Higher-order recurrences for Bernoulli numbers ☆

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Euler's well-known nonlinear relation for Bernoulli numbers, which can be written in symbolic notation as $(B_0 + B_0)^n = -nB_{n-1} - (n-1)B_n$, is extended to $(B_{k_1} + \cdots + B_{k_m})^n$ for $m \geqslant 2$ and arbitrary fixed integers $k_1, \ldots, k_m \geqslant 0$. In the general case we prove an existence theorem for Euler-type formulas, and for $m = 3$ we obtain explicit expressions. This extends the authors' previous work for $m = 2$.

1. Introduction

The Bernoulli numbers B_n, $n = 0, 1, 2, \ldots$, can be defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi. \tag{1.1}$$

The first few values are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, and $B_n = 0$ for all odd $n \geqslant 3$; we also have $(-1)^{n+1}B_{2n} > 0$ for all $n \geqslant 1$. These and many other properties can be found, for instance, in [1], [12], [14], or [18]; for a comprehensive bibliography, see [9].

One of the most remarkable identities for the Bernoulli numbers is *Euler's formula*

$$\sum_{j=0}^{n} \binom{n}{j} B_j B_{n-j} = -nB_{n-1} - (n-1)B_n \quad (n \geqslant 1), \tag{1.2}$$

which can be considered a convolution identity, or also a quadratic recurrence relation. This identity has been extended in various directions; see [3] for a summary with numerous references.

It will be convenient to use the symbolic notation (or "classical umbral calculus"; see, e.g., [10]) to write

$$(B_k + B_l)^n = \sum_{j=0}^{n} \binom{n}{j} B_{k+j} B_{l+n-j}, \tag{1.3}$$

so that Euler's formula takes the form $(B_0 + B_0)^n = -nB_{n-1} - (n-1)B_n$, $n \geqslant 1$. In [3] we extended this by giving an explicit expression for $(B_k + B_l)^n$ for arbitrary integers $k, l \geqslant 0$ and $n \geqslant 1$. At the end of that paper we indicated that similar methods could be used to also obtain Euler-type formulas for higher-order analogues of (1.3), namely for the sums

$$(B_{k_1} + \cdots + B_{k_m})^n = \sum_{\substack{i_1+\cdots+i_m=n \\ i_1,\ldots,i_m \geqslant 0}} \frac{n!}{i_1! \cdots i_m!} B_{k_1+i_1} \cdots B_{k_m+i_m}. \tag{1.4}$$

For the case $k_1 = \cdots = k_m = 0$ a variant of the problem (with even positive indices i_j and even n) was settled by the second author [8], with analogous results for Euler numbers and Bernoulli and Euler *polynomials*. Further extensions and analogues were subsequently obtained by other authors; see [7,13,15–17].

It is the purpose of this paper to deal with the sums (1.4) in general. Since this can be considered a continuation of our previous paper [3], we will quote several auxiliary results from there.

Our main result, stated in Section 2, will be the *existence* of an Euler-type formula in the most general case. In Section 3 we show how the coefficients in this main result can be determined by computation. Furthermore, if the parameters $k_1, \ldots, k_m$ are large enough (greater than $m-1$), then we will be able to explicitly state the leading coefficient in the expansion; this will be done in Section 4. In Section 5 we indicate how to obtain formulas for all triples (k_1, k_2, k_3) in the case $m = 3$, and give explicit expressions when $k_1 = k_2 = k_3$.

2. The existence result

Before we state the first and most general result of this paper, we introduce some notation. Let $m \geqslant 2$ be an integer, and $K := (k_1, \ldots, k_m)$ a vector of m nonnegative integers. Furthermore, we set $s_m := k_1 + \cdots + k_m$.

Theorem 1. *With notation as above, we have for all integers $n \geqslant m - 1$,*

$$(B_{k_1} + \cdots + B_{k_m})^n = \sum_{\nu=-m+1}^{s_m} C_\nu^K(n) B_{n+\nu}, \tag{2.1}$$

where the polynomials $C_\nu^K(x)$ have rational coefficients, depend only on the vector K (and not on n), are recursively computable, and $\deg(C_\nu^K(x)) \leqslant m - 1$ for all ν. Furthermore, $C_{-m+1}^K(n)$ vanishes unless $k_1 = \cdots = k_m = 0$, in which case

$$C_{-m+1}^K(n) = (-1)^{m-1} \frac{n!}{(n-m+1)!} = (-1)^{m-1} n(n-1) \cdots (n-m+2). \tag{2.2}$$

