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Convolution identities and lacunary recurrences for Bernoulli numbers

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Abstract

We extend Euler's well-known quadratic recurrence relation for Bernoulli numbers, which can be written in symbolic notation as $(B_0 + B_0)^n = -nB_{n-1} - (n-1)B_n$, to obtain explicit expressions for $(B_k + B_m)^n$ with arbitrary fixed integers $k, m \geqslant 0$. The proof uses convolution identities for Stirling numbers of the second kind and for sums of powers of integers, both involving Bernoulli numbers. As consequences we obtain new types of quadratic recurrence relations, one of which gives B_{6k} depending only on $B_{2k}, B_{2k+2}, \ldots, B_{4k}$.

1. Introduction

The Bernoulli numbers B_n, $n = 0, 1, 2, \ldots$, which can be defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \qquad |x| < 2\pi, \tag{1.1}$$

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have numerous important applications in number theory, combinatorics, and numerical analysis, among other areas. They have therefore been studied extensively over the last two centuries. For the most important properties see, for instance, [1,21,26], or [34]; for a comprehensive bibliography, see [12].

It is easy to find the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, and $B_n = 0$ for all odd $n \geqslant 3$. While the vanishing of odd-index Bernoulli numbers can be proved directly with (1.1) by observing that $\frac{x}{e^x-1} + \frac{x}{2}$ is an even function, individual values of B_n are best obtained by way of a recurrence relation, the most basic one being

$$\sum_{j=0}^{n} \binom{n+1}{j} B_j = 0 \quad (n \geqslant 1), \tag{1.2}$$

with $B_0 = 1$. For a brief historical discussion of such recurrence relations, with numerous references, see [6].

Next to (1.2), one of the most basic and remarkable identities for the Bernoulli numbers is the convolution identity

$$\sum_{j=0}^{n} \binom{n}{j} B_j B_{n-j} = -nB_{n-1} - (n-1)B_n \quad (n \geqslant 1), \tag{1.3}$$

which is also known in its equivalent form

$$\sum_{j=1}^{n-1} \binom{2n}{2j} B_{2j} B_{2n-2j} = -(2n+1)B_{2n}. \tag{1.4}$$

In contrast to the *linear* recurrence relation (1.2), the identities (1.3) and (1.4) can be considered *quadratic* recurrence relations. The relation (1.4) can be used, for instance, to show by induction that $(-1)^{n-1}B_{2n} > 0$ for all $n \geqslant 1$, i.e., the even-index Bernoulli numbers have alternating signs. The identities (1.3) and (1.4) are usually attributed to Euler; numerous other similar recurrences can be found in [33] or in the handbook [22].

These identities have been generalized and extended in different directions. First, Sitaramachandrarao and Davis [40] extended (1.4) to sums of products of $N = 3$ and $N = 4$ Bernoulli numbers. This was further extended to $N = 5$ by Sankaranarayanan [38], to $N \leqslant 7$ by Zhang [43], and to arbitrary positive integers N by the second author [11]. Analogues of (1.3) are also known for Bernoulli polynomials and Euler numbers and polynomials (see, e.g., [22, Chapter 50], extended in [11]), for generalized Bernoulli numbers and polynomials belonging to Dirichlet characters (see [9,28]), and for q-Bernoulli numbers ([39], extended in [27]). Other related identities can be found in [13,25] and [30].

A different type of convolution identity for Bernoulli numbers is due to Miki [31]; see also [17] for further remarks and generalizations. Lacunary versions of (1.4), for instance with $2j$ replaced by $6j$, are due to Ramanujan [36] and are listed in [22, Chapter 50]; see also [42].

It is the purpose of this paper to consider a different, but very natural, type of generalization of (1.3). To do this, it will be convenient (though not essential) to use the symbolic notation that was first introduced by Blissard in the mid-nineteenth century, was later made popular by Lucas, and is now also known as the "classical umbral calculus;" see [16] for a modern treatment. With this notation we have, for integers $k, m, n \geqslant 0$,

$$(B_k + B_m)^n = \sum_{j=0}^{n} \binom{n}{j} B_{k+j} B_{m+n-j}, \tag{1.5}$$

so that Euler's formula (1.3) can be rewritten as

$$(B_0 + B_0)^n = -nB_{n-1} - (n-1)B_n \quad (n \geqslant 1). \tag{1.6}$$

The question that arises quite naturally, but does not appear to have been considered before, is whether there exist formulas such as (1.6) also for the general sums (1.5) for arbitrary integers $k, m \geqslant 0$. We will answer this question in the affirmative and give a general formula in Section 2, followed by two special cases and a number of small instances stated explicitly.

A surprising and unusual consequence of these results are a set of quadratic recurrence relations of "Euler type" that allow us to compute B_{6k} from only $B_{2k}, B_{2k+2}, \dots, B_{4k}$, and similarly for B_{6k+2} and B_{6k+4}. These will also be stated in Section 2.

In Section 3 we prove some auxiliary results on Stirling numbers of the second kind, followed by the proof of our main theorem in Sections 4–6. We conclude this paper with some additional remarks in Section 7.

2. Results

We begin this section by stating our main result. Note that the case of identity (1.6), i.e., $k = m = 0$, is excluded. The reason for this will become clear in the proof; also, combining this case with the identity (2.1) below would make the statement of this result even more complicated.

