A unified matrix approach to the representation of Appell polynomials*

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June 6, 2014

Abstract

In this paper we propose a unified approach to matrix representations of different types of Appell polynomials. This approach is based on the creation matrix - a special matrix which has only the natural numbers as entries and is closely related to the well known Pascal matrix. By this means we stress the arithmetical origins of Appell polynomials. The approach also allows to derive, in a simplified way, the properties of Appell polynomials by using only matrix operations.

Keyword: Appell polynomials, creation matrix, Pascal matrix, binomial theorem.

MSC: 15A16, 65F60, 11B83.

1 Introduction

In the last years, the interest in Appell polynomials and their applications in different fields has significantly increased. As recent applications of Appell

*This work was supported in part by *FEDER* funds through *COMPETE*–Operational Programme Factors of Competitiveness ("Programa Operacional Factores de Competitividade") and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* and the Portuguese Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690.

polynomials in fields like probability theory and statistics we mention [6] and [24]. Generalized Appell polynomials as tools for approximating 3D-mappings were introduced for the first time in [21] in combination with Clifford Analysis methods.

As another new example we mention representation theoretic results like those of [9] and [19], that gave evidence to the central role of Appell polynomials as orthogonal polynomials. Representation theory is also the tool for their application in quantum physics as explained in [28].

Starting from Appell polynomials, but in the general framework of non-commutative Clifford algebras, one can find more traditionally motivated operational approaches to generalize Laguerre, Gould-Hopper and Chebyshev polynomials in the recent papers [10, 11, 12].

At the same time other authors were concerned with finding new characterizations of Appell polynomials themselves through new approaches. To quote some of them we mention, for instance, the novel approach developed in [26] which makes use of the generalized Pascal functional matrices introduced in [27] and also a new characterization based on a determinantal definition proposed in [15]. Both of them have permitted to derive some properties of Appell polynomials by employing only linear algebra tools and to generalize some classical Appell polynomials.

In this paper we concentrate on an unifying tool for representing Appell polynomials in one real variable in matrix form. The matrix structure of Appell polynomials is based on one and the same simple constant matrix H defined by

$$(H)_{ij} = \begin{cases} i, & i = j+1 \\ 0, & \text{otherwise}, \qquad i,j = 0,1,\ldots,m. \end{cases} \tag{1}$$

Needless to emphasize the simplicity of the structure of H: it is a subdiagonal matrix which contains as nonzero entries only the sequence of natural numbers. In the context of this paper H is called *creation matrix* because it is the matrix from which all the special types of Appell polynomials will be created. Our approach has the advantage of reducing the class of Appell polynomials to its most fundamental arithmetical origins.

At the same time the present paper can be considered as a far reaching generalization of the papers [3, 4, 5] where the authors concentrated their attention on the Pascal matrix and highlighted its connections with matrix representations of other special polynomials like, for instance, Bernoulli and Bernstein polynomials. The common Appell polynomial background was not a subject of their concern.

The paper is organized as follows. In Section 2 the concept of Appell polynomials and the generating functions of some particular cases are recalled. Section 3 explains how to obtain a matrix representation of the referred polynomials. In Section 4 we prove some properties of Appell polynomials by employing only matrix operations. We also observe that the approach works for those polynomials $p_n(x)$ which are not Appell in the strong sense, i.e., where the degree is not equal to n, using as an example the Genocchi polynomials. Finally, in Section 5 some conclusions are presented.

2 Classical equivalent characterizations of Appell polynomials

In 1880 Appell introduced in [7] sequences of polynomials $\{p_n(x)\}_{n\geq 0}$ satisfying the relation

$$\frac{d}{dx}p_n(x) = n\, p_{n-1}(x), \qquad n = 1, 2, \ldots, \tag{2}$$

in which

$$p_0(x) = c_0, \qquad c_0 \in \mathbb{R} \setminus \{0\}. \tag{3}$$

Remark 1 *Some authors use another definition for $\{p_n(x)\}_{n\geq 0}$, where the factor n on the right hand side of (2) is omitted. However, considering that the prototype of such sequences is the monomial power basis, i.e., $\{x^n\}_{n\geq 0}$, we prefer to consider Appell's original definition.*

From (2)-(3) it can easily be checked that the polynomials which form an Appell set are n-degree polynomials having the following form:

$$\begin{aligned}
p_0(x) &= c_0\\
p_1(x) &= c_1 + c_0\, x\\
p_2(x) &= c_2 + 2\, c_1\, x + c_0\, x^2\\
p_3(x) &= c_3 + 3\, c_2\, x + 3\, c_1\, x^2 + c_0\, x^3\\
&\vdots
\end{aligned}$$

This can be written in a compact form as follows:

$$p_n(x) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} c_{n-k}\, x^k, \quad n = 0, 1, \ldots, \qquad c_0 \neq 0. \tag{4}$$

