Proceedings of the Tenth International Research Conference on Fibonacci numbers and their Applications,
Applications of Fibonacci numbers 2004, vol.9: 31-38

UNIVERSAL BERNOULLI POLYNOMIALS AND P-ADIC CONGRUENCES

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1. INTRODUCTION

The *classical Bernoulli numbers* B_n are defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.$$

A fundamental property of these numbers is the Clausen-von Staudt result that the denominator of B_n is the product of distinct primes.

The *classical* l^{th} *order Bernoulli numbers* $B_n^{(l)}$ are defined by

$$\left(\frac{t}{e^t-1}\right)^l=\sum_{n=0}^{\infty}B_n^{(l)}\frac{t^n}{n!},$$

and the *classical* l^{th} *order Bernoulli polynomials* $B_n^{(l)}(x)$ are defined by

This paper is in final form and no version of it will be submitted for publication elsewhere.

F.T. Howard (ed.), Applications of Fibonacci Numbers, Volume 9: Proceedings of the Tenth International Research Conference on Fibonacci Numbers and their Applications, 1–8.

$$\left(\frac{t}{e^t-1}\right)^l e^{tx} = \sum_{n=0}^{\infty} B_n^{(l)}(x)\frac{t^n}{n!},$$

which is the associated Appel sequence.

The special case $l = 1$ gives the ordinary Bernoulli numbers and polynomials, and $B_n^{(l)}(0) = B_n^{(l)}$. It is easy to show that if $B_n^{(l)}(x) = \sum_{i=0}^n a_i x^{n-i}$ then $a_i = \binom{n}{i} B_i^{(l)}$, which is the characteristic Appel property.

L. Carlitz proved in [5, Theorem 14] and gave a more direct proof in [6, Theorem A] that if the base p representation of a positive integer l has r non-zero digits, then the highest power of the prime p in the denominator of $B_n^{(l)}$ is at most r, i.e. $p^r B_n^{(l)} \in \mathbf{Z}_p$. The proofs involve the complicated theory of Hurwitz series. The above Appel formula then shows that the same result holds for all coefficients of $B_n^{(l)}(x)$.

F. Clarke defined universal Bernoulli numbers $\hat{B}_n$ in [7] which depend on parameters $c_1, c_2, \ldots$. They are important for studying universal formal groups. We generalized these numbers in [3] to universal l^{th} order Bernoulli numbers $\hat{B}_n^{(l)}$. Our principal tool for the study of these numbers is the use of certain explicit representations coming from Lagrange inversion and related to the theory of partitions, which we first noted in [1].

In this paper we define *universal l^{th} order Bernoulli polynomials* $\hat{B}_n^{(l)}(x)$ which generalize the classical polynomials for $c_i = (-1)^i$, using the Lagrange inversion terms. We do not have a generating function for these polynomials, nor are the coefficients simple functions of l^{th} order universal Bernoulli numbers, but it is still true that $\hat{B}_n^{(l)}(0) = \hat{B}_n^{(l)}$. Other specializations that may be of interest in the context of this conference are to the Fibonacci ($c_i = F_{i+1}$) or Lucas ($c_i = L_{i+1}$) formal groups.

The basis for our definition of the universal Bernoulli polynomials is the classical inversion formula [1]

$$\frac{B_n^{(l)}(x)}{n!} = [t^n](1+t)^{x-1}\left(\frac{\log(1+t)}{t}\right)^{l-n-1}.$$

We have taken a concrete, combinatorial generalization of this formula as the definition (2), but an equivalent umbral formulation is also given by (6).

It should be noted that our universal l^{th} order Bernoulli polynomials are not an Appel sequence, e.g., they are not monic. In [10] N. Ray defined universal first order Bernoulli polynomials as an Appel sequence. This definition has some good functorial properties, but we do not believe that his polynomials are as useful as ours. In particular, our polynomials are isobaric in the variables $c_1, c_2, \ldots$, while his are not, and ours satisfy Kummer congruences for p-adic integer values and his do not (Theorem 4).

