A classical umbral view of the Riordan group and related Sheffer sequences

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Abstract

The Riordan group is the set of infinite lower triangular invertible matrices with the group operation given by a matrix multiplication that combines both the usual Cauchy product and the composition of formal power series. It is related to a broad family of polynomial sequences in one variable called Sheffer sequences. Riordan arrays and Sheffer sequences have various applications in Combinatorics, Analysis, Probability, Physics, etc.

In this talk I will present an enlightening symbolic treatment of the Riordan group and related Sheffer sequences based on a renewed approach to umbral calculus initiated by **Gian Carlo Rota** in the 90's and further developed by **Di Nardo** and **Senato** in the first decade of the present century.

Based on joint work with **Ângela Mestre** (CELC), **Pasquale Petrullo** (Università degli studi della Basilicata) and **Maria Manuel Torres** (CELC).

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Riordan arrays

[Shapiro et al. 1991]

Let g and f be two formal exponential series; namely

$$g(z) = 1 + g_1 z + g_2 \frac{z^2}{2!} + g_3 \frac{z^3}{3!} + \cdots \quad \text{and} \quad f(z) = f_1 z + f_2 \frac{z^2}{2!} + f_3 \frac{z^3}{3!} + \cdots \tag{1}$$

An (exponential) *Riordan array* is an infinite lower triangular matrix $\mathfrak{M} = (\mathfrak{m}_{n,k})_{n,k\geq 0}$, whose entries are generated by g and f as follows

$$\mathfrak{m}_{n,k} = \left[\frac{z^n}{n!}\right] \left(g(z) \frac{f(z)^k}{k!} \right) \quad \text{for} \quad n,k \geq 0\,.$$

umbral view

We shall write $\mathfrak{M} = \big(g(z), f(z)\big) = (g, f)$.

Some examples

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

Pascal array (e^z, z)

$$\begin{pmatrix} 1 \\ g_1 & 1 \\ g_2 & 2g_1 & 1 \\ g_3 & 3g_2 & 3g_1 & 1 \\ g_4 & 4g_3 & 6g_2 & 4g_1 & 1 \\ g_5 & 5g_4 & 10g_3 & 10g_2 & 5g_1 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

Appell array $(g(z), z)$

$$\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -3 & 1 \\ 0 & -6 & 11 & -6 & 1 \\ 0 & 24 & -50 & 35 & -10 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

Stirling array of 1st. kind $(1, \log(1+z))$

$$\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 7 & 6 & 1 \\ 0 & 1 & 15 & 25 & 10 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

Stirling array of 2nd. kind $(1, e^z - 1)$

Entries of a general Riordan array for $0 \leq n, k \leq 3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ g_1 & f_1 & 0 & 0 \\ g_2 & f_2 + 2f_1 g_1 & f_1^2 & 0 \\ g_3 & f_3 + 3f_2 g_1 + 3f_1 g_2 & 3f_1^2 g_1 + 3f_1 f_2 & f_1^3 \\ & \vdots & & \ddots \end{pmatrix}$$

Fundamental theorem of Riordan arrays (FTRA)

Let A and B be two exponential generating functions; that is

$$A(z) = a_0 + a_1 z + a_2 \frac{z^2}{2!} + a_3 \frac{z^3}{3!} + \cdots \quad \text{and} \quad B(z) = b_0 + b_1 z + b_2 \frac{z^2}{2!} + b_3 \frac{z^3}{3!} + \cdots$$

and let $\big(g(z), f(z)\big)$ be a Riordan array. Then

$$(g(z), f(z)) \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix} \quad \Longleftrightarrow \quad g(z)\, A\big(f(z)\big) = B(z).$$

Note that the composition $A\big(f(z)\big)$ is well defined since f has zero constant term.

umbral view

Example (FTRA in action)

Stirling numbers of 2nd. kind $S(n,j)$

$$\begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 1 & 3 & 1 & & \\ 0 & 1 & 7 & 6 & 1 & \\ 0 & 1 & 15 & 25 & 10 & 1 \\ & & & \cdots & & \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 15 \\ 52 \\ \vdots \end{pmatrix}$$

Bell numbers B_n

$$\sum_{j=0}^{n} S(n,j) = B_n \; .$$

$$(1,\, e^z - 1) \cdot e^z = 1 \cdot e^{[e^z - 1]} = e^{[e^z-1]} = B(z) \; .$$

The (exponential) Riordan group [Shapiro et al. 1991]

Let (g, f) and (h, l) be given as in (1). Consider the *multiplication*

$$\big(g(z), f(z)\big)\,\big(h(z), l(z)\big) = \Big(g(z)\,h\big(f(z)\big), l\big(f(z)\big)\Big)\,. \tag{2}$$