To begin the proof of this result we use the generating function

$$\frac{d^k}{dx^k} \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_{n+k} \frac{x^n}{n!}, \tag{2.3}$$

which follows directly from (1.1). By taking the Cauchy product of m instances of this generating function, with k replaced by $k_1, \ldots, k_m$, respectively, we get with the definition (1.4),

$$(B_{k_1} + \cdots + B_{k_m})^n = \left[\frac{d^n}{dx^n} \prod_{j=1}^{m} \left(\frac{d^{k_j}}{dx^{k_j}} \frac{x}{e^x - 1} \right) \right]_{x=0}. \tag{2.4}$$

The right-hand side of (2.4) now motivates the following auxiliary result.

Lemma 1. *Let $m \geqslant 1$ and $k_1, \ldots, k_m$ be nonnegative integers.*

(a) *There exists a unique and recursively computable sequence of polynomials $A_j(x) \in \mathbb{Q}[x]$ with $\deg(A_j(x)) \leqslant m - 1$, $j = 0, 1, \ldots, s_m + m - 1$, such that*

$$\prod_{j=1}^{m} \left(\frac{d^{k_j}}{dx^{k_j}} \frac{x}{e^x - 1} \right) = \sum_{j=0}^{s_m+m-1} A_j(x) \frac{d^j}{dx^j} \frac{x}{e^x - 1}. \tag{2.5}$$

(b) *If we set $A_j(x) = a_{j,m-1}x^{m-1} + a_{j,m-2}x^{m-2} + \cdots + a_{j,1}x + a_{j,0}$, then $a_{j,i} = 0$ whenever $j - i > s_m$.*

(c) *We have $a_{0,m-1} = 0$ unless $k_1 = \cdots = k_m = 0$, in which case $a_{0,m-1} = (-1)^{m-1}$.*

For the proof of Lemma 1, and also for Section 5, we need an explicit result from [3] which we quote here as a lemma, in a somewhat simplified form. (The corresponding result in [3] includes $k, l = 0$.)

Lemma 2. *Let k and l be positive integers, and set*

$$\left(\frac{d^k}{dx^k} \frac{x}{e^x - 1} \right) \left(\frac{d^l}{dx^l} \frac{x}{e^x - 1} \right) = \sum_{j=0}^{k+l+1} A_j^{k,l}(x) \frac{d^j}{dx^j} \frac{x}{e^x - 1}, \tag{2.6}$$

with $A_j^{k,l}(x) = b_{j,1}^{k,l}x + b_{j,0}^{k,l}$. *Then*

$$b_{j,1}^{k,l} = \begin{cases} (-1)^j[(-1)^k\binom{k}{j} + (-1)^l\binom{l}{j}]\frac{B_{k+l+1-j}}{k+l+1-j}, & 0 \leqslant j \leqslant k+l, \\ -\frac{k!l!}{(k+l+1)!}, & j = k+l+1; \end{cases} \tag{2.7}$$

$$b_{j,0}^{k,l} = \begin{cases} (-1)^j[(-1)^kl\binom{k}{j} + (-1)^lk\binom{l}{j}]\frac{B_{k+l-j}}{k+l-j}, & 0 \leqslant j \leqslant k+l-1, \\ -\frac{k!l!}{(k+l)!}, & j = k+l, \\ 0, & j = k+l+1. \end{cases} \tag{2.8}$$

The identity (2.7) *also holds when $k = 0$ or $l = 0$.*

Proof of Lemma 1. We prove this lemma by induction on m. (a) For $m = 1$ the statement is trivial. For the remainder of the proof we indicate the dependence of $A_j(x)$ on the $k_1, \ldots, k_m$ by superscripts. The case $m = 2$ is immediate from Lemma 2. Now we suppose that (2.5) holds up to some m, and we multiply both sides of (2.5) by

$$\frac{d^{k_{m+1}}}{dx^{k_{m+1}}} \frac{x}{e^x - 1}.$$

By using the result for $m = 2$ we get

$$\prod_{j=1}^{m+1}\left(\frac{d^{k_j}}{dx^{k_j}}\frac{x}{e^x-1}\right)=\sum_{j=0}^{s_m+m-1}A_j^{k_1,\dots,k_m}(x)\left(\frac{d^j}{dx^j}\frac{x}{e^x-1}\right)\left(\frac{d^{k_{m+1}}}{dx^{k_{m+1}}}\frac{x}{e^x-1}\right)$$