In addition, if $f(t)$ is any convergent power series on the whole real line with Taylor expansion given by

$$f(t) = \sum_{n=0}^{+\infty} c_n \frac{t^n}{n!}, \qquad f(0) \neq 0, \tag{5}$$

then Appell sequences can also be defined by means of the corresponding *generating function* (see [8]),

$$G(x,t) \equiv f(t)\, e^{xt} = \sum_{n=0}^{+\infty} p_n(x) \frac{t^n}{n!}. \tag{6}$$

By appropriately choosing the function $f(t)$, many of the classical polynomials can be derived. In particular, we get [1]

- the *monomials* $\{x^n\}_{n \geq 0}$ when

$$f(t) = 1;$$

- the *Bernoulli polynomials* $\{B_n(x)\}_{n \geq 0}$ when

$$f(t) \quad = \quad \frac{t}{e^t - 1} = \left(\frac{\sum_{n=0}^{+\infty} \frac{t^n}{n!} - 1}{t} \right)^{-1} = \left(E_{1,2}(t)\right)^{-1},$$

  where

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \alpha, \beta > 0, \tag{7}$$

  is the two parameter function of the Mittag-Leffler type and $\Gamma(z)$ the Gamma function [22];

[1] The expression of functions $f(t)$ for Bernoulli and Euler polynomials can be found in [25], the one of the monic Hermite polynomials in [8] and that of Laguerre polynomials in [17].

- the *Euler polynomials* $\{E_n(x)\}_{n\geq 0}$ when

$$f(t)=\frac{2}{e^t+1};$$

- the *monic Hermite polynomials* $\{\widehat{H}_n(x)\}_{n\geq 0}\equiv\{2^{-n}H_n(x)\}_{n\geq 0}$, with $H_n(x)$ the classical Hermite polynomials, when

$$f(t) \;=\; e^{-\frac{t^2}{4}}=E_{1,1}\left(-\frac{t^2}{4}\right),$$

  (see (7));

- the *"modified" generalized Laguerre polynomials* $\{(-1)^n\, n!\, L_n^{(\alpha-n)}(x)\}_{n\geq 0}$, $\alpha>-1$, when

$$f(t)=(1-t)^\alpha.$$

Some others families of special polynomials do not seem to be Appell sets since they are usually defined by a different type of generating function. However, Carlson in [14] highlights that sometimes it is possible to transform a given polynomial sequence into one of Appell type by suitable changes of variables. This is the case, for example, of the *Legendre polynomials* $\{P_n(x)\}_{n\geq 0}$ defined on the interval $(-1,1)$ whose generating function is [2]

$$J_0\left(t\sqrt{1-x^2}\right)\, e^{xt}=\sum_{n=0}^{+\infty}P_n(x)\,\frac{t^n}{n!},$$

where

$$J_0(y)=\sum_{n=0}^{+\infty}(-1)^n\,\frac{y^{2n}}{2^{2n}\,(n!)^2}$$

denotes the Bessel function of the first kind and index 0. However, setting

$$x=\frac{z}{\sqrt{z^2+1}},\qquad t=\tau\sqrt{z^2+1}, \tag{8}$$

we get

$$J_0\left(\tau\right)\, e^{z\tau}=\sum_{n=0}^{+\infty}P_n\left(\frac{z}{\sqrt{z^2+1}}\right)\,\sqrt{(z^2+1)^n}\,\frac{\tau^n}{n!},$$

[2] The generating functions of P_n, T_n, U_n here reported can be found in [25].

which is in the form (6).
Similarly, it can be verified that also the Chebyshev polynomials, both of the first and of the second kind, are Appell sequences. In fact, the *Chebyshev polynomials of the first kind* $\{T_n(x)\}_{n\geq 0}$ have as generating function

$$\cosh\left(t\sqrt{x^2-1}\right)\, e^{xt} = \sum_{n=0}^{+\infty} T_n(x)\, \frac{t^n}{n!}, \qquad x \in (-1,1), \tag{9}$$

and the *Chebyshev polynomials of the second kind* $\{U_n(x)\}_{n\geq 0}$ have as generating function

$$\frac{\sinh\left(t\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}\, e^{xt} = \sum_{n=0}^{+\infty} U_n(x)\, \frac{t^{n+1}}{(n+1)!}, \qquad x \in (-1,1). \tag{10}$$

Therefore, by using the transformations (8) in (9) as well as in (10) we get

$$\cosh(i\,\tau)\, e^{z\tau} = \sum_{n=0}^{+\infty} T_n\left(\frac{z}{\sqrt{z^2+1}}\right)\, \sqrt{(z^2+1)^n}\, \frac{\tau^n}{n!},$$

respectively

$$\frac{\sinh(i\,\tau)}{i}\, e^{z\tau} = \sum_{n=0}^{+\infty} U_n\left(\frac{z}{\sqrt{z^2+1}}\right)\, \sqrt{(z^2+1)^n}\, \frac{\tau^{n+1}}{(n+1)!}.$$