The Riordan array $(1, z)$ is the *identity* element with respect to (2). Since $g_0 \neq 0$, it follows that g has multiplicative inverse g^{-1}. Also, if $f_1 \neq 0$, then f has compositional inverse $f^{<-1>}$; that is, $f\big(f^{<-1>}(z)\big) = f^{<-1>}\big(f(z)\big) = z$. In this case, a Riordan array (g, f) is invertible with respect to (2) and its *inverse* is given by

$$(g(z), f(z))^{-1} = \left(\frac{1}{g\big(f^{<-1>}(z)\big)}\,,\, f^{<-1>}(z)\right)\,,$$

The *Riordan group* $\mathfrak{Rio}$ is the set of all invertible Riordan arrays, together with multiplication (2) as the group operation.

umbral view

Some distinguished Riordan subgroups

1. *The Appell subgroup*: $\{(g(z), z)\}$.
2. *The Associated subgroup*: $\{(1, f(z))\}$.
3. *The Bell subgroup*: $\{(g(z), zg(z))\}$.
4. *The Stochastic subgroup*:

$$\left\{ (g(z), rz) \in \mathfrak{Rio} \;\middle|\; (g(z), f(z)) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \right\}.$$

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The *magic trick*

The Bell numbers $\boldsymbol{B_n}$ are the coefficients in the Taylor series expansion of $e^{[e^z-1]}$; namely

$$e^{[e^z-1]} = 1 + \sum_{n=1}^{\infty} \boldsymbol{B_n} \frac{z^n}{n!} \simeq 1 + \sum_{n=1}^{\infty} B^n \frac{z^n}{n!} = e^{Bz}$$

The symbol $\simeq$ stresses out the purely formal character of this manipulation.

The classical umbral calculus basic data

1. a commutative integral domain R with identity 1. $R = \mathbb{C}[x, y]$.
2. a set $A = \{\alpha, \beta, \gamma, \ldots\}$ of umbrae, called *alphabet*.
3. a linear functional $E\colon R[A] \to R$ called *evaluation* such that
   - $E[1] = 1$ and $p \in R[A]$ is called umbral polynomial.
   - $E[x^n y^m \alpha^i \beta^j \cdots \gamma^k] = x^n y^m E[\alpha^i] E[\beta^j] \cdots E[\gamma^k]$ (*uncorrelation*)
4. two special umbrae: ε (*augmentation*) and υ (*unity*) such that

$$E[\varepsilon^n] = \delta_{0,n} \qquad \text{and} \qquad E[\upsilon^n] = 1\,,$$

for all $n \geq 0$.

Similarity and generating functions (g.f.)

1. α represents a sequence $(a_n)_{n\geq 1}$ if $E[\alpha^n] = a_n$ for all $n \geq 1$. We say that a_n is the n-th *moment* of α. Assume $a_0 = 1$.
2. *umbral equivalence*: $\alpha \simeq \gamma \iff E[\alpha] = E[\gamma]$.
3. *similarity*: $\alpha \equiv \gamma \iff E[\alpha^n] = E[\gamma^n],\ \forall\, n \geq 0$.
4. The generating function of α is the exponential formal series

$$e^{\alpha z} := \upsilon + \sum_{n\geq 1} \alpha^n \frac{z^n}{n!} \in R[A]\,[[z]]\,,$$

so that
$$E[e^{\alpha z}] = 1 + \sum_{n\geq 1} a_n \frac{z^n}{n!} =: f_\alpha(z) \in R[[z]].$$

We shall write $e^{\alpha z} \simeq f_\alpha(z)$. We have $\alpha \equiv \gamma \iff e^{\alpha z} \simeq e^{\gamma z}$.

Some distinguished umbrae

name	α	$e^{\alpha z} \simeq f_\alpha(z)$	$E[\alpha^n] = a_n \ (n \geq 0)$
augmentation	ε	1	$\delta_{0,n}$
Bernoulli	ι	$\frac{z}{e^z - 1}$	b_n (n-th Bernoulli number)
unity	υ	e^z	1
singleton	χ	$1+z$	$\delta_{0,n}\ ,\ \ n = 0, 1$
Bell	β	$e^{[e^z - 1]}$	B_n (n-th Bell number)

The handling of sequences of binomial type

Let $(a_n)_{n\geq 1}$ and $(l_n)_{n\geq 1}$ be two arbitrary sequences in R represented by umbrae α and λ respectively. Then, the umbra $\alpha+\lambda$ has moments

$$E\left[(\alpha+\lambda)^n\right] = \sum_{j=0}^{n}\binom{n}{j}E\left[\alpha^j\,\lambda^{n-j}\right] \underset{\text{uncorrelation}}{=} \sum_{j=0}^{n}\binom{n}{j}a_j\,l_{n-j}.$$

Thus, the umbra $\alpha+\lambda$ represents the sequence $\sum_{j=0}^{n}\binom{n}{j}a_j\,l_{n-j}$.

