp f_2, \qquad V = \frac{v_1}{v_2}, \quad v_1 \perp v_2,$$

and, in addition, $f_1 \perp v_1 \cdot E_k v_2$ and $f_2 \perp v_2 \cdot E_k v_1$, then (11) is a *strict* RNF of R. An algorithm to construct a strict RNF for a given R was discussed in [5,7] (we will refer to this algorithm as **srnf**). It is shown in [5,7] that a rational function can have several RNFs, even strict ones.

If $R \in K(k)$ where $K = \mathbb{C}(n)$, then, actually, $R \in \mathbb{C}(n,k)$. In this case we say that (11) is an RNF of $R(n,k)$ w.r.t. k and can assume that the numerators and the denominators of F and V belong to $\mathbb{C}[n,k]$.

Denote by $Z_{n,k}$ the set of all rational functions of n and k whose numerators and denominators, considered as elements from $\mathbb{C}[n,k]$, are products of integer-linear polynomials (in particular, $Z_{n,k}$ contains all the polynomials from $\mathbb{C}[n,k]$ that are such products).

Theorem 2 [6]. *For a given term $T(n,k)$, set $S = \mathcal{C}_k(T)$. Let*

$$F\frac{E_k V}{V} \tag{12}$$

be an RNF of S w.r.t. k. Then $F \in Z_{n,k}$. If $T(n,k)$ is a factorial term, then $V \in Z_{n,k}$. If $T(n,k) \in Z_{n,k}$, then $F = 1$, $V \in Z_{n,k}$.

2.3. Additive decomposition of terms in one variable

Recall that non-zero terms $T(k)$ and $T'(k)$ over K are *similar* (denoted as $T(k) \sim T'(k)$) if there exists $F(k) \in K(k)$ such that $T'(k) = F(k)T(k)$, and the sum of two non-zero terms $T(k)$ and $T'(k)$ is a term iff $T(k) \sim T'(k)$ [12]. If we apply an operator from $\mathbb{C}(n,k)[E_k]$ to a term $T(k)$, then we obtain a term which is either zero or similar to $T(k)$. Therefore, if a non-zero term $T(k)$ is represented as $T(k) = (E_k - 1)T_1(k) + T_2(k)$ where $T_1(k), T_2(k)$ are terms, then $T(k) \sim T_i(k)$ if $T_i(k)$ is non-zero, $1 \leqslant i \leqslant 2$.

Theorem 3 [5,7]. *Let $T(k)$ and $T_1(k)$ be similar terms over K and $T_2(k) = T(k) - (E_k - 1)T_1(k)$ be a non-zero term. Let*

$$F\frac{E_k V}{V}, \quad F = \frac{f_1}{f_2}, \quad f_1 \perp f_2, \qquad V = \frac{v_1}{v_2}, \quad v_1 \perp v_2, \tag{13}$$

be an RNF of the certificate of $T_2(k)$ such that for any irreducible $p \in \mathbb{C}(n)[k]$ and for $\alpha \in \mathbb{N} \setminus \{0\}$ such that $p^\alpha | v_2$, the following relations hold:

$$E_k^m p \mid v_2 \quad \Rightarrow \quad m = 0, \tag{14}$$

$$E_k^m p \mid f_1 \quad \Rightarrow \quad m < 0, \qquad E_k^m p \mid f_2 \quad \Rightarrow \quad m > 0 \tag{15}$$

(m *is assumed to be integer*). *Then for any term $T_1'(k)$, $T_1'(k) \sim T(k)$ or $T_1'(k) = 0$, the term $T_2'(k) = T(k) - (E_k - 1)T_1'(k)$ is non-zero, and for any RNF*

$$F'\frac{E_k V'}{V'}, \quad V' = \frac{v_1'}{v_2'}, \quad v_1' \perp v_2',$$

of the certificate of $T_2'(k)$, there exists an $m \in \mathbb{Z}$ such that $E_k^m p^\alpha \mid v_2'$.

For a given term $T(k)$, the algorithm from [5,7] which solves the additive decomposition problem constructs two terms $T_1(k)$, $T_2(k)$ such that $T_2(k) = T(k) - (E_k - 1)T_1(k)$ and either $T_2 = 0$ or the certificate of T_2 has an RNF of the form (13) where v_2 is shift-free (i.e., relation (14) holds), and the two relations in (15) hold for any irreducible factor p of v_2. It follows from Theorem 3 that the polynomial v_2 has minimal degree. As in the rational case, T_1, T_2 are the summable and, respectively, non-summable components of an additive decomposition $T = (E_k - 1)T_1 + T_2$.

This formulation agrees with the additive decomposition problem for rational functions [1,2,14] since if $T_2 \in \mathbb{C}(n)(k)$ then $F = 1$ and v_2 is the denominator of T_2.