Consequently, by taking into account the trigonometric identities

$$\begin{aligned} \cosh(i\,\tau) &= \frac{e^{i\,\tau}+e^{-i\,\tau}}{2} = \cos\tau, \\ \frac{\sinh(i\,\tau)}{i} &= \frac{e^{i\,\tau}-e^{-i\,\tau}}{2\,i} = \sin\tau, \end{aligned}$$

it follows that the generating functions of the Chebyshev polynomials of the first and of the second kind are

$$\cos\tau\, e^{z\tau} = \sum_{n=0}^{+\infty} T_n\left(\frac{z}{\sqrt{z^2+1}}\right)\, \sqrt{(z^2+1)^n}\, \frac{\tau^n}{n!}$$

and

$$\operatorname{sinc}\tau\, e^{z\tau} = \sum_{n=0}^{+\infty} \frac{1}{n+1}\, U_n\left(\frac{z}{\sqrt{z^2+1}}\right)\, \sqrt{(z^2+1)^n}\, \frac{\tau^n}{n!},$$

respectively, where, as usual, $\operatorname{sinc}\tau = \sin\tau/\tau$.
Summarizing, by using the changes of variables (8) we get

- the *"modified" Legendre polynomials* $\{\sqrt{(z^2+1)^n}P_n(z/\sqrt{z^2+1})\}_{n\geq 0}$ where

$$f(\tau) = J_0(\tau),$$

is the Bessel function of the first kind and index 0;

- the *"modified" Chebyshev polynomials of the first kind* $\{\sqrt{(z^2+1)^n}T_n(z/\sqrt{z^2+1})\}_{n\geq 0}$ where

$$f(\tau) = \cos\tau = E_{2,1}(-\tau^2),$$

(see (7));

- the *"modified" Chebyshev polynomials of the second kind* $\{\frac{1}{n+1}\sqrt{(z^2+1)^n}U_n(z/\sqrt{z^2+1})\}_{n\geq 0}$ when

$$f(\tau) = \text{sinc}\,\tau = E_{2,2}(-\tau^2).$$

3 Appell polynomials: the matrix approach

As mentioned in the Introduction, our unified matrix approach basically relies on the properties of the *creation matrix* (1). It is worth to observe that it is a nilpotent matrix of degree $m+1$, i.e.,

$$H^s = O, \qquad \text{for all } s \geq m+1. \tag{11}$$

This property is one of the essential ingredients for the unified matrix approach to Appell polynomials that now follows.

In order to handle the Appell sequence $\{p_n(x)\}_{n\geq 0}$ in a closed form we introduce

$$\mathbf{p}(x) = [p_0(x)\ \ p_1(x)\ \ \cdots\ \ p_m(x)]^T, \tag{12}$$

hereafter called *Appell vector*.

Due to (2), the application of the creation matrix (1) implies that the Appell vector satisfies the differential equation

$$\frac{d}{dx}\mathbf{p}(x) = H\,\mathbf{p}(x), \tag{13}$$

whose general solution is

$$\mathbf{p}(x) = e^{xH}\,\mathbf{p}(0) \equiv P(x)\,\mathbf{p}(0), \tag{14}$$

with $P(x)$ defined by

$$(P(x))_{ij} = \begin{cases} \binom{i}{j} x^{i-j}, & i \geq j \\ 0, & \text{otherwise}, \end{cases} \qquad i, j = 0, 1, \ldots, m. \tag{15}$$

The matrix (15) is called *generalized Pascal matrix* because $P(1) \equiv P$ is the lower triangular *Pascal matrix* [2, 13] defined by

$$(P)_{ij} = \begin{cases} \binom{i}{j}, & i \geq j \\ 0, & \text{otherwise}, \end{cases} \qquad i, j = 0, 1, \ldots, m. \tag{16}$$

Notice that $P(0) \equiv I$ is the identity matrix.

Consider now the vector of monomial powers

$$\xi(x) = [1 \;\; x \;\; \cdots \;\; x^m]^T$$

and the matrix M defined by

$$(M)_{ij} = \begin{cases} \binom{i}{j} c_{i-j}, & i \geq j \\ 0, & \text{otherwise}, \end{cases} \qquad i, j = 0, 1, \ldots, m. \tag{17}$$

According to (4), we have

$$\mathbf{p}(x) = M\xi(x). \tag{18}$$

This relation motivates the following definition.

Definition 1 *The matrix M defined by (17) is called the* transfer matrix *for the Appell vector (12).*

Obviously,

$$\mathbf{p}(0) = M\xi(0) = [c_0 \;\; c_1 \;\; \cdots \;\; c_m]^T. \tag{19}$$

Therefore, from (14) and (18) we conclude that to obtain the different kinds of Appell polynomials we need to specify the entries of $\mathbf{p}(0)$ or, equivalently, of the transfer matrix M. At this aim, a powerful tool is given by the following result.