$$=\sum_{j=0}^{s_m+m-1}A_j^{k_1,\dots,k_m}(x)\sum_{v=0}^{j+k_{m+1}+1}A_v^{j,k_{m+1}}(x)\frac{d^v}{dx^v}\frac{x}{e^x-1}$$

$$=\sum_{v=0}^{s_{m+1}+m}\left(\sum_{j=v-k_{m+1}-1}^{s_m+m-1}A_v^{j,k_{m+1}}(x)A_j^{k_1,\dots,k_m}(x)\right)\frac{d^v}{dx^v}\frac{x}{e^x-1},$$

where by convention we take $A_j^{k_1,\dots,k_m}(x)$ to be the zero polynomial for $j < 0$. Now the inner summation on the right-hand side is the sum of products of polynomials with rational coefficients and of degrees at most 1 and at most $m-1$, respectively. Hence the inner sum is a polynomial of degree at most m, with rational coefficients, and is recursively computable. The uniqueness of the polynomials $A_j(x)$ also follows from this induction.

(b) The case $m = 1$ is trivially true, while the statement for $m = 2$ follows from (2.8). Suppose now that the statement holds for some $m \geqslant 2$, and consider

$$A_v^{k_1,\dots,k_{m+1}}(x)=\sum_{j=v-k_{m+1}-1}^{s_m+m-1}A_v^{j,k_{m+1}}(x)A_j^{k_1,\dots,k_m}(x),\tag{2.9}$$

for $v = 0, 1, \dots, s_{m+1} + m$. Obviously it suffices to prove the statement for each summand in (2.9). So fix j, $0 \leqslant j \leqslant s_m + m - 1$, write $A_j^{k_1,\dots,k_m}(x)$ as in Lemma 1(b) and set, to simplify notation, $A_v^{j,k_{m+1}}(x) = b_{v,1}x + b_{v,0}$. Then

$$A_v^{j,k_{m+1}}(x)A_j^{k_1,\dots,k_m}(x)=\sum_{i=0}^{m}(b_{v,1}a_{j,i-1}+b_{v,0}a_{j,i})x^i,\tag{2.10}$$

where by convention we assume $a_{j,-1} = a_{j,m} = 0$. Consider now the ith coefficient in (2.10). The first summand, namely $b_{v,1}a_{j,i-1}$, vanishes by hypothesis if $v - 1 > j + k_{m+1}$ (in which case $b_{v,1} = 0$) or $j - (i - 1) > s_m$ (in which case $a_{j,i-1} = 0$). If we now add these two inequalities, we get $v - i > k_{m+1} + s_m(= s_{m+1})$; this means that at least one of the original inequalities must hold if we assume that $v - i > s_{m+1}$. Similarly, the second summand, namely $b_{v,0}a_{j,i}$, vanishes if $v > j + k_{m+1}$ or $j - i > s_m$. Again, one of these two inequalities must hold if $v - i > s_{m+1}$. This completes the proof of part (b).

(c) The case $m = 1$ is again trivial, and the statement for $m = 2$ follows from (2.7) which gives

$$b_{0,1}^{k,l}=\left[(-1)^k+(-1)^l\right]\frac{B_{k+l+1}}{k+l+1}.\tag{2.11}$$

We now see that when k and l have different parities, then $(-1)^k + (-1)^l = 0$; if k, l have the same parity then $B_{k+l+1} = 0$ since odd-index Bernoulli numbers vanish, with the only exception $B_1 = -1/2$, so that $b_{0,1}^{0,0} = -1$.

Now consider (2.9) and (2.10), and let $\bar{a}_{0,m}$ be the coefficient of x^m in the polynomial $A_0^{k_1,\dots,k_{m+1}}(x)$. Then we have

$$\bar{a}_{0,m}=\sum_{j=0}^{s_m+m-1}b_{0,1}^{j,k_{m+1}}a_{j,m-1}.$$

Table 1
The polynomials $A_j(x)$ for $K = (1, 2, 3)$.

j	8	7	6	5	4	3	2	1	0
$a_{j,2}$	$\frac{1}{3360}$	0	$\frac{-1}{720}$	0	$\frac{1}{1140}$	0	$\frac{1}{2520}$	0	0
$a_{j,1}$	0	$\frac{1}{210}$	$\frac{1}{120}$	$\frac{-1}{120}$	0	$\frac{1}{72}$	$\frac{-1}{120}$	$\frac{-1}{126}$	0
$a_{j,0}$	0	0	$\frac{1}{60}$	$\frac{1}{20}$	0	$\frac{-1}{12}$	$\frac{-1}{20}$	0	$\frac{1}{126}$