Note that for a given term $T(k)$, the mentioned algorithm from [7] which constructs an additive decomposition of $T(k)$ follows a number of steps. In the first step, the auxiliary algorithm **dcert** is applied. For a given strict RNF of $\mathcal{C}_k(T)$, it constructs RNFs of T_1, T_2 (if $T_1(k) = 0$ or $T_2(k) = 0$, then $F(k) = 0, V(k) = 1$ in the corresponding RNF of the form (11)). The algorithm **dcert** can be slightly simplified by avoiding the construction of an RNF of $\mathcal{C}_k(T_1)$. We will refer to this simplified version in Section 3.3 as **dcert**$'$.

2.4. Factorization into integer-linear polynomials

As in [4], we will face the problem of recognizing whether a given polynomial in n and k factors into integer-linear polynomials. The following theorem is the key to the solution of the problem.

Theorem 4 [4]. *A polynomial $f(n,k) \in \mathbb{C}[n,k]$ belongs to $Z_{n,k}$ iff for any irreducible factor $p(n,k)$ of $f(n,k)$, there are $I, J \in \mathbb{Z}$, $I > 0$, such that $p(n+I, k+J) \mid f(n,k)$.*

For a given polynomial $f(n,k) \in \mathbb{C}[n,k]$, Theorem 4 provides a criterion for the factorability of f into integer-linear polynomials. Additionally in [4] an algorithm to determine if f belongs to $Z_{n,k}$ was presented. This algorithm does not require a complete factorization of the input polynomial $f(n,k)$ into irreducible factors. In summary, this algorithm (we will refer to it as algorithm **ilf**) is as follows. The problem of recognizing whether a given polynomial $g(n,k)$ factors into polynomials of the form $an + bk + c$, $a, b \in \mathbb{Z}$, $c \in \mathbb{C}$ is equivalent to the possibility of factoring $g(n,k)$ into polynomials that do not depend on n and polynomials of the form $n + dk + c$, $d \in \mathbb{Q} \setminus \{0\}$, $c \in \mathbb{C}$. We can extract from $g(n,k)$ the maximal factor $v(k)$ that does not depend on n. Let $w(n,k) = g(n,k)/v(k)$. Consider d as a new variable and substitute $k - dn$ into $w(n,k)$ for k (this gives us a polynomial $\widetilde{w}(d,n,k)$) and represent the result as a polynomial in n with coefficients in $\mathbb{C}[d,k]$. Then find all rational values $d_0, \ldots, d_m$ of d such that these coefficients have a non-constant greatest common divisor, which we denote as $w_i(n,k)$ for the value d_i, $i = 0, \ldots, m$. (This can be achieved by using resultant approach.) The answer to the question under consideration is "yes" iff $\sum_{i=0}^{m} \deg_k w_i(n,k) = \deg_k w(n,k)$.

2.5. The existence of a Z-duplex for a sum of two similar terms

The notion of similarity of terms in one variable can be readily generalized to terms in two variables, i.e., $T(n,k) \sim T'(n,k)$ if $T'(n,k) = R(n,k)T(n,k)$, $R(n,k) \in \mathbb{C}(n,k)$. Similar to the univariate case, the sum $T + T'$ of non-zero terms in two variables is a term iff $T \sim T'$. The following simple theorem about the existence of a Z-duplex for a sum of two similar terms is presented in [4].

Theorem 5 [4]. *If there exist Z-duplexes for similar terms $T(n,k)$ and $T'(n,k)$, then there exists a Z-duplex for the term $T(n,k) + T'(n,k)$.*

3. Stems of rational functions and r-terms

3.1. The stem of a rational function

For any rational function $Q(n,k)$ there exists a uniquely defined monic polynomial $s(n,k)$ such that $s(n,k)$ has no integer-linear factor, and the denominator of $s(n,k)Q(n,k)$ factors into integer-linear polynomials. We call $s(n,k)$ the *stem* of $Q(n,k)$.

3.2. The stem of an r-term

Let $T(n,k)$ be an r-term. If there exists a monic $s \in \mathbb{C}[n,k]$ such that

- $s(n,k)$ has no integer-linear factor,
- $s(n,k)T(n,k)$ is a p-term,

then $s(n,k)$ is called the *stem* of the term $T(n,k)$. Hence, if $s(n,k)$ is a stem of $T(n,k)$ then

$$T(n,k) = \frac{1}{s(n,k)} T'(n,k),$$

where $T'(n,k)$ is a p-term. The following theorem shows that the stem of any r-term is uniquely defined.