Theorem 2.1. *Let $k, m, n \geqslant 0$ be integers, with k and m not both 0. Then*

$$\begin{aligned}(B_k + B_m)^n = &-\frac{k!m!}{(k+m+1)!}\big(n + \delta(k,m)(k+m+1)\big)B_{n+k+m} \\ &+ \sum_{r=1}^{k+m} (-1)^r \frac{B_{k+m+1-r}}{k+m+1-r} \left\{ (-1)^k \binom{k+1}{r} \left(\frac{k+1-r}{k+1} n - \frac{rm}{k+1} \right) \right. \\ &\left. + (-1)^m \binom{m+1}{r} \left(\frac{m+1-r}{m+1} n - \frac{rk}{m+1} \right) \right\} B_{n+r-1},\end{aligned} \tag{2.1}$$

where $\delta(k,m) = 0$ when $k = 0$ or $m = 0$, and $\delta(k,m) = 1$ otherwise.

The formula (2.1) will be simpler in special cases. We state the following two corollaries.

Corollary 2.1. *Let $k \geqslant 2$ and $n \geqslant 0$ be integers. Then*

$$(B_k + B_k)^n = -\frac{(k!)^2}{(2k+1)!}(n+2k+1)B_{n+2k} + \frac{(-1)^{k+1}}{k+1}\sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \frac{B_{2(k-r)}}{k-r}\binom{k+1}{2r+1}\bigl((k-2r)n-(2r+1)k\bigr)B_{n+2r}. \tag{2.2}$$

Corollary 2.2. *Let $k \geqslant 2$ and $n \geqslant 0$ be integers. Then*

$$(B_k + B_0)^n = -(n-1)B_k B_n - \frac{n}{k+1}\sum_{r=2}^{k-1}\binom{k+1}{r}B_{k+1-r}B_{n+r-1} - \frac{n}{2}B_{n+k-1} - \frac{n}{k+1}B_{n+k}. \tag{2.3}$$

For a different form of this last identity, see the final section. In this identity (2.3) it is clear that the first term on the right vanishes whenever k is odd. Also, note that the right-hand side of (2.2) vanishes when n is odd, and more generally, the right-hand side of (2.1) vanishes when $k \geqslant 2$, $m \geqslant 2$, $n \geqslant 2$ and the parities of n and $k+m$ are different. The cases $0 \leqslant k, m \leqslant 1$ with n and $k+m$ of different parities are also trivial, as is (2.3) with n and k of different parity. All this is best seen by considering the right-hand side of (1.5) and noting that $B_1 = -\frac{1}{2}$ is the only nonzero odd-index Bernoulli number.

It can be observed that the right-hand sides of (2.1) and (2.2) are lacunary in a certain sense if k, m, and n are appropriately chosen. In fact, this is most pronounced in (2.2) when n is close to k. The following identities result from this observation, as direct consequences of (2.2).

Corollary 2.3. *For all integers $k \geqslant 1$ we have*

$$B_{6k} = -\binom{4k}{2k}\frac{4k+1}{6k+1}\sum_{r=0}^{k}\binom{2k}{2r}\left(\frac{2k(2k-4r-1)}{(2r+1)(2k-r)}+1\right)B_{2k+2r}B_{4k-2r},$$

and for $k \geqslant 0$,

$$B_{6k+2} = \binom{4k+2}{2k+1}\frac{4k+3}{6k+3}\sum_{r=0}^{k}\binom{2k+1}{2r}\left(\frac{4k^2-8rk-2r-1}{(2k+1-r)(2r+1)} - \frac{2r}{2k+1}\right) \times B_{2k+2r}B_{4k+2-2r},$$

$$B_{6k+4} = -\binom{4k+2}{2k+1}\frac{4k+3}{6k+5}\sum_{r=0}^{k}\binom{2k+2}{2r+1}\frac{6rk+2k+4r+1}{(2k+2)(2k+1-r)}B_{2k+2+2r}B_{4k+2-2r}.$$

These identities are unusual in that B_{6k}, for instance, is obtained from only the values of $B_{2k}, B_{2k+2}, \dots, B_{4k}$. We are not aware of any other such formulas in the literature.

There are, however, some *linear* recurrences that typically require only the second half of all the Bernoulli numbers up to some B_{2n}. Such recurrences go back to von Ettingshausen [14] in 1827 and later Stern [41]. See also Nielsen's classic book [33]; for a modern treatment, see [16, p. 415f], and for some historical perspective, see [6].

The first identity in Corollary 2.3 is obtained by replacing k by $2k$ and setting $n = 2k$ in (2.2). Similarly, we get the other two identities if we replace k by $2k + 1$ and set $n = 2k$, respectively $n = 2k + 2$. The sums on both sides of (2.2) are then easily combined in all three cases. Of course we could cancel a factorial from the binomial coefficients in all three expressions, but we left them in this form to preserve the character of a binomial convolution.

In both (2.2) and (2.3) the inclusion of the case $k = 1$ would have made the identities more complicated; this is once again due to the exceptional nature of B_1. The excluded cases are listed below, along with some other cases for small k and m, including Euler's formula for completeness.

Corollary 2.4. *For all integers $n \geqslant 0$ we have*

$$
\begin{aligned}
(B_0 + B_0)^n &= -nB_{n-1} - (n-1)B_n \quad (n \geqslant 1),\\
(B_0 + B_1)^n &= -\frac{1}{2}(n+1)B_n - \frac{1}{2}nB_{n+1},\\
(B_0 + B_2)^n &= -\frac{1}{6}(n-1)B_n - \frac{1}{2}nB_{n+1} - \frac{1}{3}nB_{n+2},\\
(B_0 + B_3)^n &= -\frac{1}{4}nB_{n+1} - \frac{1}{2}nB_{n+2} - \frac{1}{4}nB_{n+3},\\
(B_1 + B_1)^n &= \frac{1}{6}(n-1)B_n - B_{n+1} - \frac{1}{6}(n+3)B_{n+2},\\
(B_1 + B_2)^n &= \frac{1}{12}nB_{n+1} - \frac{1}{2}B_{n+2} - \frac{1}{12}(n+4)B_{n+3},\\
(B_1 + B_3)^n &= -\frac{1}{30}(n-1)B_n + \frac{1}{12}(n-3)B_{n+2} - \frac{1}{2}B_{n+3} - \frac{1}{20}(n+5)B_{n+4},\\
(B_2 + B_2)^n &= \frac{1}{30}(n-1)B_n + \frac{1}{3}B_{n+2} - \frac{1}{30}(n+5)B_{n+4},\\
(B_2 + B_3)^n &= \frac{1}{60}nB_{n+1} + \frac{1}{6}B_{n+3} - \frac{1}{60}(n+6)B_{n+5},\\
(B_3 + B_3)^n &= \frac{1}{42}(n-1)B_n - \frac{1}{60}(n-9)B_{n+2} - \frac{1}{140}(n+7)B_{n+6}.
\end{aligned}
$$