Hence by (2.11) we have $\bar{a}_{0,m} = 0$ unless $k_{m+1} = 0$ and $a_{0,m-1} \neq 0$. Therefore by induction we have $\bar{a}_{0,m} = 0$ unless all $k_j = 0$, $j = 1, \ldots, m+1$, in which case $\bar{a}_{0,m} = (-1)^m$. This completes the proof. $\quad\square$

Remark. An alternative proof of the uniqueness of the polynomials $A_j(x)$ rests on the linear independence of the power series $(x/(e^x - 1))^r$ over the field $\mathbb{Q}(x)$. Indeed, if we have a linear relation

$$\sum_{j=0}^{n} f_j(x) \frac{x^j}{(e^x - 1)^j} = 0$$

with $f_n(x) \neq 0$, then there exists a positive integer k with $f_n(2\pi k i) \neq 0$. But this means that the left-hand side of the above equation has a pole of order n, which is a contradiction.

We also remark that part (b) in Lemma 1 explains the lower left triangle of zeros in Table 1, and part (c) accounts for the zero in the upper right-hand corner of the table. (In general there will not be a larger triangle of zeros in that corner.)

Proof of Theorem 1, continued. With (2.4) and (2.5) we get

$$(B_{k_1} + \cdots + B_{k_m})^n = \sum_{j=0}^{s_m+m-1} \sum_{i=0}^{m-1} a_{j,i} \left[\frac{d^n}{dx^n} \left(x^i \frac{d^j}{dx^j} \frac{x}{e^x - 1} \right) \right]_{x=0}. \tag{2.12}$$

Now by Leibniz's rule for higher derivatives of a product we have

$$\left[\frac{d^n}{dx^n} \left(x^i \frac{d^j}{dx^j} \frac{x}{e^x - 1} \right) \right]_{x=0} = \sum_{k=0}^{n} \binom{n}{k} \left[\left(\frac{d^k}{dx^k} x^i \right) \left(\frac{d^{j+n-k}}{dx^{j+n-k}} \frac{x}{e^x - 1} \right) \right]_{x=0}$$

$$= \binom{n}{i} i! \left[\frac{d^{j+n-i}}{dx^{j+n-i}} \frac{x}{e^x - 1} \right]_{x=0} = \frac{n!}{(n-i)!} B_{j+n-i},$$

where we have used (2.3) in the last step. Thus, with (2.12) and upon changing the order of summation, we get

$$(B_{k_1} + \cdots + B_{k_m})^n = \sum_{j=0}^{s_m+m-1} \sum_{i=0}^{m-1} a_{j,i} \frac{n!}{(n-i)!} B_{j+n-i}$$

$$= \sum_{v=-m+1}^{s_m+m-1} \left(\sum_{i=0}^{m-1} a_{v+i,i} \frac{n!}{(n-i)!} \right) B_{n+v}, \tag{2.13}$$

with the convention that $a_{j,i} = 0$ for $j < 0$. Now the inner sum on the right-hand side is clearly a polynomial in n of degree at most $m-1$, with rational coefficients that are computable by Lemma 1. We denote this polynomial by $C_v^K(n)$, that is,

$$C_\nu^K(n) = \sum_{i=0}^{m-1} a_{\nu+i,i} \frac{n!}{(n-i)!}. \tag{2.14}$$

Since by the last part of Lemma 1 we have $a_{\nu+i,i} = 0$ whenever $\nu > s_m$, the sum in (2.1) goes only up to s_m.

Finally, by (2.14) we have $C_{-m+1}^K(n) = a_{0,m-1}n!/(n-m+1)!$, and so the last assertion of Theorem 1 follows from Lemma 1(c). This completes the proof. $\square$

3. Connections with Stirling numbers

In this section we use some basic properties of Stirling numbers of the second kind to derive a simpler and practically feasible recurrence relation for the polynomials $A_j(x)$ in (2.5). Some useful properties and references for the Stirling numbers of the second kind, $S(n,k)$, can be found in [3]. As we did in [4] and (in a different notation) in [3], we define the linear polynomial

$$T(n, j) := (j-1)! \big[S(n+1, j)x - nS(n, j) \big] \quad (j \geqslant 1). \tag{3.1}$$

The main connection with Bernoulli numbers is then given by the following expansion, which was proved in [3] and is also used in [2].