Theorem 6. *Let $T(n,k)$ be an r-term. Let s be a monic polynomial in n,k that has no integer-linear factor, and such that sT is a p-term. Let*

$$F \frac{E_k V}{V} \tag{16}$$

be any RNF w.r.t. k of $\mathcal{C}_k(T)$. Then s is the stem of V.

Proof. We can find a polynomial $p(n,k)$ such that for $R = p/s$ the term $T' = RT$ is a fractal term. Set $S = \mathcal{C}_k(T)$, $S' = \mathcal{C}_k(T')$. Then

$$S = S' \frac{E_k R}{R}. \tag{17}$$

If S' is shift-reduced w.r.t. k, then the right-hand side of (17) is an RNF of S. Otherwise, we can transform (17) into an RNF of S by constructing an RNF of S', say $S' = G\frac{E_k W}{W}$. It follows from (17) that $G\frac{E_k(RW)}{RW}$ is an RNF of S. By Theorem 2, $G, W \in Z_{n,k}$. (Note that F in (16) belongs to $Z_{n,k}$ by Theorem 2 as well.) We have

$$\frac{F}{G} = \frac{E_k(RWV^{-1})}{RWV^{-1}}.$$

The right-hand side of the last equality is an RNF of F/G. By Theorem 2, $RWV^{-1} \in Z_{n,k}$. Since $W \in Z_{n,k}$, V differs from R by an element from $Z_{n,k}$. The claim follows. □

Corollary 1. *Let $T(n,k)$ be an r-term and* (16) *be any RNF w.r.t. k of $\mathcal{C}_k(T)$. Then the stem of V is equal to the stem of $T(n,k)$.*

3.3. An algorithm to recognize if an r-term is a p-term

The following corollary follows directly from Theorem 6.

Corollary 2. *An r-term $T(n,k)$ is a p-term iff its stem is equal to* 1.

In order to recognize if an r-term $T(n,k)$ is a p-term, one constructs an RNF w.r.t. k of $\mathcal{C}_k(T)$ of the form (11) and checks whether the denominator of V belongs to $Z_{n,k}$ by using the algorithm mentioned in Section 2.4.

So the k-certificate of an r-term $T(n,k)$ suffices to recognize if $T(n,k)$ is a p-term.

4. An additive decomposition of an r-term

Theorem 7. *Any r-term $T(n,k)$ can be represented in the form*

$$(E_k - 1)T_1(n,k) + T_2(n,k),$$

where $T_1(n,k), T_2(n,k)$ are r-terms, and either $T_2(n,k)$ is zero or the stem of $T_2(n,k)$ is shift-free w.r.t. k.

Proof. Set $S = \mathcal{C}_k(T)$. Let $\widetilde{T}$ be any non-zero term in k over the field $\mathbb{C}(n)$ with $\mathcal{C}_k(\widetilde{T}) = S$ where S is considered as an element from $\mathbb{C}(n)(k)$. Such a term can be constructed as follows. Let $k_0 \in \mathbb{Z}$ be such that $S(n,i)$ is a non-zero rational function from $\mathbb{C}(n)$ for all $i \in \mathbb{Z}$, $i \geqslant k_0$. Set

$$\widetilde{T}(k) = \prod_{i=k_0}^{k-1} S(n,i). \tag{18}$$

$\widetilde{T}$ is a term in one variable k (over $\mathbb{C}(n)$), and, therefore, it is possible to construct its additive decomposition, that was discussed in Section 2.3:

$$\widetilde{T}(k) = (E_k - 1)\widetilde{T}_1(k) + \widetilde{T}_2(k). \tag{19}$$

This means if an RNF of $\mathcal{C}_k(\widetilde{T}_2)$ has the form (13), then for any irreducible p, $p \mid v_2$, the relations in (14), (15) hold. Evidently there exist $R_1, R_2 \in \mathbb{C}(n)(k)$ such that $\widetilde{T}_1 = R_1\widetilde{T}$, $\widetilde{T}_2 = R_2\widetilde{T}$ (if $\widetilde{T}_i = 0$, then $R_i = 0$). Consider R_1, R_2 as elements of $\mathbb{C}(n,k)$. Set $T_1 = R_1 T$, $T_2 = R_2 T$. We claim that

$$T = (E_k - 1)T_1 + T_2. \tag{20}$$

Indeed, set

$$\widetilde{T}_3 = (E_k - 1)\widetilde{T}_1. \tag{21}$$

Since either $T_3 = 0$ or $\widetilde{T}_3 \sim \widetilde{T}$, there exists $R_3 \in \mathbb{C}(n)(k)$ such that $\widetilde{T}_3 = R_3\widetilde{T}$. It follows from $\widetilde{T} = \widetilde{T}_3 + \widetilde{T}_2$ that $R_3 + R_2 = 1$ and, consequently, $T = T_3 + T_2$ where $T_3 = R_3 T$. The claim is proven if we can show that