Theorem 1 *Let H be the creation matrix defined by (1). If $G(x,t) \equiv f(t)e^{xt}$ is the generating function of an Appell sequence $\{p_n(x)\}_{n\geq 0}$, then the transfer matrix M is a nonsingular matrix equal to $f(H)$.*

Proof. Since (see (6))

$$G(x,t)=\sum_{n=0}^{+\infty}p_n(x)\frac{t^n}{n!},$$

setting $x=0$, we obtain

$$f(t)=\sum_{n=0}^{+\infty}p_n(0)\frac{t^n}{n!}.$$

Substituting t by H and by taking into account (5) and (11) we get

$$f(H)=\sum_{n=0}^{m}c_n\frac{H^n}{n!},\qquad c_0\neq 0.$$

Furthermore, denoting by $\mathbf{e}_s$, $s=0,1,\ldots,m$, the standard unit basis vector in $\mathbb{R}^{m+1}$ and adopting the convention that $\mathbf{e}_s=\mathbf{0}$ whenever $s>m$ (null vector), then the entries of $f(H)$ are obtained by

$$\begin{aligned}(f(H))_{ij} &= \sum_{n=0}^{m}\frac{c_n}{n!}\mathbf{e}_i^T H^n\mathbf{e}_j=\sum_{n=0}^{m}\frac{c_n}{n!}(j+1)^{(n)}\mathbf{e}_i^T\mathbf{e}_{j+n} \\ &= \sum_{n=0}^{m}c_n\frac{(j+1)^{(n)}}{n!}\delta_{i,j+n},\end{aligned}\tag{20}$$

where $\delta_{i,j}$ is the Kronecker symbol and $(j+1)^{(n)}=(j+1)(j+2)\cdots(j+n)$ is the ascending factorial with $(j+1)^{(0)}:=1$. Thus, $(f(H))_{ij}=0$ if $i<j$, and when $i=j+n$, that is $i\geq j$,

$$(f(H))_{ij}=c_{i-j}\frac{i!}{(i-j)!j!}=\binom{i}{j}c_{i-j}$$

which completes the proof. □

Taking into account the previous theorem, we achieve for the particular cases referred in Section 2 the corresponding transfer matrices.

(i) For the monomials $\{x^n\}_{0\leq n\leq m}$,

$$M=I,$$

where I is the identity matrix of order $m+1$.

(ii) For the Bernoulli polynomials $\{B_n(x)\}_{0\le n\le m}$,

$$M = \left(\sum_{n=0}^{m} \frac{H^n}{(n+1)!}\right)^{-1}. \tag{21}$$

(iii) For the Euler polynomials $\{E_n(x)\}_{0\le n\le m}$,

$$M = 2(e^H + I)^{-1} = 2(P + I)^{-1}, \tag{22}$$

where P is the Pascal matrix of order $m+1$ (see (16)).

(iv) For the monic Hermite polynomials $\{\widehat{H}_n(x)\}_{0\le n\le m}$,

$$M = e^{-\frac{H^2}{4}} = \sum_{n=0}^{m} \frac{(-1)^n H^{2n}}{2^{2n} n!}. \tag{23}$$

In this case, introducing the diagonal matrix

$$D[\ell] = \mathrm{diag}[\ell^0, \ell^1, \ell^2, \ldots, \ell^m], \qquad \ell \neq 0, \tag{24}$$

and $\mathbf{H}(x) = [H_0(x) \;\; H_1(x) \;\; \cdots \;\; H_m(x)]^T$, the vector of the classical Hermite polynomials, we get

$$(D[2])^{-1}\, \mathbf{H}(x) = M\xi(x) \quad \Leftrightarrow \quad \mathbf{H}(x) = D[2]M\xi(x).$$

(v) For the "modified" generalized Laguerre polynomials

$$\{(-1)^n n! L_n^{(\alpha-n)}(x)\}_{0\le n\le m}$$

we have

$$M = (I - H)^{\alpha}.$$

It is worth noting that, this matrix point of view provides an easy way to relate $\{(-1)^n n! L_n^{(\alpha-n)}(x)\}_{0\le n\le m}$ with the generalized Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{0\le n\le m}$ [25]. In fact, introducing the vector

$$\mathbf{L}(x) = [L_0^{(\alpha)}(x) \;\; L_1^{(\alpha-1)}(x) \;\; \cdots \;\; L_m^{(\alpha-m)}(x)]^T$$

and the diagonal matrix

$$D_f = \mathrm{diag}[0!, 1!, 2!, \cdots, m!],$$

we obtain (see (24))

$$D[-1]D_f\mathbf{L}(x) = (I-H)^\alpha \xi(x)$$

or, equivalently,

$$\mathbf{L}(x) = (D_f)^{-1} D[-1](I-H)^\alpha \xi(x). \tag{25}$$

In addition, the recurrence relation reported in [25]

$$L_n^{(\alpha)}(x) = L_{n-1}^{(\alpha)}(x) + L_n^{(\alpha-1)}(x), \quad n>0,$$

gives, successively,

$$\begin{aligned}
L_1^{(\alpha-1)}(x) &= L_1^{(\alpha)}(x) - L_0^{(\alpha)}(x)\\
L_2^{(\alpha-2)}(x) &= L_2^{(\alpha-1)}(x) - L_1^{(\alpha-1)}(x)\\
&= (L_2^{(\alpha)}(x) - L_1^{(\alpha)}(x)) - (L_1^{(\alpha)}(x) - L_0^{(\alpha)}(x))\\
&= L_2^{(\alpha)}(x) - 2L_1^{(\alpha)} + L_0^{(\alpha)}(x)\\
&\vdots\\
L_m^{(\alpha-m)}(x) &= \sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}L_n^{(\alpha)}(x).
\end{aligned}$$