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Thus, the umbra $\alpha+\lambda$ represents the sequence $\sum_{j=0}^{n}\binom{n}{j}a_j\,l_{n-j}$.

Suppose now that $(l_n)=(a_n)$. Does $\alpha+\alpha$ represent the sequence

$$\sum_{j=0}^{n}\binom{n}{j}a_j\,a_{n-j}?$$

The handling of sequences of binomial type

Let $(a_n)_{n\geq 1}$ and $(l_n)_{n\geq 1}$ be two arbitrary sequences in R represented by umbrae α and λ respectively. Then, the umbra $\alpha+\lambda$ has moments

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Thus, the umbra $\alpha+\lambda$ represents the sequence $\displaystyle\sum_{j=0}^{n}\binom{n}{j}a_j\,l_{n-j}$.

Suppose now that $(l_n)=(a_n)$. Does $\alpha+\alpha$ represent the sequence

$\displaystyle\sum_{j=0}^{n}\binom{n}{j}a_j\,a_{n-j}$? No, since $E\left[(\alpha+\alpha)^n\right]=E\left[(2\alpha)^n\right]=2^n a_n$.

Auxiliary umbrae

Key feature: Each sequence $(a_n)_{n\geq 1}$ in R can be represented by infinitely many similar (auxiliary) umbrae. This fact is called saturation. The alphabet A will contain all possible auxiliary umbrae.

The dot operations

Let k be a nonnegative integer and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be k uncorrelated umbra similar to α (with moments a_i). The *dot-product* $\boldsymbol{k \cdot \alpha}$ is an auxiliary umbra defined to satisfy $k \cdot \alpha \equiv \alpha_1 + \alpha_2 + \cdots + \alpha_k$.

Two umbrae α and λ are said to be *inverse* to each other (with respect to addition) when $\alpha + \lambda \equiv \varepsilon$. We shall write $\lambda \equiv -1 \cdot \alpha$. We define $\boldsymbol{-k \cdot \alpha} \equiv -1 \cdot \alpha_1 + \cdots + -1 \cdot \alpha_k$. Also, we set $0 \cdot \alpha \equiv \varepsilon$.

The *dot-power* $\boldsymbol{\alpha^{\cdot k}}$ is an auxiliary umbra such that $\alpha^{\cdot k} \equiv \alpha_1 \alpha_2 \cdots \alpha_k$. We assume $\alpha^{\cdot 0} \equiv \upsilon$. We have $E[(\alpha^{\cdot k})^n] = a_n^k$ for all $n \geq 0$.

G.f. for dot product and dot power [di Nardo & Senato 2001]

For any $k \in \mathbb{Z}$, we can write

$$e^{(k\cdot\alpha)z} = \upsilon + \sum_{n=1}^{\infty} (k\cdot\alpha)^n \frac{z^n}{n!} \simeq f_{k\cdot\alpha}(z) = \left[f_\alpha(z)\right]^k = e^{k \log f_\alpha(z)}.$$

In general, for any umbra γ (with moments g_i), the auxiliary umbra $\gamma \boldsymbol{\cdot} \boldsymbol{\alpha}$ is defined to satisfy

$$e^{(\gamma\cdot\alpha)z} \simeq f_{\gamma\cdot\alpha}(z) := f_\gamma\big(\log f_\alpha(z)\big) = 1 + g_1 \log f_\alpha(z) + g_2 \frac{\left[\log f_\alpha(z)\right]^2}{2!} + \cdots$$

Similarly, for $k \geq 0$, we have $e^{(\alpha^{\cdot k})z} \simeq f_{\alpha^{\cdot k}}(z) := 1 + \sum_{n=1}^{\infty} a_n^k \frac{z^n}{n!}$.

The role played by the Bell umbra β [di Nardo & Senato 2001]

Recall that the g.f. of the Bell umbra is $e^{\beta z} \simeq e^{[e^z-1]}$. Hence, the umbra $\beta \cdot \gamma$ has g.f.

$$e^{(\beta\cdot\gamma)z} \simeq f_\beta\big(\log f_\gamma(z)\big) = e^{\left[e^{\log f_\gamma(z)}-1\right]} = e^{\left[f_\gamma(z)-1\right]}.$$

The *composition* umbra of α and γ is the umbra $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} \cdot \boldsymbol{\gamma}$ with g.f.

$$e^{(\alpha\cdot\beta\cdot\gamma)z} \simeq f_\alpha\big(\log f_\beta(\log f_\gamma(z))\big) = f_\alpha\big(f_\gamma(z)-1\big)$$

If $\gamma\cdot\beta\cdot\alpha \equiv \chi \equiv \alpha\cdot\beta\cdot\gamma$, we say that γ is the *compositional inverse* of α and viceversa. We shall write $\gamma = \boldsymbol{\alpha}^{<-1>}$.