We prove that the Carlitz bound noted above holds for the coefficients of $\hat{B}_n^{(l)}(x)$ (Theorem 3) and show that the bound is attainable, and in fact, $\nu\left(\hat{B}_n^{(l)}\right) = -r$ for suitable n. We believe that our proofs are conceptually simpler than Carlitz's, and show his result is really a special case of our Theorem 1. This is the first publication of our simple proof.

Since our analysis of the p-adic pole structure of $B_n^{(l)}(x)$ was based entirely on the Lagrange inversion terms, the statements and proofs carry over essentially without change to the universal polynomials (Theorems 1, 2). In particular, the down-sloping portion of the Newton polygon of $\hat{B}_n^{(l)}(x)$ is identical with that of $B_n^{(l)}(x)$, and indeed with all specializations where the c_i are p-adic units.

Finally, we consider the universal (ordinary) Bernoulli polynomials, where $l = 1$. We show that if $a \in \mathbf{Z}_p$ and $p - 1 \nmid n$ then $\hat{B}_n(a)/n \in \mathbf{Z}_p[c_1, c_2, \ldots, c_n]$, and extend the Kummer congruences for $n \not\equiv 0,\ 1 (\text{mod } p-1)$ to the universal case (Theorem 4) for values of $\hat{B}_n(x)/n$. This generalizes our work on universal Kummer congruences in [3, 4].

2. UNIVERSAL ARBITRARY ORDER BERNOULLI POLYNOMIALS

Consider the formula from [1, §3]

$$B_n^{(l)}(x) = n! \sum_{i=0}^{n} (-1)^{n-i} \binom{x-1}{i} \sum_{w(u)=n-i} \frac{\binom{s}{d}\binom{d}{u}}{\Lambda^u} \tag{1}$$

where $s = l - n - 1$, the summation is over all non-negative integer sequences $(u) = (u_1, u_2, \ldots, u_n)$, with $w(u) = \sum_i iu_i$ and $d = d(u) = \sum_i u_i$,

$$\binom{d}{u} = \binom{d}{u_1\ u_2 \ldots u_n}, \ \Lambda^u = 2^{u_1} 3^{u_2} \ldots (n+1)^{u_n}, \text{ and } \binom{s}{d} \text{ and } \binom{x-1}{i}$$

are binomial coefficients.

Note that (u) is a partition of $w(u)$ into $d(u)$ parts, where u_i = number of occurrences of i in the partition. With the same notations, let $\tau_u(s) = \binom{s}{d}\binom{d}{u}/\Lambda^u$.

Definition: Let $c_1, c_2, \ldots$ be indeterminates over $\mathbf{Q}$, let $c_0 = 1$, and let $c^u = c_1^{u_1} c_2^{u_2} \ldots c_n^{u_n}$. Then

$$\hat{B}_n^{(l)}(x) = n! \sum_{i=0}^{n} (-1)^i c_i \binom{x-1}{i} \sum_{w(u)=n-i} \tau_u(l-n-1) c^u. \tag{2}$$

From (1), the specialization $c_i = (-1)^i$ gives the classical $B_n^{(l)}(x)$.

Some low degree examples are $\hat{B}_0^{(l)}(x) = 1, \hat{B}_1^{(l)}(x) = -c_1(x - l/2)$, and $\hat{B}_2^{(l)}(x) = c_2x^2 - (3c_2 + (l-3)c_1^2)x + (2lc_2/3 + l(l-3)c_1^2/4)$.

Note that all coefficients of $\hat{B}_n^{(l)}(x)$ are isobaric polynomials of weight n in $\mathbf{Q}[l][c_1, c_2, \ldots, c_n]$, where $wt(c_i) = i$, and the highest coefficient is $(-1)^n c_n$. For most of our applications, l will be a positive integer, a p-adic integer, or a variable.