$$T_3 = (E_k - 1)T_1. \tag{22}$$

By (21) we have $\widetilde{T}_3 = (E_k - 1)R_1\widetilde{T}$, or

$$R_3 = (E_k R_1)B - R_1, \tag{23}$$

where $B = \mathcal{C}_k(\widetilde{T})$. It follows from (23) and from $\mathcal{C}_k(\widetilde{T}) = \mathcal{C}_k(T)$ that relation (22) holds. Hence, relation (20) also holds. Evidently $\mathcal{C}_k(T_2) = \mathcal{C}_k(\widetilde{T}_2)$ and, therefore, any RNF w.r.t. k of $\mathcal{C}_k(T_2)$, written in the form (13), has v_2 shift-free w.r.t. k. □

5. The applicability of $\mathcal{Z}$

5.1. r-terms whose stems are shift-free w.r.t. k

Let Λ be a field of characteristic 0. A polynomial $f(x) \in \Lambda(x)$ is *spread* if for any irreducible $p(x)$ which divides $f(x)$ there is $m \in \mathbb{Z} \setminus \{0\}$ such that $p(x+m) | f(x)$.

Theorem 8. *Let $\hat{T}(n,k)$ be an r-term whose stem is not spread. Then there does not exist a term $\check{T}(n,k)$ such that $\hat{T} = (E_k - 1)\check{T}$.*

Proof. Let $F\frac{E_k V}{V}$ be an RNF of $\mathcal{C}_k(\hat{T})$, $F, V \in \mathbb{C}(n,k)$, and are represented as

$$F = \frac{f_1}{f_2}, \quad f_1 \perp f_2, \qquad V = \frac{v_1}{v_2}, \quad v_1 \perp v_2.$$

It follows from the hypothesis of the theorem that the stem of V has an irreducible factor $p(n,k)$ such that $p(n,k+m)$ is not a factor of v_2 for any $m \in \mathbb{Z} \setminus \{0\}$. By Theorem 2, $F \in Z_{n,k}$. Hence, f_1, f_2 have no factor of the form $p(n,k+m)$, $m \in \mathbb{Z}$. Since the hypothesis Theorem 3 (including the relations in (14) and (15)) is satisfied, the claim follows. □

For the case where a given $F(n,k) \in \mathbb{C}(n,k)$ is also a polynomial in k over $\mathbb{C}(n)$ or over $\mathbb{C}[n]$, we denote $F(n,k)$ as $F(n;k)$. If some polynomial substitutions $n = \varphi(n,k)$, $k = \psi(n,k)$ for n and k are applied to $F(n;k)$, then the expression

$$F\big(\varphi(n,k); \psi(n,k)\big) \tag{24}$$

is also considered as a polynomial in k over $\mathbb{C}(n)$ or over $\mathbb{C}[n]$.

Theorem 9. *Let $T(n,k)$ be an r-term which is not a p-term. Let the stem of T be shift-free w.r.t. k. Then for any operator $L \in \mathbb{C}[n, E_n]$, the stem of the term LT is not spread w.r.t. k.*

Proof. Let $s(n,k)$ be the stem of $T(n,k)$. We can find a rational function $R(n,k)$ whose denominator is the stem of $T(n,k)$, the numerator has no integer-linear factor and additionally $T = RT'$ where T' is a factorial term. Let the operator $L \in \mathbb{C}[n, E_n]$ be of the form (4). Then LT is the product of MR and T', where

$$M = a_\rho(n)t_\rho E_n^\rho + \cdots + a_1(n)t_1 E_n + a_0(n), \quad t_1, \ldots, t_\rho \in Z_{n,k},$$

that is, M is an operator from $\mathbb{C}(n,k)[E_n]$ whose coefficients belong to $Z_{n,k}$. Recall that the denominator of R is shift-free w.r.t. k.

Suppose that the stem of MR is spread w.r.t. k. It is shown in [4, Lemma 3], that if $M = b_\rho E_n^\rho + \cdots + b_0 \in \mathbb{C}[n, E_n]$, then this implies the following: for any irreducible factor $p(n,k)$ of the denominator of $R(n,k)$ there exists a factor $q(n,k)$ of this denominator such that

$$p(n,k) = q(n+I, k+J), \quad I, J \in \mathbb{Z},\ I > 0. \tag{25}$$

As shown in the proof [4], we consider the partial fraction decomposition of R over $\mathbb{C}(n)$, and use the fact that if $b_m \in \mathbb{C}[n]$, then the application of $b_m E_n^m$ to a simple fraction with the denominator $p(n;k)^\mu$ results in a simple fraction with the denominator

