REPRESENTATIONS OF STIRLING NUMBERS OF THE FIRST KIND BY MULTIPLE INTEGRALS

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Abstract

The Stirling numbers of the first kind $s(n,k)$, for $k \geq 2$ and $n \geq k$, are expressed in two different ways as $(k-1)$-fold integrals of certain symmetric polynomials in $k-1$ variables. This extends a well-known integral for the harmonic numbers.

1. Introduction

Stirling numbers of both kinds belong to the most basic and important objects in combinatorics, with numerous applications also in other areas of mathematics. They have therefore been studied extensively and continue to attract a great deal of attention.

In this paper we consider the Stirling numbers of the *first* kind, $s(n,k)$, which can be defined, among other equivalent definitions, by the exponential generating function

$$\frac{(\log(1+x))^k}{k!} = \sum_{n=k}^{\infty} \frac{s(n,k)}{n!} x^n; \tag{1}$$

see, e.g., [8, Ch. 26]. The main combinatorial interpretation of $|s(n,k)|$ is the number of ways to arrange n objects into k cycles; see, e.g., [6, p. 259].

Our point of departure is a well-known integral for the harmonic number $H_n :=$

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$1 + \frac{1}{2} + \cdots + \frac{1}{n}$, namely

$$H_n = \int_0^1 \frac{1 - x^n}{1 - x}\, dx, \tag{2}$$

which is easy to verify. Because of the connection $s(n, 2) = (-1)^n(n-1)!H_{n-1}$ (see, e.g., [8, eq. (26.8.15)]), we have the integral representation

$$s(n, 2) = (-1)^n(n - 1)! \int_0^1 \frac{1 - x^{n-1}}{1 - x}\, dx. \tag{3}$$

Using the substitution $x \to 1 - x$ and adding the integral thus obtained to the original integral in (3), we get

$$s(n, 2) = (-1)^n \frac{(n - 1)!}{2} \int_0^1 \frac{1 - x^n - (1 - x)^n}{x(1 - x)}\, dx. \tag{4}$$

This integral was used in [2] to obtain certain convolution identities for Bernoulli numbers.

It is the purpose of this paper to obtain extensions of (3) and (4) for any $s(n, k)$, with $n \geq k \geq 2$, in terms of multiple integrals. We begin with the extension of the integral (4).

Theorem 1. *Let $k \geq 2$ be an integer, and $x_1, \ldots, x_k$ be real variables with $x_1 + \cdots + x_k = 1$. Define the sum*

$$S_n(x_1, \ldots, x_k) := 1 + \sum_{r=1}^{k-1} (-1)^{k-r} \sum_{1 \leq i_1 < \ldots < i_r \leq k} (x_{i_1} + \cdots + x_{i_r})^n. \tag{5}$$

Then for all $n \geq k$ we have

$$\begin{aligned} s(n, k) =& (-1)^{n-k} \frac{(n - 1)!}{k!} \\ & \times \int_0^1 \int_0^{1-x_1} \ldots \int_0^{1-x_1-\cdots-x_{k-2}} \frac{S_n(x_1, \ldots, x_k)}{x_1 \ldots x_k}\, dx_{k-1} \ldots dx_1. \end{aligned} \tag{6}$$

In the case $k = 2$ the multiple integral in (6) is interpreted as the single integral from 0 to 1. With $k = 2$, then, and setting $x_1 = x$ and $x_2 = 1-x$, we clearly obtain (4). As next smallest example we set $k = 3$ and $x_1 = x$, $x_2 = y$, and $x_3 = 1 - x - y$. Then (6) reduces to the following double integral expression.

Corollary 1. *For $n \geq 3$ we have*

$$s(n, 3) = (-1)^{n-1} \frac{(n - 1)!}{3!} \int_0^1 \int_0^{1-x} \frac{\overline{S}_n(x, y)}{xy(1 - x - y)}\, dy\, dx,$$

where

$$\overline{S}_n(x, y) = 1 - (x + y)^n - (1 - x)^n - (1 - y)^n + x^n + y^n + (1 - x - y)^n.$$

Our second result generalizes the integral expression (3).

Theorem 2. *Let $k \geq 2$ be an integer, and $x_1, \dots, x_{k-1}$ be real variables. Define the sum*

$$R_n(x_1, \dots, x_{k-1}) := 1 + \sum_{r=1}^{k-1} (-1)^r \sum_{1 \leq i_1 < \dots < i_r \leq k-1} \frac{(x_{i_1} + \dots + x_{i_r})^n - 1}{x_{i_1} + \dots + x_{i_r} - 1}. \tag{7}$$

Then for all $n \geq k$ we have

$$\begin{aligned} s(n,k) =& (-1)^{n-1} \frac{(n-1)!}{(k-1)!} \\ & \times \int_0^1 \int_0^{1-x_1} \dots \int_0^{1-x_1-\dots-x_{k-2}} \frac{R_n(x_1, \dots, x_{k-1})}{x_1 \dots x_{k-1}} \, dx_{k-1} \dots dx_1. \end{aligned} \tag{8}$$

In the case $k = 2$ the multiple integral reduces again to the single integral from 0 to 1, and with $x_1 = x$ we have

$$\frac{R_n(x)}{x} = \frac{1}{x} \left(1 - \frac{x^n - 1}{x - 1} \right) = -\frac{1 - x^{n-1}}{1 - x},$$

so (8) reduces to (3) in this case. Once again we state the next simplest case, $k = 3$, as a corollary which follows immediately from Theorem 2 with $x_1 = x$ and $x_2 = y$.