The critical cases are when l is an integer in the range $0 \le l \le n+1$. For $l = n+1$, obviously

$$\hat{B}_n^{(n+1)}(x) = (-1)^n c_n (x-1)_n = (-1)^n c_n (x-1)(x-2)\ldots(x-n). \tag{3}$$

In the classical case $B_n^{(0)}(x) = x^n$, but there is no corresponding simple formula for $\hat{B}_n^{(0)}(x)$. It is true that $\hat{B}_n^{(0)}(0) = 0$ if $n > 0$, and we will show (Corollary 2 to Theorem 3) that all coefficients of $\hat{B}_n^{(0)}(x)$ are in $\mathbf{Z}[c_1, c_2, \ldots, c_n]$.

The classical $B_n^{(l)}(x)$ is skew-symmetric about $x = l/2$, but there is no symmetry in the universal case, and no obvious root if $n > 1$ and n is odd, unlike the classical case.

From [3, Corollary 2.3], we get the key formulas

$$\hat{B}_n^{(l)}(0) = \hat{B}_n^{(l)} \tag{4}$$

and

$$\hat{B}_n^{(l+1)}(1) = \frac{l-n}{l} \hat{B}_n^{(l)}. \tag{5}$$

There is an umbral way of representing $\hat{B}_n^{(l)}(x)$, namely let $F(t) = \sum_{i=0}^{\infty} c_i t^{i+1}/(i+1)$ be the logarithm of the universal formal group law, so $(t/F^{-1}(t))^l$ generates the universal l^{th} order Bernoulli numbers [3]. Then

$$\frac{\hat{B}_n^{(l)}(x)}{n!} = [t^n](1 - Ct)^{x-1} \left(\frac{F(t)}{t}\right)^{l-n-1}, \tag{6}$$

where $C^i = c_i$.

We do not take (6) as the definition, because of the presence of the index n in the "umbral generating function." We suspect that an umbral form of Lagrange inversion is involved. Our definition gives us the concrete terms that we need for analysis.

If J is the ideal generated by all $c_i - c_1^i$ in $\mathbf{Q}[l][c_1, \dots, c_n]$, the umbral representation leads naturally to the easily verified recursive congruence

$$\hat{B}_n^{(l)}(x+1) - \hat{B}_n^{(l)}(x) \equiv -nc_1\hat{B}_{n-1}^{(l-1)}(x) \pmod{J}.$$

However, since the specialization $c_i = c_1^i$ clearly implies $\hat{B}_n^{(l)}(x) = (-c_1)^n B_n^{(l)}(x)$, this is equivalent to the classical recursion

$$B_n^{(l)}(x+1) - B_n^{(l)}(x) = nB_{n-1}^{(l-1)}(x).$$

3. P-ADIC CONSIDERATIONS

Let p be prime. Extend the standard p-adic valuation $\nu = \nu_p$ of $\mathbf{Q}_p$ to $\mathbf{Q}_p[c_1, c_2, \cdots]$ by $\nu(\sum a_I c^I) = \min\{\nu(a_I)\}$, so in particular a polynomial $f(c)$ is integral if and only if all coefficients of f are integral. If $b > 0$ and $\nu(f(c)) = -b$, we say $f(c)$ has a *pole of order* b.

Since the analysis of the pole structure of $B_n^{(l)}(x)$ carried out in [1] entirely involved single terms $\tau_u(s)$ where (u) is concentrated in place $p-1$, i.e., such that $u_i = 0$ if $i \neq p-1$, all the results carry over without change to $\hat{B}_n^{(l)}(x)$. For ease of reference, we summarize these results using the versions stated in [2].

Let $n = \sum_{i=0}^{m} a_i p^i$ be the base p expansion of n. Then $S(n) = \sum_i a_i$ is the digit sum. If $S(n) \geq p-1$, then $N(n)$ is the smallest $t > 0$ such that $p-1|t$ and $p\nmid\binom{n}{t}$. Thus $N(n)$ is the smallest t such that $p\nmid\binom{n}{t}$ and $S(t) = p-1$.