Of course, this list could easily be extended by using (2.1)–(2.3) with other small values of k and m.

3. Stirling numbers

The most important tool in this paper are the *Stirling numbers of the second kind* $S(n,k)$. They can be defined by the generating function

$$\prod_{j=1}^{k} \frac{x}{1 - jx} = \sum_{n=k}^{\infty} S(n,k)x^n, \tag{3.1}$$

or as coefficients in the change between the two standard bases of the vector space of polynomials:

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\cdot\ldots\cdot(x-k+1). \tag{3.2}$$

For a combinatorial interpretation see again, e.g., [21], where numerous other properties can be found. The book [10] is another good reference. The most basic recurrence relation is the *triangular* or *Pascal-type* relation

$$S(n+1,k) = S(n,k-1) + kS(n,k). \tag{3.3}$$

We will also use the well-known finite sum

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}(k-j)^n, \tag{3.4}$$

rewritten as

$$S(n,k) = \frac{(-1)^k}{(k-1)!}\sum_{j=1}^{k}(-1)^j\binom{k-1}{j-1}j^{n-1}. \tag{3.5}$$

Some explicit values are

$$S(0,0) = 1, \qquad S(n,0) = 0 \quad \text{for } n \geqslant 1, \tag{3.6}$$

$$S(n,1) = S(n,n) = 1, \tag{3.7}$$

$$S(n,2) = 2^{n-1} - 1, \tag{3.8}$$

$$S(n,3) = \frac{1}{2}\left(3^{n-1} - 2^n + 1\right), \tag{3.9}$$

$$S(n,n-1) = \binom{n}{2}. \tag{3.10}$$

Note that (3.7)–(3.9) are just special cases of (3.4). In spite of some advantages to the bracket notation used in the book [21] (see also [29]), we chose the competing notation $S(n,k)$ as used, for instance, in the book [10] or in recent papers such as [16] or [17].

The principal connection between the Bernoulli numbers and the Stirling numbers of the second kind, and in fact the key to the proof of our main theorem, is given by the following lemma. A proof was already given in [6], but for the sake of completeness we repeat it here.

Lemma 3.1. *For any $m \geqslant 0$ we have*

$$\frac{d^m}{dx^m}\frac{1}{e^x-1} = (-1)^m\sum_{j=1}^{m+1}(j-1)!\frac{S(m+1,j)}{(e^x-1)^j}, \tag{3.11}$$

$$\frac{d^m}{dx^m}\frac{x}{e^x-1}=(-1)^m\sum_{j=1}^{m+1}(j-1)!\frac{S(m+1,j)x-mS(m,j)}{(e^x-1)^j}. \tag{3.12}$$

Proof. We prove (3.11) by induction. The case $m=0$ is obvious by (3.7). Now note that

$$\frac{d}{dx}\left(e^x-1\right)^{-j}=-j\left(e^x-1\right)^{-j-1}e^x=-\frac{j(e^x-1)+j}{(e^x-1)^{j+1}},$$

so that from (3.11) we get

$$\begin{aligned}(-1)^{m+1}\frac{d^{m+1}}{dx^{m+1}}\frac{1}{e^x-1}&=\sum_{j=1}^{m+1}(j-1)!\frac{jS(m+1,j)}{(e^x-1)^j}+\sum_{j=1}^{m+1}j!\frac{S(m+1,j)}{(e^x-1)^{j+1}}\\&=\sum_{j=1}^{m+1}(j-1)!\frac{jS(m+1,j)}{(e^x-1)^j}+\sum_{j=2}^{m+2}(j-1)!\frac{S(m+1,j-1)}{(e^x-1)^j}\\&=\sum_{j=1}^{m+2}(j-1)!\frac{S(m+2,j)}{(e^x-1)^j},\end{aligned}$$

where we have used (3.3). This proves (3.11) by induction. Finally we use Leibniz's rule in the form

$$\frac{d^m}{dx^m}\frac{x}{e^x-1}=x\frac{d^m}{dx^m}\frac{1}{e^x-1}+m\frac{d^{m-1}}{dx^{m-1}}\frac{1}{e^x-1}.$$

With this, the identity (3.12) follows immediately from (3.11). □

A lemma very similar to (3.11) was derived and applied in [2]. Problem 209 in [35, p. 44] is also relevant in this connection.

Convolution identities for Stirling numbers (of the second kind) will also be important for this paper. Such identities, including those for Stirling numbers of the first kind, were recently studied by the authors in [6]; others can be found, in somewhat different form, in [21, p. 272]. However, the following type of convolution does not seem to be covered by either reference.