$$p(n+m;k)^\mu. \tag{26}$$

If $b_m \in Z_{n,k}$, then since $p(n,k)$ divides the stem of T, $p(n,k)$ is not integer-linear, and the application of $b_m E_n^m$ to a simple fraction with the denominator $p(n,k)^\mu$ results in a rational function, considered as a rational function in k over $\mathbb{C}(n)$, whose partial fraction decomposition contains a simple fraction with the denominator (26) and no simple fraction with the denominator of the form $p(n+m_1;k)^{\mu_1}$, $m_1 \neq m$, $\mu_1 > 0$. Consequently, for the simple fractions with the denominators of the form (26), the logic from [4] remains valid for the case where the coefficients of the difference operator M belong to $Z_{n,k}$. Therefore, if $p(n,k)$ is an irreducible factor of the denominator of R, then this denominator also has a factor $q(n,k)$ such that equality (25) is satisfied. It follows from Theorem 4 that all irreducible factors of the denominator of $R(n,k)$ are integer-linear, a contradiction. □

5.2. An algorithm to recognize the applicability of $\mathcal{Z}$ to an arbitrary term

Let $T(n,k)$ be a term. By Theorem 1, there exists an r-term $T_0(n,k)$ that has the same certificates as those of the original term. By Proposition 1 we can now consider T_0 instead of T. Let T_1, T_2 be terms such that

$$T_0 = (E_k - 1)T_1 + T_2$$

and $T_2 = 0$ or the stem of T_2 is shift-free w.r.t. k. Suppose that $T_2 \neq 0$. By Theorem 5 the term T_0 has a Z-duplex iff T_2 has a Z-duplex. By Theorem 8, if T_2 has a Z-duplex (L, G), then the stem of LT_2 should be spread w.r.t. k. By Theorem 9, this condition is not satisfied unless the stem of T_2 is 1, i.e., T_2 is a p-term. By combining this information with the "fundamental theorem," we arrive at the following theorem.

Theorem 10. *For a given term $T(n,k)$, let $T_0(n,k)$ be an r-term that has the same certificates. Let the terms T_1, T_2 be such that $T_2 \neq 0$, the stem of T_2 is shift-free and*

$$T_0 = (E_k - 1)T_1 + T_2. \tag{27}$$

Then $\mathcal{Z}$ is applicable to $T(n,k)$ iff T_2 is a p-term.

This gives a criterion for the applicability of $\mathcal{Z}$ to a given term. We have mentioned, however, that by Corollaries 1, 2, the k-certificate $\mathcal{C}_k(T_2)$ suffices to recognize if T_2 is a p-term. In turn, $\mathcal{C}_k(T_2)$ can be constructed by algorithm **dcert**′ (Section 2.3) starting with $\mathcal{C}_k(T_0)$ only. But $\mathcal{C}_k(T_0) = \mathcal{C}_k(T)$. This way the answer to the question "is $\mathcal{Z}$ applicable to $T(n,k)$?" can be provided algorithmically, starting with the k-certificate of $T(n,k)$:

1. Construct by algorithm **srnf** (Section 2.2) a strict RNF $D(n,k)U(n,k+1)/U(n,k)$ of $\mathcal{C}_k(T)$. Considering $\mathcal{C}_k(T)$ as the k-certificate of a term $\widetilde{T}$ in k over the rational function field of n, construct by **dcert**′ (Section 2.3) an RNF $F(n,k)V(n,k+1)/V(n,k)$ of the k-certificate of the non-summable component of an additive decomposition of $\widetilde{T}$ (if the non-summable component is 0, then set $F = 0$, $V = 1$).
2. By algorithm **ilf** (Section 2.4) recognize if the denominator of V factors into integer-linear factors (the answer is "yes," if, in particular, V is a polynomial). $\mathcal{Z}$ is applicable to $T(n,k)$ iff such factorization is feasible.

Notice that in spite of the non-uniqueness of an additive decomposition (as a consequence, a possible k-certificate of the non-summable component is not unique in general), and non-uniqueness of RNF of the k-certificate, by Theorem 10 this algorithm gives the one-valued answer to the question on the applicability of $\mathcal{Z}$.