Note that $E[\alpha] \neq 0$ for $\alpha^{<-1>}$ to exist.

Useful identities and dictionary

1. $\alpha \cdot \varepsilon \equiv \varepsilon \equiv \varepsilon \cdot \alpha$.
2. $\beta \cdot \chi \equiv \upsilon \equiv \chi \cdot \beta$.
3. $\alpha\,\eta \equiv \eta\,\alpha$.
4. $r\alpha \equiv \alpha \cdot (r\upsilon)$. (In general, note that $r \cdot \alpha \equiv r\upsilon \cdot \alpha \not\equiv \alpha \cdot r\upsilon \equiv r\alpha$)

Umbrae	**formal power series**
$\alpha + \eta$	Cauchy product
$\alpha\,\eta$	Hadamard product
$\alpha \dot{+} \eta$	usual addition
$\alpha \cdot \beta \cdot \gamma$	formal composition

$(A, +, \cdot)$ is a left distributive algebra

1. $(A, +)$ is an abelian group.

$$\begin{array}{c} \alpha + \eta \equiv \eta + \alpha \\ (\alpha + \eta) + \gamma \equiv \alpha + (\eta + \gamma) \end{array} \quad \text{and} \quad \begin{array}{c} \alpha + \varepsilon \equiv \alpha \equiv \varepsilon + \alpha \\ \alpha + (-1 \cdot \alpha) \equiv \varepsilon \equiv (-1 \cdot \alpha) + \alpha \end{array}$$

2. $(A, \cdot)$ is a monoid.

$$\alpha \cdot (\eta \cdot \gamma) \equiv (\alpha \cdot \eta) \cdot \gamma \quad \text{and} \quad \alpha \cdot \upsilon \equiv \alpha \equiv \upsilon \cdot \alpha$$

3. The scalar product.

$$\begin{array}{c} 1\alpha \equiv \alpha \\ r(s\alpha) \equiv (rs)\alpha \end{array} \quad \text{and} \quad \begin{array}{c} r(\alpha + \eta) \equiv r\alpha + r\eta \\ (r + s)\alpha \equiv r\alpha + s\alpha \end{array}$$

4. The left distributive laws.

$$\begin{array}{c} (\alpha + \eta) \cdot \gamma \equiv \alpha \cdot \gamma + \eta \cdot \gamma \\ \gamma \cdot (\alpha + \eta) \not\equiv \gamma \cdot \alpha + \gamma \cdot \eta \end{array} \quad \text{and} \quad \begin{array}{c} \alpha \cdot (r\eta) \equiv r(\alpha \cdot \eta) \\ \alpha \cdot (r\eta) \not\equiv (r\alpha) \cdot \eta \end{array}$$

The constant umbrae

Let $r \in R$. The *constant* umbra ς_r has moments $E[\varsigma_r^n] = r$ for all $n \geq 1$. We have

1. $\varsigma_0 \equiv \varepsilon$ and $\varsigma_1 \equiv \upsilon$.
2. $E[(\varsigma_r \alpha)^n] = r a_n$ while $E[(r\alpha)^n] = r^n a_n$.
3. $\varsigma_r \, \varsigma_s \equiv \varsigma_s \, \varsigma_r \equiv \varsigma_{rs}$ for any $r, s \in R$.
4. $\varsigma_r \cdot \alpha \equiv \varsigma_r \, \alpha \equiv \alpha \, \varsigma_r$ while $\alpha \cdot \varsigma_r \not\equiv \alpha \, \varsigma_r$.
5. $e^{(\varsigma_r \alpha) z} \simeq 1 + r\big(f_\alpha(z) - 1\big)$ while $e^{(r\alpha)z} \simeq f_\alpha(rz)$.

In fact, we have $\varsigma_r \equiv \chi \cdot r \cdot \beta \cdot \alpha$.

The primitive and derivative umbrae. [di Nardo & Niederhausen & Senato 2001, 2009]

Let α be an umbra with moments a_i. The *derivative* umbra $\boldsymbol{\alpha}_{\mathcal{D}}$ of α is the umbra whose powers satisfy $\alpha_{\mathcal{D}}^n \simeq n\alpha^{n-1}$ for $n \geq 1$. In particular $E[\alpha_{\mathcal{D}}] = 1$ and this implies that $\alpha_{\mathcal{D}}$ has compositional inverse $\alpha_{\mathcal{D}}^{<-1>}$.

The g.f. of $\alpha_{\mathcal{D}}$ satisfy $\quad e^{\alpha_{\mathcal{D}} z} \simeq 1 + z\, e^{\alpha z}$.

The *primitive* umbra $\boldsymbol{\alpha}_{\mathcal{P}}$ of α is the umbra whose powers satisfy $\alpha_{\mathcal{P}}^n \simeq \frac{\alpha^{n+1}}{a_1(n+1)}$ for $n \geq 0$. Therefore $a_1 \neq 0$, so that only umbrae with compositional inverse have primitive umbra.