Lemma 3.2. *For integers $k,m\geqslant 1$ and $d\geqslant 0$ we have*

$$\sum_{i=0}^{d-1}\frac{S(k,i+1)S(m,d-i)}{d\binom{d-1}{i}}=\sum_{a=2}^{d+1}\frac{(-1)^{d+1-a}}{(a-1)!(d+1-a)!}\sum_{r=1}^{a-1}r^{k-1}(a-r)^{m-1}. \tag{3.13}$$

Proof. For $d=0$ both sides of (3.13) clearly vanish; so let $d\geqslant 1$. Using (3.5) and the obvious identity $d\binom{d-1}{i}i!(d-i-1)!=d!$, we rewrite (3.13) as

$$\sum_{i=0}^{d-1}\left(\sum_{r=1}^{i+1}(-1)^r\binom{i}{r-1}r^{k-1}\right)\left(\sum_{j=1}^{d-i}(-1)^j\binom{d-i-1}{j-1}j^{m-1}\right)$$

$$=\sum_{a=2}^{d+1}(-1)^a\binom{d}{a-1}\sum_{r=1}^{a-1}r^{k-1}(a-r)^{m-1}. \tag{3.14}$$

We now introduce the variable x and consider the following polynomial expansion, where we collect terms belonging to equal powers of x:

$$\begin{aligned}
&\sum_{i=0}^{d-1}\left(\sum_{r=1}^{i+1}\binom{i}{r-1}r^{k-1}x^r\right)\left(\sum_{j=1}^{d-i}\binom{d-i-1}{j-1}j^{m-1}x^j\right)\\
&\quad=\sum_{i=0}^{d-1}\sum_{a=2}^{d+1}\left[\sum_{r=1}^{a-1}\binom{i}{r-1}r^{k-1}\binom{d-i-1}{a-r-1}(a-r)^{m-1}\right]x^a\\
&\quad=\sum_{a=2}^{d+1}\left[\sum_{i=0}^{d-1}\sum_{r=1}^{i+1}\binom{i}{r-1}\binom{d-i-1}{a-r-1}r^{k-1}(a-r)^{m-1}\right]x^a\\
&\quad=\sum_{a=2}^{d+1}\left[\sum_{r=1}^{a-1}\left(\sum_{i=r-1}^{d-1}\binom{i}{r-1}\binom{d-1-i}{a-r-1}\right)r^{k-1}(a-r)^{m-1}\right]x^a.
\end{aligned}$$

Now the inner-most sum of products of binomial coefficients has the very simple closed expression $\binom{d}{a-1}$. This is just an instance of a well-known combinatorial identity related to the "Vandermonde convolution;" see, e.g., identity (5.26) in [21, p. 169], or identity (3.3) in [20]. Finally, if we replace x by -1, we immediately have (3.14), and this completes the proof. □

For a different evaluation of the left-hand side of (3.13), see Section 5.

4. Proof of Theorem 2.1

Binomial convolutions such as the one in (1.3) are usually obtained by taking the Cauchy product of suitable exponential generating functions. Indeed, Euler's formula (1.3) can be proved by multiplying (1.1) with itself, and suitably manipulating the square of the generating function on the left. In analogy, to prove our main theorem, we use the generating function

$$\frac{d^m}{dx^m}\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_{n+m}\frac{x^n}{n!}, \tag{4.1}$$

which follows immediately from (1.1). By taking the Cauchy product of two such expressions, with the kth and the mth derivative, respectively, we obtain with (1.5) the expression

$$(B_k+B_m)^n=\left[\frac{d^n}{dx^n}\left(\left(\frac{d^k}{dx^k}\frac{x}{e^x-1}\right)\left(\frac{d^m}{dx^m}\frac{x}{e^x-1}\right)\right)\right]_{x=0}. \tag{4.2}$$

Thus we need to study the product of derivatives. To simplify notation, we set

$$g_{n+1,j}:=(j-1)!\big[S(n+1,j)x-nS(n,j)\big], \tag{4.3}$$

where the variable x in $g_{n+1,j}$ is implied. Now, using (3.12) we get

$$\left(\frac{d^k}{dx^k}\frac{x}{e^x-1}\right)\left(\frac{d^m}{dx^m}\frac{x}{e^x-1}\right) = (-1)^{k+m}\left(\sum_{j=1}^{k+1}\frac{g_{k+1,j}}{(e^x-1)^j}\right)\left(\sum_{j=1}^{m+1}\frac{g_{m+1,j}}{(e^x-1)^j}\right)$$
$$= (-1)^{k+m}\sum_{r=1}^{k+m+2}\left(\sum_{i=1}^{r-1}g_{k+1,i}\,g_{m+1,r-i}\right)\frac{1}{(e^x-1)^r}. \tag{4.4}$$

We now state and then apply the following central result which will be proved in the next section.

Proposition 4.1. *For each $r = 1, 2, \dots, k+m+2$ we have*

$$\sum_{i=1}^{r-1} g_{k+1,i}\,g_{m+1,r-i} = \sum_{j=r-1}^{k+m+1}(-1)^{k+m-j}\left(a_j^{k,m}x + b_j^{k,m}\right)g_{j+1,r}, \tag{4.5}$$

where the sum on the left is considered to be 0 *when $r = 1$, and the constants $a_j^{k,m}$, $b_j^{k,m}$ are rational numbers given by*

$$a_j^{k,m} = \begin{cases} (-1)^j\left[(-1)^k\binom{k}{j} + (-1)^m\binom{m}{j}\right]\frac{B_{k+m+1-j}}{k+m+1-j}, & 0 \leqslant j \leqslant k+m,\\ -\frac{k!m!}{(k+m+1)!}, & j = k+m+1, \end{cases} \tag{4.6}$$

$$b_j^{k,m} = \begin{cases} (-1)^j\left[(-1)^k m\binom{k}{j} + (-1)^m k\binom{m}{j}\right]\frac{B_{k+m-j}}{k+m-j}, & 0 \leqslant j \leqslant k+m-1,\\ 0, & j = k+m+1, \end{cases} \tag{4.7}$$

$$b_{k+m}^{k,m} = \begin{cases} -\frac{k!m!}{(k+m)!} & \text{when } k>0 \text{ and } m>0,\\ 0 & \text{when one of } k \text{ and } m \text{ is } 0,\\ 1 & \text{when } k=m=0. \end{cases} \tag{4.8}$$