Example 1. For the hypergeometric term [9, 3.112]

$$T(n,k) = (-1)^k \binom{n+1}{k}\binom{2n-2k-1}{n-1},$$

$$S = \mathcal{C}_k(T) = \frac{(k-n-1)(2k-n)(2k-n+1)}{2(k+1)(k-n+1)(2k-2n+1)}.$$

By algorithm **srnf** we get a strict RNF w.r.t. k of S in the form $D\frac{E_k U}{U}$:

$$D = \frac{(2k-n)(2k-n+1)}{2(k+1)(2k-2n+1)}, \qquad U = \frac{1}{(k-n)(k-n-1)}.$$

By algorithm **dcert**′ we get an RNF of the k-certificate of the non-summable component $\widetilde{T}_2$ from the additive decomposition (18). This RNF is of the form $F\frac{E_k V}{V}$:

$$F = \frac{(2k-n-2)(2k-n-1)}{2(k+1)(2k-2n+1)}, \qquad V = \frac{v_1}{v_2} = \frac{-n^2-3n+4k-2}{4(k-n-1)}.$$

Since v_2 can be written as a product of integer-linear polynomials, $\mathcal{Z}$ is applicable to $T(n,k)$. Notice that in this example, the given term T itself is a p-term.

Remark. We could construct the complete additive decomposition (18). This yields

$$\widetilde{T} = \prod_{i=0}^{k-1} \frac{(i-n-1)(2i-n)(2i-n+1)}{2(i+1)(i-n+1)(2i-2n+1)},$$

$$\widetilde{T}_1 = \frac{kn^2(1-n)(2nk+2k-2n^2-3n-1)}{(1-2n)(k-n-1)(2k-n-2)(2k-n-1)} \times \prod_{i=1}^{k-1} \frac{(2i-n)(2i-n+1)}{2(i+1)(2i-2n+1)},$$

$$\widetilde{T}_2 = -\frac{1}{2}\frac{n(4nk-5n+4k-n^3-4n^2-2)}{(1-2n)(k-n-1)} \prod_{i=1}^{k-1} \frac{(2i-n-2)(2i-n-1)}{2(i+1)(2i-2n+1)}$$

but for our goal we do not need this, and the RNF of $\mathcal{C}_k(T_2)$ as given before this remark is sufficient.

Example 2.

$$T(n,k) = (-1)^k \frac{1}{nk+1}\binom{n+1}{k}\binom{2n-2k-1}{n-1}.$$

(Notice that this term is a product of the term T in Example 1 and the rational function $1/(nk+1)$.)

We have

$$S = \mathcal{C}_k(T) = \frac{(nk+1)(k-n-1)(2k-n)(2k-n+1)}{2(nk+n+1)(k+1)(k-n+1)(2k-2n+1)}.$$

By algorithm **srnf** we get

$$D = \frac{(2k-n)(2k-n+1)}{2(k+1)(2k-2n+1)}, \qquad U = \frac{n}{(nk+1)(k-n)(k-n-1)}.$$

By algorithm **dcert**′ we get

$$F = \frac{(2k-n-1)(2k-n-2)}{2(k+1)(2k-2n+1)},$$

$$V = \frac{v_1}{v_2} = \frac{8nk^2-n^3k-7n^2k-8nk+4k+n^3+2n^2-n-2}{8(k-n-1)(nk-n+1)}.$$

Since v_2 cannot be written as a product of integer-linear polynomials, $\mathcal{Z}$ is not applicable to $T(n,k)$. This is an example where the given term T is not a p-term, and $\mathcal{Z}$ is not applicable to T, either.

Example 3.

$$T(n,k) = (-1)^k \frac{n^2k^2+n^2k-1}{(nk+1)(nk+n+1)} \binom{2n-2k-3}{n-1}.$$

We have $S = \mathcal{C}_k(T) = s_1/s_2$ where

$$\begin{aligned} s_1 = {} & -(nk+1)(2k-n)(2k-n+1) \\ & \times \big(-6+14nk+51n^2k+4k^2n^4-8k^3n^3-38k^2n^3+4k^4n^2 \\ & \quad + 26k^3n^2+12kn^4-55n^3k+4k^2n+58k^2n^2+14n^2+11n \\ & \quad - 25n^3+8n^4-10k-4k^2\big), \\ s_2 = {} & 2(nk+2n+1)(k-n+2)(2k-2n+3) \\ & \times \big(6nk+4k^2n-3n^2k-2k-4k^2-n^2+n-3n^3k+4k^2n^2 \\ & \quad + 4k^2n^4-8k^3n^3-14k^2n^3+10k^3n^2+4k^4n^2+4kn^4\big). \end{aligned}$$

By algorithm **srnf** we get

$$D = -\frac{(2k-n)(2k-n+1)}{2(k-n+2)(2k-2n+3)}, \qquad U = \frac{u_1}{u_2}$$

where

$$\begin{aligned} u_1 = {} & -2k+6nk-3n^2k+4n^4k^2+4k^2n-3n^3k+4n^2k^2-8n^3k^3 \\ & - 14n^3k^2+4n^2k^4-n^2+n-4k^2+10n^2k^3+4n^4k, \\ u_2 = {} & 4(nk+1)(nk+n+1). \end{aligned}$$

By algorithm **dcert**$'$ we have

$$F = \frac{(2k-n-1)(2k-n-2)}{2(k-n+3)(2k-2n+5)}, \qquad V = \frac{v_1}{v_2} = \frac{4k-3n+2}{4}.$$

Therefore $\mathcal{Z}$ is applicable to $T(n,k)$, even though the given term T is not a p-term.