The g.f. of $\alpha_{\mathcal{P}}$ satisfy $\quad e^{\alpha z} \simeq 1 + a_1 z\, e^{\alpha_{\mathcal{P}} z}$.

Straightforward identities

Lemma

Let $\alpha \in A$ *be any umbra and let* $r \in R$ *be a nonzero scalar. Then*

1. $(\alpha_{\mathcal{D}})_{\mathcal{P}} \equiv \alpha$. *In addition, if* $a_1 = E[\alpha] \neq 0$ *then* $(\alpha_{\mathcal{P}})_{\mathcal{D}} \equiv \varsigma_{1/a_1}\alpha$.
2. $(\varsigma_r\alpha)_{\mathcal{P}} \equiv \alpha_{\mathcal{P}}$, $(r\alpha)_{\mathcal{P}} \equiv r\alpha_{\mathcal{P}}$, $(\varsigma_r\alpha)_{\mathcal{D}} \equiv \varsigma_r\alpha_{\mathcal{D}}$ *and* $(r\alpha)_{\mathcal{D}} \equiv \varsigma_{1/r}(r\alpha_{\mathcal{D}})$.

Theorem (A.M.P.T. 2010)

Let α *and* γ *be two umbrae with first moments* $a_1, g_1 \neq 0$. *Then*

$$(\alpha \cdot \beta \cdot \gamma)_{\mathcal{P}} \equiv \gamma_{\mathcal{P}} + \alpha_{\mathcal{P}} \cdot \beta \cdot \gamma.$$

▸ Extended Bell subgroup

Corollary

If $g_1 = {}^1\!/_{a_1}$ *then* $\alpha \cdot \beta \cdot \gamma \equiv (\gamma_{\mathcal{P}} + \alpha_{\mathcal{P}} \cdot \beta \cdot \gamma)_{\mathcal{D}}$.

Applications of Corollary

Taking $\gamma = \alpha^{<-1>}$ yields:

1. $\chi \equiv \alpha \cdot \beta \cdot \alpha^{<-1>} \equiv \left((\alpha^{<-1>})_{\mathcal{P}} + \alpha_{\mathcal{P}} \cdot \beta \cdot \alpha^{<-1>}\right)_{\mathcal{D}}$,
2. $\varepsilon \equiv \chi_{\mathcal{P}} \equiv (\alpha^{<-1>})_{\mathcal{P}} + \alpha_{\mathcal{P}} \cdot \beta \cdot \alpha^{<-1>}$,
3. $(\alpha^{<-1>})_{\mathcal{P}} \equiv -1 \cdot \alpha_{\mathcal{P}} \cdot \beta \cdot \alpha^{<-1>}$,
4. $\alpha_{\mathcal{P}} \equiv -1 \cdot (\alpha^{<-1>})_{\mathcal{P}} \cdot \beta \cdot \alpha$.

If $E[\alpha] = 1$ then

1. $\alpha^{<-1>} \equiv \left(-1 \cdot \alpha_{\mathcal{P}} \cdot \beta \cdot \alpha^{<-1>}\right)_{\mathcal{D}}$
2. $\alpha \equiv \left(-1 \cdot (\alpha^{<-1>})_{\mathcal{P}} \cdot \beta \cdot \alpha\right)_{\mathcal{D}}$.

Lagrange's inversion formula I

Let $f(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$ and suppose that $a_0 = 0$ and $a_1 \neq 0$. Then $f^{<-1>}$ is well defined. One version of Lagrange's inversion formula states that for any integer $n \geq 1$, it holds

$$\left[\frac{z^n}{n!}\right] f^{<-1>}(z) = \left[\frac{z^{n-1}}{(n-1)!}\right] \left(\frac{f(z)}{z}\right)^{-n} .$$

Let α be an umbra such that $e^{\alpha z} \simeq 1 + f(z)$. Then $f(z) \simeq a_1 z\, e^{\alpha_{\mathcal{P}} z}$ and $e^{\alpha^{<-1>} z} \simeq (a_1 z\, e^{\alpha_{\mathcal{P}} z})^{<-1>}$.

Proposition (di Nardo & Niederhausen & Senato 2009)

Let α be an umbra with compositional inverse $\alpha^{<-1>}$ (hence $a_1 = E[\alpha] \neq 0$). For any $n \geq 1$, we have $(\alpha^{<-1>})^n \simeq \frac{1}{a_1^n}(-n \cdot \alpha_{\mathcal{P}})^{n-1}$ or equivalently, $\alpha^{\cdot n}(\alpha^{<-1>})^n \simeq (-n \cdot \alpha_{\mathcal{P}})^{n-1}$. In particular, $(\alpha_{\mathcal{D}}^{<-1>})^n \simeq (-n \cdot \alpha)^{n-1}$.