Denoting by $\mathcal{L}(x) = [L_0^{(\alpha)}(x) \ \ L_1^{(\alpha)}(x) \ \cdots \ L_m^{(\alpha)}(x)]^T$ the vector of the first $m+1$ generalized Laguerre polynomials and taking into account (15), we have

$$\mathbf{L}(x) = P(-1)\mathcal{L}(x).$$

Finally, from (25) it follows that

$$\mathcal{L}(x) = P(D_f)^{-1}D[-1](I-H)^\alpha \xi(x).$$

In particular, when $\alpha = 0$ we get the ordinary *Laguerre polynomials.*

(vi) For the modified Legendre polynomials $\{\sqrt{(z^2+1)^n}P_n(z/\sqrt{z^2+1})\}_{0\le n\le m}$ we get

$$M = J_0(H) = \sum_{n=0}^{m}(-1)^n \frac{H^{2n}}{2^{2n}(n!)^2}.$$

(vii) For the modified Chebyshev polynomials of the first kind $\{\sqrt{(z^2+1)^n}T_n(z/\sqrt{z^2+1})\}_{0\le n\le m}$,

$$M = \cos H = \sum_{n=0}^{m}(-1)^n \frac{H^{2n}}{(2n)!}.$$

(viii) For the modified Chebyshev polynomials of the second kind $\{\frac{1}{n+1}\{\sqrt{(z^2+1)^n}U_n(z/\sqrt{z^2+1})\}_{0\le n\le m}$,

$$M \equiv M_{\mathbf{U}} = \sum_{n=0}^{m}(-1)^n \frac{H^{2n}}{(2n+1)!}. \tag{26}$$

Of course, considering

$$z = \frac{x}{\sqrt{1-x^2}}, \qquad x \in (-1,1),$$

we obtain the first $m+1$ elements of the classical sequences $\{P_n(x)\}_{n\ge 0}$, $\{T_n(x)\}_{n\ge 0}$ and $\{U_n(x)\}_{n\ge 0}$. Collecting these elements into the vectors $\mathbf{P}(x)$, $\mathbf{T}(x)$, and $\mathbf{U}(x)$, respectively, we get

$$\begin{aligned}
\mathbf{P}(x) &= D[\sqrt{1-x^2}]\, J_0(H) D^{-1}[\sqrt{1-x^2}]\, \xi(x),\\
\mathbf{T}(x) &= D[\sqrt{1-x^2}]\, \cos H D^{-1}[\sqrt{1-x^2}]\, \xi(x),\\
\mathbf{U}(x) &= D_{m+1} D[\sqrt{1-x^2}]\, M_{\mathbf{U}} D^{-1}[\sqrt{1-x^2}]\, \xi(x),
\end{aligned}$$

where $D_{m+1} = \mathrm{diag}[1,2,\ldots,m+1]$.

Remark 2 *The matrix $M_{\mathbf{U}}$ given in (26) satisfies $HM_{\mathbf{U}} = \sin H$.*

4 Some properties of Appell polynomials

In order to establish several identities of Appell polynomials in a friendly and unified way, we now use the transfer matrix and some properties of the generalized Pascal matrix. It is worth mentioning that some of them were derived in [15, 16] by using the determinantal approach.

Lemma 1 *Let $P(x)$ be the generalized Pascal matrix and $\xi(x)$ the vector containing the ordinary monomials as previously defined. Then,*

$$\xi(x+y) = P(x)\,\xi(y), \qquad \forall\, x,y \in \mathbb{R}.$$

Proof. The result is a consequence of the binomial theorem. In fact, (see (15))

$$(\xi(x+y))_i = (x+y)^i = \sum_{k=0}^{i} \binom{i}{k} x^{i-k}\, y^k = (P(x)\,\xi(y))_i \,.$$

□

According to Carlson [14, Theorem 1, p. 545], it is known that if $\{p_n(x)\}_{n\geq 0}$ is an Appell sequence, then it satisfies a binomial theorem of the form

$$p_n(x+y) = \sum_{k=0}^{\infty} \binom{n}{k} p_k(x) y^{n-k}, \qquad n = 0, 1, \ldots . \tag{27}$$

This property is necessary but not sufficient to define an Appell sequence in the sense of (2)-(3). In fact, $\{p_n(x)\}_{n\geq 0}$ verifies (27) if and only if

$$\frac{d}{dx} p_n(x) = n p_{n-1}(x), \quad n = 0, 1, \ldots, \tag{28}$$

where the right hand-side is taken to be zero in the case of $n = 0$. This means that the binomial theorem property does not require the condition that $p_n(x)$ should be exactly of degree n as we mentioned in the beginning of Section 2 as consequence of (2)-(3).