Next we substitute (4.5) into (4.4) and change the order of summation:

$$\left(\frac{d^k}{dx^k}\frac{x}{e^x-1}\right)\left(\frac{d^m}{dx^m}\frac{x}{e^x-1}\right)$$
$$= \sum_{r=1}^{k+m+2}\left(\sum_{j=r-1}^{k+m+1}(-1)^j\left(a_j^{k,m}x + b_j^{k,m}\right)g_{j+1,r}\right)\frac{1}{(e^x-1)^r}$$
$$= \sum_{j=0}^{k+m+1}(-1)^j\left(a_j^{k,m}x + b_j^{k,m}\right)\sum_{r=1}^{j+1}\frac{g_{j+1,r}}{(e^x-1)^r}$$
$$= \sum_{j=0}^{k+m+1}\left(a_j^{k,m}x + b_j^{k,m}\right)\frac{d^j}{dx^j}\frac{x}{e^x-1},$$

where we have used (3.12). Hence with (4.2) we have by Leibniz's generalized product formula, followed by (4.1),

$$
\begin{aligned}
(B_k+B_m)^n &= \left[\frac{d^n}{dx^n}\sum_{j=0}^{k+m+1}\left(a_j^{k,m}x+b_j^{k,m}\right)\frac{d^j}{dx^j}\frac{x}{e^x-1}\right]_{x=0}\\
&= \left[n\sum_{j=0}^{k+m+1}a_j^{k,m}\frac{d^{j+n-1}}{dx^{j+n-1}}\frac{x}{e^x-1}+\sum_{j=0}^{k+m+1}b_j^{k,m}\frac{d^{j+n}}{dx^{j+n}}\frac{x}{e^x-1}\right]_{x=0}\\
&= n\sum_{j=0}^{k+m+1}a_j^{k,m}B_{n+j-1}+\sum_{j=0}^{k+m+1}b_j^{k,m}B_{n+j}.
\end{aligned}
$$

Using the convention $b_{-1}^{k,m}=0$ and the fact that $b_{k+m+1}^{k,m}=0$ by (4.7), we have therefore proved the following

Proposition 4.2. *For all integers $k,m,n\geqslant 0$ we have*

$$
(B_k+B_m)^n=\sum_{j=0}^{k+m+1}\left(na_j^{k,m}+b_{j-1}^{k,m}\right)B_{n+j-1}, \tag{4.9}
$$

with the coefficients $a_j^{k,m}$ and $b_j^{k,m}$ as in (4.6)–(4.8)*, and $b_{-1}^{k,m}=0$ by convention.*

Let us first consider Euler's case $k=m=0$. In this case (4.6)–(4.8) show that $a_0^{0,0}=2B_1=-1$, $a_1^{0,0}=-1$, $b_0^{0,0}=1$, and $b_1^{0,0}=0$. Then (4.9) clearly gives us (1.6).

When $k+m\geqslant 1$, then (4.6)–(4.8) substituted into (4.9) easily leads to the right-hand side of (2.1). This completes the proof of Theorem 2.1, pending the proof of Proposition 4.1.

5. Proof of Proposition 4.1

In preparation for dealing with some special cases we use (4.3) with (3.6), (3.7), and (3.10) to find the particular evaluations

$$
g_{n+1,1}=x-n \quad (n\geqslant 0), \tag{5.1}
$$

$$
g_{n+1,n}=n!\left(\frac{n+1}{2}x-1\right) \quad (n\geqslant 1), \tag{5.2}
$$

$$
g_{n+1,n+1}=n!x \quad (n\geqslant 0), \tag{5.3}
$$

$$
g_{n+1,j}=0 \quad \text{for } j>n+1\ (n\geqslant 0). \tag{5.4}
$$

We first prove (4.5) for the special case $k=m=0$. With (5.1), (5.3) and the appropriate values of $a_j^{0,0}$ and $b_j^{0,0}$ the case $r=1$ reduces to $0=(-x+1)x-(-x)(x-1)$, which is clearly true. The case $r=2$ is covered by the next case.

Next we consider the case $r = k + m + 2$ for all $k, m \geqslant 0$. Then because of the property (5.4), the identity (4.5) reduces to

$$g_{k+1,k+1}g_{m+1,m+1} = -\left(a_{k+m+1}^{k,m}x + b_{k+m+1}^{k,m}\right)g_{k+m+2,k+m+2},$$

and with (5.3) we immediately get

$$a_{k+m+1}^{k,m}x + b_{k+m+1}^{k,m} = -\frac{k!m!}{(k+m+1)!}x; \tag{5.5}$$

this leads to the second values in both (4.6) and (4.7).

The next case, $r = k + m + 1$, is slightly more complicated. Again by (5.4) our identity (4.5) reduces to

$$\begin{aligned} &g_{k+1,k}g_{m+1,m+1} + g_{k+1,k+1}g_{m+1,m} \\ &\quad = \left(a_{k+m}^{k,m}x + b_{k+m}^{k,m}\right)g_{k+m+1,k+m+1} - \left(a_{k+m+1}^{k,m}x + b_{k+m+1}^{k,m}\right)g_{k+m+2,k+m+1}, \end{aligned} \tag{5.6}$$

provided we have both $k \geqslant 1$ and $m \geqslant 1$. All terms except the first term in parentheses on the right are known by (5.2), (5.3), and (5.5). An easy calculation then gives

$$a_{k+m}^{k,m}x + b_{k+m}^{k,m} = -\frac{k!m!}{(k+m)!},$$

which is consistent with the first part of (4.6) (since both binomial coefficients vanish when $j = k + m$ and $k > 0$, $m > 0$), and with the first part of (4.8).