5.3. Remarks on the field K; parameterized terms

So far we considered rational functions and terms over the fields $\mathbb{C}$ and $\mathbb{C}(n)$. In this sense the field $\mathbb{C}$ played the role of the ground field.

Algorithmically speaking, this choice of the ground field is not completely appropriate (it was made for simplicitys sake) because, for example, algorithms **srnf**, **dcert**′, **ilf** involve the search for integer and rational roots of algebraic equations with coefficients from the ground field. On the other hand, it is known that $\mathcal{Z}$ can be applied to some parameterized terms, since the "fundamental theorem" is valid for the case where the coefficients of $P(n,k)$ and u, v, c_is, c_i's, involved in (7), depend on parameters. The problem is how to avoid the difficulties associated with the root computation and, additionally, to cover the interesting parameterized case.

Let Λ be a field of characteristic 0 (this is actually equal to $\mathbb{Q} \subset \Lambda$). It is easy to show that if there is an algorithm to compute rational roots of any polynomial $f(x)$ over Λ, then there exists a corresponding algorithm for any simple extension $\Lambda(\theta)$, algebraic or transcendental. This implies that we can consider as the ground field (instead of $\mathbb{C}$) any field of the form $\mathbb{Q}(\theta_1,\dots,\theta_m)$, where for each θ_i either it is known that θ_i is transcendental over $\mathbb{Q}(\theta_1,\dots,\theta_{i-1})$, or an irreducible polynomial $P_i(x)$ over $\mathbb{Q}(\theta_1,\dots,\theta_{i-1})$ such that $P_i(\theta_i)=0$ is given. (In the first case θ_i can be considered as a parameter.)

Let K be a field of such form, $\overline{K}$—the algebraic closure of K. We can consider an integer-linear polynomial as a polynomial of the form $an+bk+c$, where $a,b\in\mathbb{Z}$, $c\in\overline{K}$ (the definitions of $Z_{n,k}$, p-terms and r-terms have to be adjusted accordingly). The "fundamental theorem" and Theorems 1, 4 still hold. Besides this there is no problem with computing integer and rational roots of algebraic equations over K and $K(n)$ and algorithms **srnf**, **dcert**′, **ilf** can be used. This gives an opportunity to apply the proposed algorithm to $\mathcal{C}_k(T)\in K(n,k)$ to determine in advance whether $\mathcal{Z}$ will succeed on $T(n,k)$.

Example 4.

$$T(n,k) = \left(m-\sqrt{2}\right)^k \left(\frac{m-\sqrt{2}}{mn+k}\binom{n}{k+1}^2 - \frac{2}{mn+k-1}\binom{n}{k}^2\right).$$

We consider $\mathbb{Q}(m,\sqrt{2})$ as the ground field: m is transcendental over $\mathbb{Q}$, while $\sqrt{2}$ is algebraic over $\mathbb{Q}(m)$. We have $S=\mathcal{C}_k(T)=s_1/s_2$ where

$$\begin{aligned} s_1 = {} & (-n+k)^2\left(m-\sqrt{2}\right)(mn+k-1)\\ & \times \big(-8-16k-8mn-k\sqrt{2}n^2+2k^2\sqrt{2}n+2k^2m+k^3m+m^2n\\ & \quad -2m^2n^2+m^2n^3+km-mn\sqrt{2}-\sqrt{2}mn^3-10mnk-4mnk^2\\ & \quad +mn^2k+k^2nm^2+2knm^2-2kn^2m^2-k\sqrt{2}-k^3\sqrt{2}-2k^2\sqrt{2}\\ & \quad -2knm\sqrt{2}+2kn^2m\sqrt{2}-k^2nm\sqrt{2}+2k\sqrt{2}n-10k^2-2k^3\\ & \quad +2n^2m\sqrt{2}\big), \end{aligned}$$

$$\begin{aligned} s_2 = {} & (mn+k+1)(k+2)^2 \\ & \times \bigl(m^2n^3 - \sqrt{2}mn^3 - 2kn^2m^2 + 2kn^2m\sqrt{2} + k^2nm^2 - k^2nm\sqrt{2} \\ & \quad + mn^2k - k\sqrt{2}n^2 - 4mnk^2 + 2k^2\sqrt{2}n + k^3m - k^3\sqrt{2} - mn^2 \\ & \quad + n^2\sqrt{2} - 2mnk - 2k\sqrt{2}n - k^2m + k^2\sqrt{2} - 2mn - 2k^3 - 4k^2 - 2k\bigr). \end{aligned}$$