Lemma 3. *For any $m \geqslant 0$ we have*

$$\frac{d^m}{dx^m} \frac{x}{e^x - 1} = (-1)^m \sum_{j=1}^{m+1} \frac{T(m, j)}{(e^x - 1)^j}. \tag{3.2}$$

While the proof of Lemma 1 allows us, in principle, to compute the polynomials $A_j(x)$, and thus also the $C_\nu^K(n)$, this would be rather cumbersome in practice. The main significance of Lemma 1 lies in the fact that it shows us that the $A_j(x)$ are polynomials over $\mathbb{Q}$ of degree at most $m-1$. This is used in the following result which will lead to easier computations.

Theorem 2. *Let $m \geqslant 1$ and $k_1, \ldots, k_m$ be nonnegative integers. Then for each $r = 1, 2, \ldots, s_m + m$ we have polynomials $A_j(x)$ of degree at most $m-1$, with*

$$\sum_{\substack{i_1+\cdots+i_m=r \\ i_1,\ldots,i_m \geqslant 1}} \prod_{j=1}^{m} T(k_j, i_j) = \sum_{j=r-1}^{s_m+m-1} (-1)^{s_m-j} A_j(x) T(j, r), \tag{3.3}$$

and the $A_j(x)$ are the same as in Lemma 1.

Proof. With (3.2) we get

$$\prod_{j=1}^{m} \left(\frac{d^{k_j}}{dx^{k_j}} \frac{x}{e^x - 1} \right) = (-1)^{s_m} \prod_{j=1}^{m} \left(\sum_{i=1}^{k_j+1} \frac{T(k_j, i)}{(e^x - 1)^i} \right)$$

$$= (-1)^{s_m} \sum_{r=1}^{s_m+m} \left(\sum_{\substack{i_1+\cdots+i_m=r \\ i_1,\ldots,i_m \geqslant 1}} \prod_{j=1}^{m} T(k_j, i_j) \right) \frac{1}{(e^x - 1)^r}, \tag{3.4}$$

where the inner sum on the right is empty, and thus zero, for $r < m$. On the other hand, we have from (2.5) and (3.2),

$$\prod_{j=1}^{m}\left(\frac{d^{k_j}}{dx^{k_j}}\frac{x}{e^x-1}\right) = \sum_{j=0}^{s_m+m-1} A_j(x)(-1)^j \sum_{r=1}^{j+1}\frac{T(j,r)}{(e^x-1)^r}$$

$$= \sum_{r=1}^{s_m+m}\left(\sum_{j=r-1}^{s_m+m-1}(-1)^j A_j(x)T(j,r)\right)\frac{1}{(e^x-1)^r}. \qquad (3.5)$$

Since the functions $(e^x-1)^{-r}$, $r = 1, 2, \ldots, s_m+m$, are linearly independent over $\mathbb{Q}(x)$ (see the Remark following the proof of Lemma 1), we immediately get (3.3) from comparing the right-hand sides of (3.4) and (3.5). $\square$

If we set $r = s_m + m$ in (3.3), the only nonzero term on the left corresponds to $i_j = k_j + 1$ for $j = 1, 2, \ldots, m$. Then we use the fact that $S(n,n) = 1$ and $S(n,k) = 0$ for $k > n$, then with (3.1) we get $k_1! \cdots k_m! x^m = (-1)^{m-1}(s_m + m - 1)! x A_{s_m+m-1}(x)$, and thus

$$A_{s_m+m-1}(x) = (-1)^{m-1}\frac{k_1!\cdots k_m!}{(s_m+m-1)!}x^{m-1}. \qquad (3.6)$$

We can now use this as the beginning of a recurrence relation for the $A_j(x)$; just rewrite (3.3) as

$$(-1)^{s_m-r-1}(r-1)! x A_{r-1}(x) = \sum_{\substack{i_1+\cdots+i_m=r \\ i_1,\ldots,i_m\geqslant 1}}\prod_{j=1}^{m} T(k_j,i_j) + \sum_{j=r}^{s_m+m-1}(-1)^{s_m-j-1}A_j(x)T(j,r). \qquad (3.7)$$

This can be used as a "downwards" recursion, successively for $r = s_m + m - 1, s_m + m - 2, \ldots, 1$. Computations are facilitated through the fact that major computer algebra systems, such as Maple or Mathematica, have the Stirling numbers (of both kinds) as built-in functions.

As an example we take $k_1 = 1$, $k_2 = 2$, $k_3 = 3$. If we set, as in Lemma 1(b), $A_j(x) = a_{j,2}x^2 + a_{j,1}x + a_{j,0}$ $(j = 0, 1, \ldots, 8)$, then (3.6) leads to the column for $j = 8$ in Table 1 (note that $1!2!3!/(6 + 3 - 1)! = 1/3360$), and (3.7) gives all the successive columns.