Now suppose that one of k, m is zero. By symmetry we may suppose that $m = 0$. Then $r = k + 1$, and the left-hand side of (5.6) changes to the single summand $g_{k+1,k}g_{1,1}$, while the right-hand side remains unchanged (but with $m = 0$). Again, solving as before we get

$$a_k^{k,0}x + b_k^{k,0} = -\frac{1}{2}x;$$

this is consistent with the first part of (4.6) which gives $a_k^{k,0} = -\frac{1}{2} = B_1$ when $k \geqslant 1$, and with the second part of (4.8).

For the main part of the proof of Proposition 4.1 we note that the summands on both sides of (4.5) are products of two linear polynomials; see (4.3). We multiply these polynomials and compare coefficients of the powers of x. It will be convenient to replace r by $d + 1$ in (4.5) and for the remainder of this section, so that $d = 0, 1, \ldots, k + m + 1$; but note that the cases $d = k + m, k + m + 1$ have already been settled.

First we compare the constant coefficients on both sides of (4.5), and we get

$$\begin{aligned} &(d-1)!km\sum_{i=0}^{d-1}\frac{S(k,i+1)S(m,d-i)}{\binom{d-1}{i}} \\ &\quad = (-1)^{k+m}d!\sum_{j=d+1}^{k+m+1}(-1)^{j+1}b_j^{k,m}jS(j,d+1), \end{aligned} \tag{5.7}$$

where we have used the fact that $S(d, d+1) = 0$. If we divide both sides by $d!km$ and use (4.7) and (4.8), we easily see that (5.7) is equivalent to the following identity which we state as a proposition, to be proved later.

Proposition 5.1. *For all integers $k, m \geqslant 1$ and $d = 0, 1, \dots, k+m-1$ we have*

$$\begin{aligned}\sum_{j=d+1}^{k+m-1} &\left((-1)^m \binom{k-1}{j-1} + (-1)^k \binom{m-1}{j-1}\right) \frac{B_{k+m-j}}{k+m-j} S(j, d+1) \\ &= \frac{(k-1)!(m-1)!}{(k+m-1)!} S(k+m, d+1) - \sum_{i=0}^{d-1} \frac{S(k, i+1) S(m, d-i)}{d\binom{d-1}{i}}. \qquad (5.8)\end{aligned}$$

Next we equate the coefficients of x^2 on both sides of (4.5). Then, after dividing both sides by $d!$, we have

$$\sum_{i=0}^{d-1} \frac{S(k+1, i+1) S(m+1, d-i)}{d\binom{d-1}{i}} = (-1)^{k+m} \sum_{j=d}^{k+m+1} (-1)^j a_j^{k,m} S(j+1, d+1).$$

With (4.6), replacing $k+1$ by k and $m+1$ by m, and shifting the order of summation on the right-hand side, we see that this last identity follows again from (5.8).

Finally we equate the coefficients of x on both sides of (4.5). Then, again after dividing by $d!$,

$$\begin{aligned}&-\sum_{i=0}^{d-1} \frac{mS(k+1, i+1) S(m, d-i) + kS(k, i+1) S(m+1, d-i)}{d\binom{d-1}{i}} \\ &= (-1)^{k+m} \left(-\sum_{j=d+1}^{k+m+1} (-1)^j j a_j^{k,m} S(j, d+1) - \sum_{j=d+1}^{k+m+1} (-1)^j b_{j-1}^{k,m} S(j, d+1)\right). \qquad (5.9)\end{aligned}$$

First we consider the two sums on the right and note that for $j \leqslant k+m$ we get from the first parts of (4.6) and (4.7),

$$\begin{aligned}(-1)^j \left(j a_j^{k,m} + b_{j-1}^{k,m}\right) &= \left((-1)^k k \binom{k-1}{j-1} + (-1)^m m \binom{m-1}{j-1}\right. \\ &\qquad \left. - (-1)^k m \binom{k}{j-1} - (-1)^m k \binom{m}{j-1}\right) \frac{B_{k+m-j+1}}{k+m-j+1} \\ &= -k\left((-1)^{k-1} \binom{k-1}{j-1} + (-1)^m \binom{m}{j-1}\right) \frac{B_{k+m-j+1}}{k+m-j+1} \\ &\qquad - m\left((-1)^k \binom{k}{j-1} + (-1)^{m-1} \binom{m-1}{j-1}\right) \frac{B_{k+m-j+1}}{k+m-j+1}.\end{aligned}$$

For $j = k+m+1$ we have by (4.6) and (4.8),

$$
\begin{aligned}
(k+m+1)a_{k+m+1}^{k,m}+b_{k+m}^{k,m} &= -\frac{k!m!}{(k+m)!}-\frac{k!m!}{(k+m)!} \\
&= -k\frac{(k-1)!m!}{((k-1)+m+1)!}-m\frac{k!(m-1)!}{(k+(m-1)+1)!}.
\end{aligned}
$$

We now see that (5.9) is obtained by adding two copies of (5.8), one with k replaced by $k+1$, and the other with m replaced by $m+1$. This completes the proof of Proposition 4.1, with Proposition 5.1, however, still remaining to be proved.

6. Convolved powers and the proof of Proposition 5.1

If we consider Proposition 5.1 in relation with the representation (3.5) of the Stirling number $S(n,k)$, it is plausible that (5.8) should be related to the following identity.