Remark 3 *The sequence of the* Genocchi polynomials, *whose generating function is* [3]

$$f(t)e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \qquad f(t) = \frac{2t}{e^t + 1}, \tag{29}$$

is an example of a sequence that satisfies (27). Nevertheless, by virtue of (28) and (29), it is possible to get the corresponding transfer matrix $M = M_{\mathbf{G}}$ in the same way as in the Section 3 for the case of Appell polynomials, but in this case M is singular. Actually, from Theorem 1,

$$M_{\mathbf{G}} = 2H(e^H + I)^{-1} = 2H(P + I)^{-1}.$$

Now, from (27) we can derive the following result.

Theorem 2 *Let $\{p_n(x)\}_{n\geq 0}$ be an Appell sequence and $P(x)$ the generalized Pascal matrix defined by (15). For the corresponding Appell vector we have*

$$\mathbf{p}(x+y) = P(y)\,\mathbf{p}(x), \qquad \forall\, x, y \in \mathbb{R}. \tag{30}$$

[3] The expression of the function $f(t)$ for the Genocchi polynomials can be found in [20].

Proof. From (18) and Lemma 1 one has

$$\mathbf{p}(x+y) = M\xi(x+y) = MP(y)\,\xi(x).$$

The proof is completed by observing that $P(y)$ and M commute since they are both functions of the creation matrix H. □

Corollary 1 *Let $\{p_n(x)\}_{n\geq 0}$ be an Appell sequence. Then, for any constant a and for all $x \in \mathbb{R}$, the Appell vector of the given sequence satisfies the following relations:*

(i) forward difference:

$$\mathbf{p}(x+1) - \mathbf{p}(x) = (P - I)\,\mathbf{p}(x);$$

(ii) multiplication theorem:

$$\mathbf{p}(ax) = P((a-1)\,x)\,\mathbf{p}(x), \tag{31}$$

$$\mathbf{p}(ax) = MD[a]\,\xi(x), \tag{32}$$

where $D[a]$ is defined by (24).

Proof.

(i) The result follows from (30) with $y = 1$ and by recalling that $P(1) \equiv P$.

(ii) The relation (31) can be immediately deduced from (30) with $y = (a-1)x$. Concerning (32), it follows from (18) and by observing that $\xi(ax) = D[a]\,\xi(x)$. □

It is worth noting that (31) generalizes, for all kinds of Appell polynomials, the well-known properties for the Bernoulli and the Euler polynomials (see, e.g., [1, 15, 23])

$$B_n(ax) = \sum_{i=0}^{n} \binom{n}{i} B_i(x)(a-1)^{n-i}x^{n-i},$$

$$E_n(ax) = \sum_{i=0}^{n} \binom{n}{i} E_i(x)(a-1)^{n-i}x^{n-i}.$$

In addition, there are some identities involving Appell polynomials which at a first glance are not equivalent. To provide an example we consider the following ones involving the Bernoulli polynomials:

$$B_n(1-x) = (-1)^n B_n(x), \qquad B_n(1) = (-1)^n B_n(0).$$

It is trivial to check that the relation on the left hand-side implies the one on the right, while it is not clear that the opposite implication also holds true. Similar arguments can be used about the following relations involving Euler polynomials:

$$E_n(1-x) = (-1)^n E_n(x), \qquad E_n(1) = (-1)^n E_n(0).$$

However, the referred equivalences will become evident from the next theorem.

Theorem 3 *Let $\{p_n(x)\}_{n\geq 0}$ be an Appell sequence. For the corresponding Appell vector the following equivalence holds*

$$(\, \mathbf{p}(h-x) = D[-1]\,\mathbf{p}(x),\, \forall\, h, x \in \mathbb{R}\,) \Leftrightarrow (\, \mathbf{p}(h) = D[-1]\,\mathbf{p}(0),\, \forall\, h \in \mathbb{R}\,), \tag{33}$$

where $D[-1]$ is defined by (24).

Proof. ($\Rightarrow$) This implication is trivial from the hypothesis with $x = 0$.
($\Leftarrow$) Using (30) and by observing that $P(-x) = D[-1]P(x)D[-1]$ (see (15)) we get

$$\begin{aligned} \mathbf{p}(h-x) &= P(-x)\mathbf{p}(h) = D[-1]P(x)D[-1]D[-1]\,\mathbf{p}(0) \\ &= D[-1]P(x)\,\mathbf{p}(0) = D[-1]\,\mathbf{p}(x). \end{aligned}$$

□

It is interesting to ask for the consequences of Theorem 3 for the coefficients of the Appell polynomials.