Lagrange's inversion formula II

Another version of Lagrange's inversion formula states that for any integer $n \geq 1$ and $f(z)$ as before, it holds

$$\left[\frac{z^n}{n!}\right] \Phi\left(f^{<-1>}(z)\right) = \left[\frac{z^{n-1}}{(n-1)!}\right] D\Phi(z) \left(\frac{f(z)}{z}\right)^{-n} ,$$

where $\Phi(z)$ is any formal exponential series and D is the usual differential operator on formal power series.

Theorem (A.M.P.T. 2010)

Let α be any umbra and γ an umbra with compositional inverse $\gamma^{<-1>}$ (hence $g_1 = E[\gamma] \neq 0$). For any $n \geq 1$ we have

$$(\alpha \cdot \beta \cdot \gamma^{<-1>})^n \simeq \frac{1}{g_1^n}\, \alpha(\alpha - n \cdot \gamma_{\mathcal{P}})^{n-1} .$$

Abel's identity

The *adjoint* umbra of γ is $\boldsymbol{\gamma}^{*} = \beta \cdot \gamma^{<-1>}$.

Theorem (A.M.P.T. 2010)

Let $\gamma \in A$ be any umbra with $g_1 = E[\gamma] \neq 0$. For any umbrae α and δ we have

$$(\alpha + \delta)^n \simeq \sum_{k=0}^{n} \binom{n}{k} \gamma^{\cdot k} (\alpha + k \cdot \gamma_{\mathcal{P}})^{n-k} (\delta \cdot \gamma^{*})^k.$$

FTRA

Contents

The Riordan group revisited

Definition

Given two umbrae α and γ, we say that (α, γ) represent the Riordan array (g, f) if

$$e^{\alpha z} \simeq g(z) \qquad \text{and} \qquad e^{\gamma z} \simeq 1 + f(z)\,.$$

Thus, the Riordan group is given by

$$\mathfrak{Rio} = \Big\{ (\alpha, \gamma) \in A \times A \,:\, E[\gamma] \neq 0 \Big\}\,.$$

The group operation reads as $(\alpha, \gamma)\,(\zeta, \eta) \equiv (\alpha + \zeta \cdot \beta \cdot \gamma\,,\, \eta \cdot \beta \cdot \gamma)$.

The inverse of (α, γ) is $(\alpha, \gamma)^{-1} = (-1 \cdot \alpha \cdot \beta \cdot \gamma^{<-1>}, \gamma^{<-1>})$ and the identity is (ε, χ) .

classical view

Entries of an invertible Riordan array

Proposition (A.M.P.T., di Nardo & Niederhausen & Senato 2010, $\gamma \rightsquigarrow \gamma^{<-1>}$)

Let $\mathfrak{M} = (\alpha, \gamma) \in \mathfrak{Rio}$. *Then*

$$\mathfrak{m}_{n,k} \simeq \binom{n}{k} \gamma^{\cdot k} (\alpha + k \cdot \gamma_{\mathcal{P}})^{n-k} \quad \textit{for} \quad n, k \geq 0.$$

▸ classical view

▸ FTRA

Fundamental theorem of Riordan arrays (FTRA)

Let $(\alpha, \gamma) \in \mathfrak{Rio}$ and $\delta \in A$. The group $\mathfrak{Rio}$ acts over A by

$$(\alpha, \gamma) \bullet \delta = \alpha + \delta \cdot \beta \cdot \gamma \, .$$

Given any two umbrae $\delta, \eta \in A$, the FTRA is equivalent to saying that there exists an invertible Riordan array $(\alpha, \gamma) \in \mathfrak{Rio}$ such that $(\alpha, \gamma) \bullet \delta = \eta$. That is, the $\mathfrak{Rio}$-action is transitive. ▸ classical view

By replacing δ with $\delta \cdot \beta \cdot \gamma$ in ▸ Abel's identiy and using the umbral characterization for the ▸ entries of an invertible Riordan array, we obtain

$$(\alpha + \delta \cdot \beta \cdot \gamma)^n \simeq \sum_{k=0}^{n} \mathfrak{m}_{n,k} \, \delta^k \qquad \text{for all} \quad n \geq 0 \, .$$

Some important Riordan subgroups

1. *The Appell subgroup*: $\{(\alpha,\chi)\}$.

$$(\alpha,\chi)\,(\zeta,\chi) \equiv (\alpha+\zeta,\chi) \qquad \text{and} \qquad (\alpha,\chi)^{-1} \equiv (-1\cdot\alpha,\chi)\ .$$

2. *The Associated subgroup*: $\{(\varepsilon,\gamma)\}$.

$$(\varepsilon,\gamma)\,(\varepsilon,\eta) \equiv (\varepsilon,\eta\cdot\gamma) \qquad \text{and} \qquad (\varepsilon,\gamma)^{-1} \equiv (\varepsilon,\gamma^{<-1>})\ .$$