Theorem 1: Let $l \in \mathbf{Z}_p$. If $S(n) < p-1$, then the coefficients of $\hat{B}_n^{(l)}(x)$ have no poles. If $S(n) \geq p-1$, the successively higher order poles of the coefficients, from degree n down, are determined as follows: the first pole is simple (order one) and occurs in degree $n - N_1$, where N_1 is minimal satisfying $N_1 = N(n - l_1)$ for some bottom segment l_1 of n (possibly $l_1 = 0$) such that $p\nmid\binom{l-n-1}{N_1/(p-1)}$. Similarly, the next higher order pole is double, and occurs in degree $n - N_1 - N_2$ where N_2 is minimal satisfying $N_2 = N(n - l_1 - N_1 - l_2)$ for some bottom segment l_2 of $n - l_1 - N_1$ such that $p\nmid\binom{l-n-1}{N_2/(p-1)}$, etc.

If $f(x) = \sum_{i=0}^{n} a_i x^{n-i} \in \mathbf{Q}_p[c_1, c_2, \ldots][x]$, consider the *spots* $(i, \nu(a_i))$ where $a_i \neq 0$ as lattice points in the (x,y)-plane. The *Newton polygon* of $f(x)$ is the lower boundary of the convex null of the set of spots [2].

The preceding theorem characterizes the downward-sloping portion of the Newton polygon of $\hat{B}_n^{(l)}(x)$ as follows.

Theorem 2: Let $l \in \mathbf{Z}_p$. The negative slope sides of the Newton polygon of $\hat{B}_n^{(l)}(x)$ all satisfy $\Delta y_i = -1$, for $1 \leq i \leq b$, where $\nu\left(\hat{B}_n^{(l)}(x)\right) = -b$. The corresponding $\Delta x_i = N_i$ are determined algorithmically as in the preceding theorem, so in particular $p-1|\Delta x_i$ for all i and $p|\Delta x_i$ for all $i > 1$ (and also for $i = 1$ if $p|n$).

Before deducing the Carlitz bound for the coefficients, recall the basic fact about p-divisibility of binomial coefficients [8].

$$\text{(Lucas's Theorem)} \qquad p \nmid \binom{n}{m} \text{ iff } n_i \geq m_i \text{ for all the base } p \text{ digits.} \tag{7}$$

Since $\binom{-n}{m} = (-1)^m \binom{n+m-1}{m}$, we deduce that $p \nmid \binom{-n}{m}$ iff the base p sum of $n-1$ and m has no carries. Thus if $p-1|N$, then $N + N/(p-1) = Np/(p-1)$, so $p|\binom{-N-1}{N/(p-1)}$. Also, if a_i and b_i are the lowest digits of N and $N/(p-1)$ respectively (which occur in the same place), then $a_i = p - b_i$.

Theorem 3: Let $l \in \mathbf{N}$, and suppose that the base p expansion of l has r non-zero digits. Then $\nu\left(\hat{B}_n^{(l)}(x)\right) \geq -r$, i.e., all coefficients a_i satisfy $\nu(a_i) \geq -r$.

Proof: Since $\hat{B}_n^{(n+1)}(x)$ is p-integral by (3), we have only the two cases $0 \leq l \leq n$ and $l > n+1$ to consider.

Case 1: $0 \leq l \leq n$. With the notations of Theorem 1 and by the preceding remarks, since $p \nmid \binom{l-n-1}{N_1/(p-1)}$, l must have a digit in the lowest place of N_1 or in a lower place, causing a carry from $n-l$ to that place. We prove inductively that l has at least i digits up to the bottom digit of N_i. This is true because, as above, if l has no digit in the lowest place of N_i, then there must be a carry from below for $n-l$. Thus l has a digit between the bottom places of N_i and N_{i-1} or n has a succession of digits all $p-1$ below N_i, and l has an extra digit below the succession, which causes the carry for $n-l$ to the lowest place of N_i.