Corollary 2 *Let $\{p_n(x)\}_{n\geq 0}$ be an Appell sequence. For the Appell vector we get*

$$(\, \mathbf{p}(-x) = D[-1]\,\mathbf{p}(x), \quad \forall\, x \in \mathbb{R}\,) \quad \Leftrightarrow \quad (\, c_{2n+1} = 0, \quad n = 0, 1, \ldots\,).$$

Proof. From Theorem 3, by fixing $h = 0$ one has that

$$(\, \mathbf{p}(-x) = D[-1]\,\mathbf{p}(x), \quad \forall\, x \in \mathbb{R}\,) \quad \Leftrightarrow \quad (\, \mathbf{p}(0) = D[-1]\,\mathbf{p}(0)\,).$$

The relation on the right of this equivalence together with the fact that $\mathbf{p}(0) = [c_0 \;\; c_1 \;\; \cdots \;\; c_m]^T$ (see (19)) lead to the desired result. □

We observe that the polynomials of Hermite, Legendre and Chebyshev of the first and of the second kind belong to the Appell class and verify the equivalence of the previous corollary. This is due to the fact that their transfer matrices are defined as expansions of even powers of H whose entries satisfy (see (20))

$$\mathbf{e}_i^T H^{2n} \mathbf{e}_j = (j+1)^{(2n)} \mathbf{e}_i^T \mathbf{e}_{j+2n} = 0, \qquad i-j \neq 2n.$$

Among the properties of Appell polynomials we now refer to the ones stated in Theorem 11 of [15], which were proved applying a determinantal approach. Here we propose an alternative proof based on our matrix approach and making use of the transfer matrix.

Theorem 4 *Let $\{v_n(x)\}_{n\geq 0}$ and $\{u_n(x)\}_{n\geq 0}$ be two sequences of Appell polynomials and $\mathbf{v}(x)$ and $\mathbf{u}(x)$ their corresponding Appell vectors. Then,*

- (i) *for all $\lambda, \mu \in \mathbb{R}$, $\lambda\mathbf{v}(x) + \mu\mathbf{u}(x)$ is an Appell vector for the sequence $\{\lambda v_n(x) + \mu u_n(x)\}_{n\geq 0}$;*
- (ii) *replacing in $v_n(x)$ the powers $x^0, x^1, \ldots, x^n$ by $u_0(x), u_1(x), \ldots, u_n(x)$ and denoting the resulting polynomial by $w_n(x)$, the vector $\mathbf{w}(x) = [w_0(x)\, w_1(x)\, \ldots\, w_m(x)]^T$ is the Appell vector of $\{w_n(x)\}_{n\geq 0}$.*

Proof. Let M_v and M_u be the transfer matrices for $\mathbf{v}(x)$ and $\mathbf{u}(x)$, respectively, i.e.,

$$\mathbf{v}(x) = M_v \xi(x), \qquad \mathbf{u}(x) = M_u \xi(x).$$

Then,

- (i) $\lambda\mathbf{v}(x) + \mu\mathbf{u}(x) = (\lambda M_v + \mu M_u)\, \xi(x)$,
- (ii) $\mathbf{w}(x) = M_v \mathbf{u}(x)$.

To prove that $\lambda\mathbf{v}(x) + \mu\mathbf{u}(x)$ and $\mathbf{w}(x)$ are Appell vectors, we need to check if they satisfy the relation (13). The result follows by recalling that M_v and M_u are both functions of H (see Theorem 1). Consequently, they commute with H as well any of their linear combination. □

In particular, the last theorem allows us to obtain in a straightforward way some classes of Appell polynomials recently introduced in [18]. In fact, the transfer matrix

- (i) for the 2-iterated Bernoulli polynomials $\{B_n^{[2]}(x)\}_{0\leq n\leq m}$ is (see (21))

$$M = \left(\sum_{n=0}^{m} \frac{H^n}{(n+1)!} \right)^{-2};$$

(*ii*) for the 2-iterated Euler polynomials $\{E_n^{[2]}(x)\}_{0\leq n\leq m}$ is (see (22))

$$M = 4(e^H + I)^{-2} \equiv 4(P+I)^{-2};$$

(*iii*) for the Bernoulli-Euler polynomials $\{{}_EB_n(x)\}_{0\leq n\leq m}$ (or Euler-Bernoulli polynomials $\{{}_BE_n(x)\}_{0\leq n\leq m}$) is (see (21) and (22))

$$M = 2\left((P+I)\sum_{n=0}^{m}\frac{H^n}{(n+1)!}\right)^{-1}.$$

Some other properties of Appell polynomials can be obtained by making use of the inverse of the transfer matrix. To achieve them, we recall that (see Theorem 1 and (5))

$$M = \sum_{k=0}^{m} c_k \frac{H^k}{k!}, \qquad c_0 \neq 0.$$

Setting

$$M^{-1} = \sum_{k=0}^{m} \gamma_k \frac{H^k}{k!}, \tag{34}$$

we have

$$\gamma_0 = \frac{1}{c_0}, \qquad \gamma_k = -\frac{1}{c_0}\sum_{s=0}^{k-1}\binom{k}{s} c_{k-s}\gamma_s, \qquad k = 1, 2, \ldots, m. \tag{35}$$