By algorithm **srnf** we get

$$D = \frac{(m-\sqrt{2})(k-n)^2}{(k+2)^2}, \qquad U = \frac{u_1}{u_2}$$

where

$$\begin{aligned} u_1 = {} & m^2n^3 - mn^3\sqrt{2} - 2m^2n^2k + 2mn^2k\sqrt{2} \\ & + m^2nk^2 - mnk^2\sqrt{2} + mn^2k - kn^2\sqrt{2} - 4mnk^2 \\ & + 2k^2n\sqrt{2} + k^3m - k^3\sqrt{2} - mn^2 + n^2\sqrt{2} - 2mnk \\ & - 2nk\sqrt{2} - mk^2 + k^2\sqrt{2} - 2mn - 2k^3 - 4k^2 - 2k, \\ u_2 = {} & \bigl(-2 + m - \sqrt{2}\bigr)(mn+k)(mn+k-1). \end{aligned}$$

By algorithm **dcert**′ we have

$$F = \frac{(k-n-1)^2}{(k+3)^2}, \qquad V = \frac{v_1}{(mn+k-1)v_2}$$

where $v_1 \in \mathbb{Q}(m,\sqrt{2})[n,k]$, $v_2 \in \mathbb{Q}(m,\sqrt{2})[n]$. (We do not show v_1 and v_2 due to their sizes.) Therefore $\mathcal{Z}$ is not applicable to $T(n,k)$.

Example 5.

$$T(n,k) = \bigl(m-\sqrt{2}\bigr)^k \left(\frac{m-\sqrt{2}}{mn+k}\binom{n}{k+1}^2 - \frac{1}{mn+k-1}\binom{n}{k}^2\right).$$

As in Example 4, we consider $\mathbb{Q}(m,\sqrt{2})$ as the ground field. We have $S = \mathcal{C}_k(T) = s_1/s_2$, where

$$\begin{aligned} s_1 = {} & (-n+k)^2\bigl(m-\sqrt{2}\bigr)(nk-1) \\ & \times \bigl(4 + 4k - m - 8n + 3mn + \sqrt{2} - 12nk + mn^3 - 2mk \\ & \quad + 5mnk + mn^3k - 2mn^2k^2 + mnk^3 + k^2 - 3mn^2 - mk^2 - nk^3 \\ & \quad - 6nk^2 + 3mnk^2 - 4mn^2k - \sqrt{2}n^3k + 2\sqrt{2}n^2k^2 - \sqrt{2}nk^3 - 5\sqrt{2}nk \\ & \quad - 3\sqrt{2}nk^2 + 4\sqrt{2}n^2k + 3\sqrt{2}n^2 + \sqrt{2}k^2 + 2\sqrt{2}k - 3\sqrt{2}n - \sqrt{2}n^3\bigr), \end{aligned}$$

$$
\begin{aligned}
s_2 = {} & (nk + 2n - 1)(k + 2)^2 \\
& \times \bigl(mn^3k - 2mn^2k^2 + mnk^3 - mn^2 + 2mnk - mk^2 - \sqrt{2}n^3k \\
& \quad + 2\sqrt{2}n^2k^2 - \sqrt{2}nk^3 + \sqrt{2}n^2 - 2\sqrt{2}nk + \sqrt{2}k^2 - nk^3 \\
& \quad - 3nk^2 - 3nk - n + k^2 + 2k + 1\bigr).
\end{aligned}
$$

By algorithm **srnf** we get

$$
D = \frac{(m - \sqrt{2})(k - n)^2}{(k + 2)^2}, \qquad U = \frac{u_1}{u_2}
$$

where

$$
\begin{aligned}
u_1 = {} & \bigl(mn^3k - 2mn^2k^2 + mnk^3 - mn^2 + 2mnk - mk^2 - \sqrt{2}k^3k \\
& + 2\sqrt{2}n^2k^2 - \sqrt{2}nk^3 + \sqrt{2}n^2 - 2\sqrt{2}nk + \sqrt{2}k^2 - nk^3 \\
& - 3nk^2 - 3nk - n + k^2 + 2k + 1\bigr)n, \\
u_2 = {} & \bigl(m - \sqrt{2} - 1\bigr)(nk + n - 1)(nk - 1).
\end{aligned}
$$

By algorithm **dcert**$'$, the non-summable component is 0, i.e., $F = 0$, $V = 1$. Therefore, $\mathcal{Z}$ is applicable to $T(n, k)$.

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