3. *The Bell subgroup*: $\{(\alpha,\alpha_{\mathcal{D}})\}$.

$$(\alpha,\alpha_{\mathcal{D}})\,(\zeta,\zeta_{\mathcal{D}}) \equiv (\alpha+\zeta\cdot\beta\cdot\alpha_{\mathcal{D}}\,,\,\zeta_{\mathcal{D}}\cdot\beta\cdot\alpha_{\mathcal{D}})$$

and

$$(\alpha,\alpha_{\mathcal{D}})^{-1} \equiv (-1\cdot\alpha\cdot\beta\cdot\alpha_{\mathcal{D}}^{<-1>}\,,\,\alpha_{\mathcal{D}}^{<-1>})\ .$$

Note that $\zeta_{\mathcal{D}}\cdot\beta\cdot\alpha_{\mathcal{D}} \equiv (\alpha+\zeta\cdot\beta\cdot\alpha_{\mathcal{D}})_{\mathcal{D}}$ and $\alpha_{\mathcal{D}}^{<-1>} \equiv (-1\cdot\alpha\cdot\beta\cdot\alpha_{\mathcal{D}}^{<-1>})_{\mathcal{D}}$.

The extended Bell subgroup [A.M.P.T. 2010]

4. *The extended Bell subgroup*: $\{(\alpha, \gamma_{\mathcal{D}})\}$.

$$(\alpha, \gamma_{\mathcal{D}})\,(\zeta, \eta_{\mathcal{D}}) \equiv (\alpha + \zeta\cdot\beta\cdot\gamma_{\mathcal{D}}\,,\, \eta_{\mathcal{D}}\cdot\beta\cdot\gamma_{\mathcal{D}})$$

and

$$(\alpha, \gamma_{\mathcal{D}})^{-1} \equiv (-1\cdot\alpha\cdot\beta\cdot\gamma_{\mathcal{D}}^{<-1>}\,,\, \gamma_{\mathcal{D}}^{<-1>})\,.$$

Note that $\eta_{\mathcal{D}}\cdot\beta\cdot\gamma_{\mathcal{D}} \equiv (\gamma+\eta\cdot\beta\cdot\gamma_{\mathcal{D}})_{\mathcal{D}}$ and $\gamma_{\mathcal{D}}^{<-1>} \equiv (-1\cdot\gamma\cdot\beta\cdot\gamma_{\mathcal{D}}^{<-1>})_{\mathcal{D}}$.

This subgroup clearly contains the Bell subgroup.

▸ Straightforward identities

The Stabilizer subgroups

5. *The Stabilizer subgroups*: Given any $\delta \in A$, the stabilizer *Stab*(δ) of δ (with respect to the $\mathfrak{Rio}$-action) is

$$Stab(\delta) = \Big\{(\alpha, \gamma) \in \mathfrak{Rio} \;:\; \alpha + \delta \cdot \beta \cdot \gamma \equiv \delta\Big\}.$$

Since $(\alpha + \delta \cdot \beta \cdot \gamma)^n \simeq \sum_{k=0}^{n} \mathfrak{m}_{n,k}\ \delta^k$, the identity $\alpha + \delta \cdot \beta \cdot \gamma \equiv \delta$ is equiv. to

$$\sum_{k=0}^{n-1} \mathfrak{m}_{n,k}\ \delta^k + (\mathfrak{m}_{n,n} - 1)\ \delta^n = 0, \quad \text{for all} \quad n \geq 1.$$

In particular, we have

$$Stab(\varepsilon) = \Big\{(\alpha, \gamma) \in \mathfrak{Rio} \;:\; \alpha \equiv \varepsilon\Big\} = \textit{Associated subgroup}.$$

$$Stab(\upsilon) = \Big\{(\alpha, \gamma) \in \mathfrak{Rio} \;:\; \alpha + \beta \cdot \gamma \equiv \upsilon\Big\} = \textit{Stochastic subgroup}.$$

$$Stab(\chi) = \Big\{(\alpha, \gamma) \in \mathfrak{Rio} \;:\; \alpha + \gamma \equiv \chi\Big\}.$$

Entries for some Riordan subgroups

Subgroup	$\mathfrak{m}_{n,k} \simeq \binom{n}{k} \gamma^{\cdot k} \left(\alpha + k \cdot \gamma_{\mathcal{P}}\right)^{n-k}$ $\quad n, k \geq 0$
Appell (α, χ)	$\binom{n}{k} \alpha^{n-k}$
Associated (ε, γ)	$\binom{n}{k} \gamma^{\cdot k} \left(k \cdot \gamma_{\mathcal{P}}\right)^{n-k}$
Bell $(\alpha, \alpha_{\mathcal{D}})$	$\binom{n}{k} \left((k+1) \cdot \alpha\right)^{n-k}$
extended Bell $(\alpha, \gamma_{\mathcal{D}})$	$\binom{n}{k} (\alpha + k \cdot \gamma)^{n-k}$