In (2.14) we saw how the polynomials $C_\nu^K(n)$ are related to the $A_j(x)$. Here it reduces to

$$C_\nu^{(1,2,3)}(n) = a_{\nu+2,2}n(n-1) + a_{\nu+1,1}n + a_{\nu,0}, \quad \nu = -2, -1, \ldots, 6. \qquad (3.8)$$

Thus the highest term in (2.1) for $K = (1, 2, 3)$ is

$$\left(\frac{1}{3360}n(n-1) + \frac{1}{210}n + \frac{1}{60}\right)B_{n+6} = \frac{(n+8)(n+7)}{3360}B_{n+6}.$$

For the other terms, and for other parameter vectors K, see Corollary 1 below.

Corollary 1. *For all $n \geqslant 2$ we have*

$$(B_1 + B_1 + B_1)^n = \frac{(n+5)(n+4)}{120}B_{n+3} + \frac{n+3}{4}B_{n+2} - \frac{n^2-n-24}{24}B_{n+1}$$

$$- \frac{n-1}{4}B_n + \frac{n(n-2)}{30}B_{n-1};$$

$$(B_1 + B_1 + B_2)^n = \frac{(n+6)(n+5)}{360} B_{n+4} + \frac{n+4}{12} B_{n+3} - \frac{n^2+n-24}{72} B_{n+2}$$
$$- \frac{n}{12} B_{n+1} + \frac{(n+1)(n-1)}{90} B_n;$$

$$(B_1 + B_2 + B_2)^n = \frac{(n+7)(n+6)}{1260} B_{n+5} + \frac{n+5}{60} B_{n+4} - \frac{(n+5)(n+4)}{360} B_{n+3}$$
$$- \frac{1}{6} B_{n+2} - \frac{n(n-13)}{360} B_{n+1} - \frac{n-1}{60} B_n + \frac{n(n-2)}{210} B_{n-1};$$

$$(B_1 + B_1 + B_3)^n = \frac{(n+7)(n+6)}{840} B_{n+5} + \frac{n+5}{20} B_{n+4} - \frac{n^2-n+50}{120} B_{n+3}$$
$$- \frac{n-3}{12} B_{n+2} + \frac{n(n-3)}{60} B_{n+1} + \frac{n-1}{30} B_n - \frac{n(n-2)}{105} B_{n-1};$$

$$(B_2 + B_2 + B_2)^n = \frac{(n+8)(n+7)}{5040} B_{n+6} - \frac{n+5}{60} B_{n+4} - \frac{n^2-n-32}{240} B_{n+2}$$
$$+ \frac{(5n+26)(n-1)}{1260} B_n;$$

$$(B_1 + B_2 + B_3)^n = \frac{(n+8)(n+7)}{3360} B_{n+6} + \frac{n+6}{120} B_{n+5} - \frac{n(n+5)}{720} B_{n+4} - \frac{1}{12} B_{n+3}$$
$$+ \frac{(6n-19)(n+18)}{6840} B_{n+2} - \frac{1}{120} B_{n+1} + \frac{(n-1)(n-20)}{2520} B_n;$$

$$(B_1 + B_1 + B_4)^n = \frac{(n+8)(n+7)}{1680} B_{n+6} + \frac{n+6}{30} B_{n+5} - \frac{n^2-n-75}{180} B_{n+4} - \frac{n-4}{12} B_{n+3}$$
$$+ \frac{11n^2-31n+48}{720} B_{n+2} + \frac{n}{20} B_{n+1} - \frac{(13n-8)(n-1)}{1260} B_n.$$

Proof. Use (3.6) and (3.7) to create the equivalent of Table 1 for each parameter vector K. Then use (3.8) to compute the coefficients $C_\nu^K(n)$ in (2.1). $\quad\square$

4. The leading coefficient

In this short section we use a certain convolution formula for Stirling numbers of the second kind, proved elsewhere, to find an explicit expression for the leading coefficient $C_{s_m}^K(n)$ in the expansion (2.1).

If we were to set up Table 1 for $K = (2, 2, 2)$, we would find that $a_{8,2} = 1/5040$, $a_{7,1} = 1/315$, and $a_{6,0} = 1/90$, with all the other coefficients for $j = 6, 7, 8$ vanishing. This is actually true in general: If $k_1, \ldots, k_m$ are sufficiently large then in addition to (3.6) there are explicit formulas for $A_{s_m}(x)$, $A_{s_m+1}(x), \ldots, A_{s_m+m-2}(x)$ as monomials.