Proposition 6.1. *For integers $k, m \geqslant 0$ and $a \geqslant 1$ we have*

$$
\begin{aligned}
&(-1)^{m+1}\sum_{j=0}^{k}\binom{k}{j}\frac{B_{m+1+j}}{m+1+j}a^{k-j}+(-1)^{k+1}\sum_{j=0}^{m}\binom{m}{j}\frac{B_{k+1+j}}{k+1+j}a^{m-j} \\
&\quad=\frac{k!m!}{(k+m+1)!}a^{k+m+1}-\sum_{r=1}^{a-1}r^k(a-r)^m. \qquad (6.1)
\end{aligned}
$$

A few remarks are in order before we prove and apply this result. The identity (6.1) is not new, and has a long and interesting history. It can be seen as an evaluation of the convolved power sum on the right-hand side. Such sums were already studied by Glaisher, first for $k=m$ in [18], and then in general in [19]. There is also an expression for the general case in [24], but it is not explicit. Later the general case was again considered by Neuman and Schonbach [32] from the point of view of numerical analysis. In fact, it is pointed out that (6.1) can be considered a discrete analogue of the interesting integral convolution

$$
\int_0^a r^k(a-r)^m\,dr=\frac{k!m!}{(k+m+1)!}a^{k+m+1};
$$

see [32] for further remarks and references. At about the same time Carlitz [8] generalized (6.1) to a certain "shifted" power convolution; his result involves Bernoulli *polynomials* in place of the Bernoulli numbers. Such a result was also later obtained by the first author [5, p. 205] in a somewhat different but very simple form, using the symbolic notation discussed in the introduction. *Alternating* convolved sums were also obtained in [5, p. 205]; they involve Euler instead of Bernoulli polynomials.

The *Bernoulli polynomials* $B_k(x)$ can be defined by

$$
B_k(x)=\sum_{j=0}^{k}\binom{k}{j}B_jx^{k-j}. \qquad (6.2)
$$

Their most important property is the difference equation $B_k(x+1) - B_k(x) = kx^{k-1}$ for $k \geqslant 1$, which gives rise to the famous summation formula

$$\sum_{r=1}^{a-1} r^k = \frac{1}{k+1}\big(B_{k+1}(a) - B_{k+1}(1)\big). \tag{6.3}$$

The identity (6.1) can actually be seen as a generalization of (6.3). Indeed, if we set $m = 0$ in (6.1) and use the fact that $\binom{k}{j}\frac{1}{j+1} = \binom{k+1}{j+1}\frac{1}{k+1}$ and the definition (6.2) with $k+1$ instead of k, we easily obtain (6.3).

From a different point of view, the identity (6.1) can also be seen as a generalization of a result of Saalschütz [37], later rediscovered by M.B. Gelfand [15], namely

$$(-1)^{m+1}\sum_{j=0}^{k}\binom{k}{j}\frac{B_{m+1+j}}{m+1+j} + (-1)^{k+1}\sum_{j=0}^{m}\binom{m}{j}\frac{B_{k+1+j}}{k+1+j} = \frac{k!m!}{(k+m+1)!}.$$

This is obviously just (6.1) with $a = 1$.

Although (6.1) is known, we present here a proof by induction which appears to be shorter than other known proofs.

Proof of Proposition 6.1. We proceed by induction on m. The case $m = 0$ was already considered above, following (6.3). We now assume that (6.1) holds for a certain $m \geqslant 0$ and for all $k \geqslant 0$; we will show that it then holds for $m+1$ and all k. To do this, we replace k by $k+1$ in (6.1) and then subtract a times the original identity (6.1) from this changed identity. It suffices to consider the four terms separately. First,

$$\begin{aligned}
&(-1)^{m+1}\sum_{j=0}^{k+1}\left[\binom{k+1}{j} - \binom{k}{j}\right]\frac{B_{m+1+j}}{m+1+j}a^{k+1-j} \\
&\quad = (-1)^{m+1}\sum_{j=0}^{k+1}\binom{k}{j-1}\frac{B_{m+1+j}a^{k+1-j}}{m+1+j} = -(-1)^{m+2}\sum_{j=0}^{k}\binom{k}{j}\frac{B_{(m+1)+1+j}a^{k-j}}{(m+1)+1+j}.
\end{aligned}$$

Next,

$$\begin{aligned}
&(-1)^{k}\sum_{j=0}^{m}\binom{m}{j}\frac{B_{k+2+j}}{k+2+j}a^{m-j} - (-1)^{k+1}\sum_{j=0}^{m}\binom{m}{j}\frac{B_{k+1+j}}{k+1+j}a^{m+1-j} \\
&\quad = (-1)^{k}\sum_{j=0}^{m+1}\left[\binom{m}{j-1} + \binom{m}{j}\right]\frac{B_{k+1+j}}{k+1+j}a^{m+1-j} \\
&\quad = -(-1)^{k+1}\sum_{j=0}^{m+1}\binom{m+1}{j}\frac{B_{k+1+j}}{k+1+j}a^{(m+1)-j}.
\end{aligned}$$

The last two terms are even easier:

$$\left(\frac{(k+1)!m!}{(k+m+2)!}-\frac{k!m!}{(k+m+1)!}\right)a^{k+m+2}$$
$$=\frac{k!m!}{(k+m+2)!}\big(k+1-(k+m+2)\big)a^{k+m+2}=-\frac{k!(m+1)!}{(k+m+2)!}a^{k+m+2},$$

and finally,

$$-\sum_{r=1}^{a-1}r^{k+1}(a-r)^m+a\sum_{r=1}^{a-1}r^k(a-r)^m=\sum_{r=1}^{a-1}r^k(a-r)^{m+1}.$$