Sheffer umbrae

[di Nardo & Niederhausen & Senato 2009, 2010]

Let $\big(g(z), f(z)\big) \in \mathfrak{Rio}$. A polynomial sequence $s_n(x)$ is said to be Sheffer for $\big(g(z), f(z)\big)$ if they satisfy

$$\sum_{n=0}^{\infty} s_n(x) \frac{z^n}{n!} = \frac{1}{g\big(f^{<-1>}(z)\big)} e^{x\, f^{<-1>}(z)} .$$

Representing $\big(g(z), f(z)\big)$ by the pair of umbrae (α, γ), the *Sheffer umbra* $\boldsymbol{\sigma_x^{(\alpha,\gamma)}}$ for (α, γ) is defined as

$$\sigma_x^{(\alpha,\gamma)} \equiv -1 \cdot \alpha \cdot \beta \cdot \gamma^{<-1>} + x \cdot \upsilon \cdot \beta \cdot \gamma^{<-1>} \equiv (-1 \cdot \alpha + x \cdot \upsilon) \cdot \gamma^{*} .$$

By construction, the g.f. of $\sigma_x^{(\alpha,\gamma)}$ is

$$e^{\sigma_x^{(\alpha,\gamma)} z} \simeq \sum_{n=0}^{\infty} s_n(x) \frac{z^n}{n!} ,$$

so that its moments $\left(\sigma_x^{(\alpha,\gamma)}\right)^n \simeq s_n(x)$ form a Sheffer sequence.

Characterization of Sheffer sequences

Proposition (A.M.P.T., di Nardo & Niederhausen & Senato 2010, $-1 \cdot \alpha \cdot \gamma^* \rightsquigarrow \alpha$)

The umbral expression for Sheffer polynomials $s_n(x) \simeq \left(\sigma_x^{(\alpha,\gamma)}\right)^n$ *coming from a Riordan array* (α, γ) *is given by*

$$s_n(x) \simeq \sum_{k=0}^{n} \left[\binom{n}{k} \gamma^{<-1>\cdot k} \Big(-1 \cdot \alpha \cdot \gamma^* + k \cdot (\gamma^{<-1>})_{\mathcal{P}} \Big)^{n-k} \right] x^k$$

$$\simeq \sum_{k=0}^{n} \mathfrak{m}_{n,k}^{-1}\, x^k \,,$$

where $\mathfrak{m}_{n,k}^{-1}$ *is the* (n,k)*-th entry of* $(\alpha,\gamma)^{-1} = (-1 \cdot \alpha \cdot \gamma^*, \gamma^{<-1>})$.

Some distinguished Sheffer sequences

Name	$s_n(x)$
Appell (α, χ)	$\sum_{k=0}^{n} \binom{n}{k} \left(-1 \cdot \alpha \right)^{n-k} x^k$
Associated (ε, γ)	$\sum_{k=0}^{n} \binom{n}{k} \left(\gamma^{<-1>} \right)^{\cdot k} \left(k \cdot \left(\gamma^{<-1>} \right)_{\mathcal{P}} \right)^{n-k} x^k$
Bell $(\alpha, \alpha_{\mathcal{D}})$	$\sum_{k=0}^{n} \binom{n}{k} \left((k+1) \cdot \left(\alpha_{\mathcal{D}}^{<-1>} \right)_{\mathcal{P}} \right)^{n-k} x^k$
extended Bell $(\alpha, \gamma_{\mathcal{D}})$	$\sum_{k=0}^{n} \binom{n}{k} \left(-1 \cdot \alpha \cdot \gamma_{\mathcal{D}}^{*} + k \cdot \left(\gamma_{\mathcal{D}}^{<-1>} \right)_{\mathcal{P}} \right)^{n-k} x^k$

Bi-parameterized Sheffer umbrae [A.M.P.T. 2010]

Definition

Let $(\alpha, \gamma) \in \mathfrak{Rio}$ and let $x, y \in A$. The *bi-parameterized Sheffer umbra* corresponding to (α, γ) is given by

$$\sigma_{x,y}^{(\alpha,\gamma)} \equiv (y \cdot \alpha + x \cdot \upsilon) \cdot \gamma^*$$

Note that $\sigma_{x,-1}^{(\alpha,\gamma)} \equiv \sigma_x^{(\alpha,\gamma)}$ (di Nardo and Senato Sheffer umbra), $\sigma_{x,0}^{(\alpha,\gamma)} \equiv x \cdot \gamma^* \equiv \sigma_x^{(\varepsilon,\gamma)}$ (Associated umbra with respect to γ), etc.

Thank you!