Theorem 3. *Let $m \geqslant 2$ and $k_j \geqslant m - 1$ for $j = 1, \ldots, m$. Then we have*

$$A_{s_m+m-\nu}(x) = (-1)^{m-1} \frac{k_1! \cdots k_m!}{(s_m+m-\nu)!} \binom{m-1}{\nu-1} x^{m-\nu} \tag{4.1}$$

for $\nu = 1, 2, \ldots, m$.

The identity (3.6) is obviously a special case of (4.1). The main ingredient in the proof of Theorem 3 is the following result, proved in [4]; see also [6].

Lemma 4. *Let $m \geqslant 2$ and $k_j \geqslant m - 1$ for $j = 1, \ldots, m$. Then for all integers $r \geqslant s_m + 1$ we have*

$$\sum_{\substack{i_1 + \cdots + i_m = r \\ i_1, \ldots, i_m \geqslant 1}} \prod_{j=1}^{m} \frac{T(k_j, i_j)}{k_j!} = \sum_{\nu=1}^{s_m + m + 1 - r} (-1)^{\nu-1} \binom{m-1}{\nu-1} \frac{T(s_m + m - \nu, r)}{(s_m + m - \nu)!} x^{m-\nu}. \qquad (4.2)$$

Proof of Theorem 3. Using the uniqueness of $A_j(x)$ and changing the order of summation on the right-hand side of (3.3), we see that (4.1) follows immediately from (3.3) and (4.2). $\square$

If we use the notation of Lemma 1(b) and set $i := m - \nu$, then we get from (4.1) for $i = 0, 1, \ldots, m - 1$,

$$a_{s_m + i, i} = (-1)^{m-1} \frac{k_1! \cdots k_m!}{(s_m + i)!} \binom{m-1}{i}. \qquad (4.3)$$

This, substituted into (2.14), gives

$$C_{s_m}^K(n) = (-1)^{m-1} k_1! \cdots k_m! n! \sum_{i=0}^{m-1} \binom{m-1}{i} \frac{1}{(s_m + i)!(n - i)!}$$

$$= (-1)^{m-1} \frac{k_1! \cdots k_m! n!}{(s_m + n)!} \sum_{i=0}^{m-1} \binom{m-1}{i} \binom{s_m + n}{n - i}.$$

The sum on the right-hand side has the explicit evaluation $\binom{s_m + m - 1 + n}{n}$. This follows from a variant of the well-known "Vandermonde convolution"; see, e.g., identity (3.4) in [11], or (5.27) in [12, p. 170]. Hence we have

$$C_{s_m}^K(n) = (-1)^{m-1} \frac{k_1! \cdots k_m!}{(s_m + n)!} \frac{(s_m + m - 1 + n)!}{(s_m + m - 1)!},$$

which proves the following result.

Corollary 2. *Let $m \geqslant 2$ and $k_j \geqslant m - 1$ for $j = 1, \ldots, m$. Then the leading term in the expansion* (2.1) *is*

$$(-1)^{m-1} \frac{k_1! \cdots k_m!}{(s_m + m - 1)!} \left(\prod_{i=1}^{m-1} (n + s_m + i) \right) B_{n+s_m}. \qquad (4.4)$$

We see that (4.4) is consistent with Corollary 1 for $K = (2, 2, 2)$, and with the list of specific expansions for $m = 2$ and $k_1 \geqslant 1$, $k_2 \geqslant 1$ given in [3, Corollary 2.4]. However, it appears that (4.4) remains true for all the other special cases listed in Corollary 1. Thus, while computations show that (4.1) is not valid for $\nu = m$ unless $k_j \geqslant m - 1$ for all j, it appears that (4.3) remains valid for $i = 0$. We will not consider this possible improvement here.

5. The case $m = 3$

The proof of Lemma 1 indicates that in general the determination of the polynomial $A(x)$ in (2.5) through iterating (2.9) would be very cumbersome, and we cannot expect reasonable closed expressions. However, it is still possible to find explicit expressions for $m = 3$, generalizing those listed in Corollary 1.