Case 2: $l > n+1$. In this case, l must have a digit in the lowest place of N_1 or a digit in a lower place to prevent a carry for $l-n-1$ to that place. We have a similar proof to case 1, except that the digits of l are now required to prevent carries.

Thus $b \le r$ in both cases, i.e., the pole has order at most r. □

Corollary 1: With the same notations, $\nu\left(\hat{B}_n^{(l)}\right) \ge -r$.

We can verify that the Carlitz bound is best possible, e.g., if $l = \sum_{i=1}^r a_i p^{s_i}$ with $s_1 < s_2 < \cdots < s_r$ and $1 \le a_i \le p-1$, then if $n = (p-1)\sum_{i=1}^r p^{s_i}$, taking $N_i = (p-1)p^{s_i}$ shows that $\nu\left(\hat{B}_n^{(l)}\right) = -r$, hence also $\nu\left(\hat{B}_n^{(l)}(x)\right) = -r$.

Note that our proof of Carlitz's bound gives us a new proof for the classical case, as well as for all specializations where the c_i are p-adic integers, and the bound is then attained as long as c_{p-1} is a p-adic unit.

If $l = 0$, the preceding theorem shows that for every prime p, the coefficients of $\hat{B}_n^{(0)}(x)$ are p-integral. Thus we deduce

Corollary 2: The coefficients of $\hat{B}_n^{(0)}(x)$ are in $\mathbf{Z}[c_1, c_2, \ldots, c_n]$.

Finally we turn to the case $l = 1$, i.e., the ordinary universal Bernoulli numbers $\hat{B}_n$ and polynomials $\hat{B}_n(x)$. In this case, as in [3, Corollary 2.3]

$$(n-1)!\tau_u(-n) = (-1)^d(n+d-1)!/(u!\Lambda^u) \tag{8}$$

where $u! = u_1!u_2!\ldots u_n!$ and Λ^u is as before. Hence

$$\frac{\hat{B}_n(x)}{n} = \sum_{i=0}^{n}(-1)^i c_i \binom{x-1}{i} \sum_{w(u)=n-i} (-1)^d \frac{(n+d-1)!c^u}{u!\Lambda^u}. \tag{9}$$

The following corollary follows immediately from Theorem 3, taking $l = 1$.
Corollary 3: The coefficients of $\hat{B}_n(x)$ have square-free denominators, so in particular if $a \in \mathbf{Z}$, $\hat{B}_n(a)$ has square-free denominator.

The proofs of [3, Lemma 3.1 and Theorem 3.2] give the following result.
Theorem 4: Let $a \in \mathbf{Z}_p$. Then

(1) If $p-1 \nmid n$, then $\hat{B}_n(a)/n \in \mathbf{Z}_p[c_1, c_2, \ldots, c_n]$.

(2) If $n \not\equiv 0,\ 1 (\text{mod } p-1)$, then $\hat{B}_{n+p-1}(a)/(n+p-1) \equiv \hat{B}_n(a)c_{p-1}/n \pmod p$, where mod p is an abbreviation for mod $p\mathbf{Z}_p[c_1, c_2, \dots]$.

The first assertion with the preceding corollary gives a modest generalization of the von Staudt result to values of universal ordinary Bernoulli polynomials, while the second generalizes the Kummer congruences. As noted in [3], the hypotheses are essential, even for the special case $a = 0$ of the ordinary universal Bernoulli numbers. See [4] for the extension of the case $a = 0$ to $n \equiv 1 \pmod{p-1}$, where we have found an explicit formula for $\hat{B}_n/n$ mod p.

ACKNOWLEDGMENT

The author would like to thank the anonymous referee for several useful suggestions.

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AMS Classification Numbers: 11B63, 11B68