This completes the proof by induction. □

We are now ready to prove Proposition 5.1. To begin, we write (6.1) in a more convenient form by replacing k and m by $k-1$ and $m-1$, respectively:

$$\sum_{j=1}^{k+m+1}\left[(-1)^m\binom{k-1}{j-1}+(-1)^k\binom{m-1}{j-1}\right]\frac{B_{k+m-j}}{k+m-j}a^{j-1}$$
$$=\frac{(k-1)!(m-1)!}{(k+m-1)!}a^{k+m-1}-\sum_{r=1}^{a-1}r^{k-1}(a-r)^{m-1}. \tag{6.4}$$

Now for each $a=1,2,\dots,d+1$ we multiply both sides of (6.4) by $(-1)^{d+1+a}\binom{d}{a-1}\frac{1}{d!}$ and sum over all these a. Then by (3.5) we get $S(j,d+1)$ in place of a^{j-1} in (6.4), and $S(k+m,d+1)$ in place of a^{k+m-1}; note that the first sum now starts only at $j=d+1$ since $S(j,d+1)=0$ for $j\leqslant d$. Finally, the sum of the last term is just (3.13), so altogether we get (5.8), and the proof is complete.

7. Further remarks

7.1. We could interpret our main result (2.1) also as a relationship between two convolutions involving Bernoulli numbers. Here the length of the left-hand convolution increases with n (if indeed we consider (2.1) as an Euler-type formula), while the length of the right-hand convolution remains fixed and depends on the given parameters k and m.

This relationship between two convolutions is even more obvious in the special case (2.3). We can carry this further by changing the right-hand side of (2.3), using the fact that $B_1=-\frac{1}{2}$ and $B_0=1$, and replacing k by $k-1$. Then we get the symmetric expression

$$k\sum_{j=0}^{n-1}\binom{n}{j}B_{k+n-1-j}B_j+n\sum_{r=0}^{k-1}\binom{k}{r}B_{k+n-1-r}B_r=-nkB_{k+n-2}. \tag{7.1}$$

This identity was actually proved in [3] already. A similar identity, involving both the Bernoulli numbers and the numbers $B_n':=\frac{1}{n}(1-2^n)B_n$ for $n\geqslant 1$, which are closely related to the Genocchi numbers (see remark 7.3 below), was obtained in [4].

Other similar identities can be obtained from our main theorem. For instance, if we set $n = 0$ in (2.1) and note that $(B_k + B_m)^0 = B_k B_m$ by (1.5), then after some easy manipulation we get, for $k \geqslant 1$ and $m \geqslant 1$,

$$m\sum_{r=0}^{k}\frac{(-1)^r}{m+r}\binom{k}{r}B_{m+r}B_{k-r} + k\sum_{r=0}^{m}\frac{(-1)^r}{k+r}\binom{m}{r}B_{k+r}B_{m-r} = B_kB_m + \frac{k!m!}{(k+m)!}B_{k+m}, \tag{7.2}$$

or equivalently (after replacing k by $k+1$ and m by $m+1$),

$$\sum_{r=0}^{k}(-1)^r\binom{k}{r}\frac{B_{m+1+r}}{m+1+r}\cdot\frac{B_{k+1-r}}{k+1-r} + \sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{B_{k+1+r}}{k+1+r}\cdot\frac{B_{m+1-r}}{m+1-r} = \frac{B_{k+1}}{k+1}\cdot\frac{B_{m+1}}{m+1} + \left(\frac{k!m!}{(k+m+1)!} + \frac{(-1)^k}{k+1} + \frac{(-1)^m}{m+1}\right)\frac{B_{k+m+2}}{k+m+2}. \tag{7.3}$$

The factors $(-1)^r$ in (7.2) and (7.3) can be removed if we pay appropriate attention to the exceptional number $B_1 = -1/2$. Also note that the identities are meaningful, as are most others we have dealt with, only when k and m have the same parity, although they do hold for all pairs k and m. The identity (7.2) is also not really new. It can be obtained from a formula for Bernoulli polynomials that was derived in [33, p. 75] and can be found in modern notation, and more accessibly, in [7].

These last relations, especially (7.3), are somehow reminiscent of a remarkable identity of Miki [31] that connects a binomial convolution and an ordinary convolution of Bernoulli numbers, but in contrast to (7.3) also involves harmonic numbers. See also [17] for further remarks and generalizations.

7.2. In view of the various generalizations of Euler's formula to sums of products of more than two Bernoulli numbers that were mentioned in the introduction, it is natural to ask whether Theorem 2.1 can also be extended in this direction. The proof in Section 4 indicates that such an extension is indeed possible. However, this would lead too far for this paper, and will be the subject of a separate study.

7.3. We are grateful to one of the anonymous referees for the following remark. The *Genocchi numbers* G_n, $n = 0, 1, \dots$, can be defined by the generating function

$$\frac{2x}{e^x+1} = \sum_{n=0}^{\infty} G_n\frac{x^n}{n!}, \quad |x| < 2\pi.$$

By comparing this with the generating function (1.1) it can be seen that

$$G_n = 2\left(1-2^n\right)B_n,$$

and with the von Staudt–Clausen theorem (see, e.g., [23, p. 91]) we see that the G_n are integers. Using expansions analogous to (3.11) and (3.12) we obtain results analogous to Propositions 4.1 and 4.2, and then to all the results in Section 2.

This generalizes the most basic Euler-type convolution for the Genocchi numbers, namely

$$\sum_{j=0}^{n} \binom{n}{j} G_j G_{n-j} = 2\big(nG_{n-1} + (n-1)G_n\big), \quad n \geqslant 1, \tag{7.4}$$

which can be found in [4] in a slightly different form. Compare this with (1.3) and note the factor -2 on the right-hand side of (7.4). This is true in general: The Genocchi analogue of (2.1) is obtained by replacing B_{n+r-1} and B_{n+k+m} by G_{n+r-1} and G_{n+k+m}, respectively, and then multiplying the right-hand side by -2.

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