We rewrite (2.9) for $\nu = 0, 1, \ldots, k_1 + k_2 + k_3 + 2$:

$$A_\nu^{k_1,k_2,k_3}(x) = \sum_{j=\nu-k_3-1}^{k_1+k_2+1} A_\nu^{j,k_3}(x) A_j^{k_1,k_2}(x)$$

$$= \sum_{j=\nu-k_3-1}^{k_1+k_2+1} \left(a_\nu^{j,k_3} x + b_\nu^{j,k_3}\right)\left(a_j^{k_1,k_2} x + b_j^{k_1,k_2}\right). \tag{5.1}$$

We can now combine this with Lemma 2 and (2.14) to obtain an expression for (2.1) for any triple $K = (k_1, k_2, k_3)$. For instance, in this way we obtain the following special formulas which supplement Corollary 1.

Corollary 3. *For all $n \geqslant 2$ we have*

$$(B_0 + B_0 + B_0)^n = \frac{(n-1)(n-2)}{2} B_n + 3\frac{n(n-2)}{2} B_{n-1} + n(n-1)B_{n-2},$$

$$(B_0 + B_0 + B_1)^n = \frac{n(n-1)}{6} B_{n+1} + \frac{(n-1)(n+1)}{2} B_n + \frac{n(n+1)}{3} B_{n-1},$$

$$(B_0 + B_1 + B_1)^n = \frac{n(n+3)}{24} B_{n+2} + \frac{n(n+8)}{12} B_{n+1} - \frac{n^2 - 19n - 6}{24} B_n - \frac{n(n-2)}{12} B_{n-1},$$

$$(B_0 + B_0 + B_2)^n = \frac{n(n-1)}{12} B_{n+2} + \frac{n(n-1)}{3} B_{n+1} + \frac{(5n-2)(n-1)}{12} B_n + \frac{n(n-2)}{6} B_{n-1}.$$

While in general the method just outlined is not a very satisfactory result, in the special case $k_1 = k_2 = k_3$ an explicit general formula can be obtained. We set $k := k_1 = k_2 = k_3$, so that $K = (k, k, k)$.

Theorem 4. *For all $k \geqslant 1$ and $n \geqslant 2$ we have with $K = (k, k, k)$,*

$$(B_k + B_k + B_k)^n = \sum_{j=-1}^{3k} C_j^K(n) B_{n+j}, \tag{5.2}$$

where

$$C_{3k}^K(n) = \frac{k!^3}{(3k+2)!}(n+3k+1)(n+3k+2), \tag{5.3}$$

$$C_j^K(n) = 0, \quad 2k+1 \leqslant j \leqslant 3k-1, \tag{5.4}$$

$$C_j^K(n) = \frac{3(-1)^j k!^2}{(j+2)!(2k-j)!}\left(n+(j+1)\right)\left((2k-j)n - (j+2)k\right)\frac{B_{3k-j}}{3k-j}$$

$$+ \frac{3}{k+1}\binom{k+1}{j+1}\sum_{i=j}^{k}\binom{k-j}{i-j}\left[n(n-1)(k-i)\frac{i-j}{j+2}\right.$$

$$\left. - nk(k-j) + k^2(j+1)\right]\frac{B_{2k-i}B_{k-j+i}}{(2k-i)(k-j+i)}, \quad -1 \leqslant j \leqslant 2k, \tag{5.5}$$

where for $j \geqslant k+1$ the summation on the right is considered to be 0.

The proof rests on (5.1) and Lemma 2, and Theorem 1 in [5] is also used. We skip the details which are long and tedious.

Finally, we also list two more specific expansions, directly obtained from Theorem 4. They supplement Corollaries 1 and 3.

Corollary 4. *For all $n \geqslant 2$ we have*

$$
(B_3 + B_3 + B_3)^n = \frac{(n+11)(n+10)}{184\,800} B_{n+9} + \frac{(n+6)(n-21)}{5600} B_{n+5}
$$
$$
- \frac{(n+4)(3n-15)}{1680} B_{n+3} + \frac{n(n-1)}{300} B_{n+1} - 2\frac{n(n-2)}{1155} B_{n-1};
$$
$$
(B_4 + B_4 + B_4)^n = \frac{(n+14)(n+13)}{6\,306\,300} B_{n+12} + \frac{n+9}{6300} B_{n+8} + \frac{(n+7)(n-16)}{5880} B_{n+6}
$$
$$
- \frac{n^2 - n - 32}{600} B_{n+4} + \frac{63n^2 + 65n - 768}{13\,860} B_{n+2} - \frac{(n-1)(437n + 1646)}{143\,325} B_n.
$$

We have used the computer algebra system Maple to check and verify the expansions in Corollaries 1, 3 and 4.

Acknowledgment

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