In fact,

$$\begin{aligned} I &= MM^{-1} = \left(\sum_{k=0}^{m} c_k \frac{H^k}{k!}\right)\left(\sum_{r=0}^{m}\gamma_r\frac{H^r}{r!}\right) = \sum_{n=0}^{m}\left(\sum_{k+r=n} c_k\gamma_r\frac{H^{k+r}}{k!r!}\right) \\ &= \sum_{n=0}^{m}\left(\sum_{r=0}^{n}\frac{n!c_{n-r}\gamma_r}{(n-r)!r!}\right)\frac{H^n}{n!} = \sum_{n=0}^{m}\left(\sum_{r=0}^{n}\binom{n}{r}c_{n-r}\gamma_r\right)\frac{H^n}{n!}. \end{aligned}$$

Consequently, (18) implies that

$$M^{-1}\mathbf{p}(x) = \xi(x)$$

or, equivalently,

$$\sum_{k=0}^{n}\binom{n}{k}\gamma_{n-k}\, p_k(x) = x^n, \qquad n = 0, 1, \ldots, m,$$

from which we deduce a *general recurrence relation* for Appell polynomials:

$$p_n(x) = \frac{1}{\gamma_0}\left(x^n - \sum_{k=0}^{n-1}\binom{n}{k}\gamma_{n-k}\,p_k(x)\right), \qquad n = 0, 1, \ldots .$$

By taking into account (34), it is an easy matter to notice from (21), (22) and (23), that

- for the Bernoulli polynomials:

$$\gamma_k = \frac{1}{k+1}, \qquad k = 0, 1, \ldots, m;$$

- for the Euler polynomials:

$$\gamma_0 = 1, \quad \gamma_k = \frac{1}{2}, \qquad k = 1, \ldots, m;$$

- for the monic Hermite polynomials:

$$\gamma_k = \begin{cases} \frac{1}{2^k}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}, \qquad k = 0, 1, \ldots, m.$$

Furthermore, for the *generalized Euler polynomials* introduced in [15] we have

$$\gamma_0 = 1, \quad \gamma_k = \bar{\gamma}, \qquad k = 1, \ldots, m,$$

which leads to

$$M^{-1} = I + \sum_{k=1}^{m}\bar{\gamma}\,\frac{H^k}{k!} = (1-\bar{\gamma})I + \sum_{k=0}^{m}\bar{\gamma}\,\frac{H^k}{k!} = (1-\bar{\gamma})\,I + \bar{\gamma}\,P.$$

Knowing the relationship between the coefficients of M and its inverse, we can prove the following result which relates the coefficients of an Appell polynomial with those of the general recurrence relation.

Proposition 1 *The elements of the sets $\{c_n\}_{0\le n\le m}$ and $\{\gamma_n\}_{0\le n\le m}$ characterizing the transfer matrix M and its inverse, respectively, satisfy the following equivalences:*

$$c_{2j+1} = 0 \quad \Leftrightarrow \quad \gamma_{2j+1} = 0, \qquad j = 0, 1, \ldots, \frac{m-1}{2}.$$

Proof. We proceed by induction on j. If $j = 0$, the statement is verified directly using (35). Next, let us suppose that it is true for $j-1$ and we prove that it is true also for j. From (35)

$$\begin{aligned}\gamma_{2j+1} = & -\frac{1}{c_0}\sum_{s=0}^{2j}\binom{2j+1}{s}c_{2j+1-s}\gamma_s = -\frac{1}{c_0}c_{2j+1}\gamma_0 + \\ & -\frac{1}{c_0}\sum_{\substack{1\leq s\leq 2j-1\\ s\ odd}}\binom{2j+1}{s}c_{2j+1-s}\gamma_s - \frac{1}{c_0}\sum_{\substack{2\leq s\leq 2j\\ s\ even}}c_{2j+1-s}\binom{2j+1}{s}\gamma_s.\end{aligned}$$

By using the induction hypothesis the last two sums vanish, and this completes the proof. □

5 Conclusion

For almost all classical polynomials defined in the ordinary way as, for instance, in [25], the corresponding generating functions are well known. In some cases, like Bernoulli polynomials or Euler polynomials, the usual generating functions are already given in a form that reveals their property of being Appell polynomials due to the inclusion of the exponential function (see (6) and [8]).

But this is not the case for all classical polynomials, like Legendre or Chebyshev (both of the first and second kind) polynomials. In these cases, this paper shows how the Appell polynomial nature can be disclosed by some substitution in a way that they can be treated as such. In this sense the paper tries to call attention also to important and well known polynomials that normally are not known as Appell polynomials.

Being the central ingredients of the presented unified matrix approach to Appell polynomials, the roles of the creation matrix H as well as of the transfer matrix M are studied.

Furthermore, the paper confirmed the effectiveness of the unified matrix representation by showing that some new types of recently introduced Appell polynomials can immediately be deduced.

Finally, the special role of the transfer matrix is also stressed and advantageously used for deriving, in an easy and compact way, the relationship between the coefficients of Appell polynomials and their general recurrence relations.

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