Corollary 2. *For $n \geq 3$ we have*

$$s(n,3) = (-1)^{n-1} \frac{(n-1)!}{2} \int_0^1 \int_0^{1-x} \frac{R_n(x,y)}{xy} \, dy \, dx,$$

where

$$R_n(x,y) = 1 - \frac{x^n - 1}{x - 1} - \frac{y^n - 1}{y - 1} + \frac{(x+y)^n - 1}{x + y - 1}.$$

In Section 2 we state and partly prove some lemmas which are interesting in their own rights; the first two of them are known. These will then be used in Section 3 to prove Theorems 1 and 2. We conclude this paper with some additional remarks in Section 4.

2. Some Lemmas

The Stirling numbers of the first kind have two well-known multiple sum expressions that are similar in appearance. The first of these is

$$s(n,k) = (-1)^{n-k} (n-1)! \sum_{1 \leq j_1 < \dots < j_{k-1} \leq n-1} \frac{1}{j_1 \dots j_{k-1}}; \tag{9}$$

see, e.g., [5] or [7]. Some further remarks can be found at the end of Section 4. The second such sum will be required in the next section, and we state it as a lemma.

Lemma 1. *For all $n \geq k \geq 1$ we have*

$$s(n,k) = (-1)^{n-k}\frac{n!}{k!}\sum_{\substack{j_1+\ldots+j_k=n\\ j_1,\ldots,j_k\geq 1}}\frac{1}{j_1\ldots j_k}. \tag{10}$$

This identity can be found, for instance, in [7] or [4, p. 291 ff.]. The proof follows from (1) by taking the k-fold Cauchy product of the series $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$, and equating coefficients.

The next lemma is an extension of the well-known beta function, or beta integral,

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{11}$$

While this is valid for complex parameters m, n with $\Re(m), \Re(n) > 0$, we will require m and n only to be positive integers. The following extension to multiple integrals is true in similar generality, although again we will need it only for positive integer parameters.

Lemma 2. *For positive real $j_1,\ldots,j_k$ with $k \geq 2$ we have*

$$\int_0^1\int_0^{1-x_1}\cdots\int_0^{1-x_1-\cdots-x_{k-2}} x_1^{j_1-1}\ldots x_{k-1}^{j_{k-1}-1}(1-x_1-\cdots-x_{k-1})^{j_k-1} \times dx_{k-1}\ldots dx_1 = \frac{\Gamma(j_1)\ldots\Gamma(j_k)}{\Gamma(j_1+\cdots+j_k)}. \tag{12}$$

Once again, for $k = 2$ we interpret the multiple integral in (12) as the single integral from 0 to 1. Lemma 2 can be proved by induction on k, with (11) as the base case. This multivariate beta integral is not new; it can be found, for instance, in [10], where it was further generalized.

For the next lemma and for Section 3 we introduce the following notation which is related to the multiple integrals in (6), (8) and (12). For any $d \geq 1$, let Δ_d be the section of the d-dimensional unit cube defined by

$$\Delta_d := \{(x_1,\ldots,x_d) \in \mathbb{R}^n \;|0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 - x_1, \ldots, 0 \leq x_d \leq 1 - x_1 - \cdots - x_{d-1}\}.$$

Thus, Δ_1 is the unit interval, Δ_2 is the right-angled triangle with area 1/2, and Δ_3 is the solid of volume 1/6 obtained by cutting the appropriate triangular object from the 3-dimensional unit cube. Also, the multiple integrals in (6), (8) and (12) are taken over Δ_{k-1}.

The following lemma could be proved in greater generality. However, for simplicity we restrict ourselves to continuous functions.

Lemma 3. *Let $d \geq 1$ be an integer and $f(x_1, \ldots, x_d)$ a continuous function in d real variables defined on Δ_d. For $1 \leq j \leq d$ let $f_j(x_1, \ldots, x_d)$ be the function obtained from $f(x_1, \ldots, x_d)$ by replacing x_j by $1 - x_1 - \cdots - x_d$. Then for all $1 \leq j \leq d$ we have*

$$\int_{\Delta_d} f_j(x_1, \ldots, x_d)\mathbf{dx} = \int_{\Delta_d} f(x_1, \ldots, x_d)\mathbf{dx}, \tag{13}$$

where $\mathbf{dx} := dx_d \ldots dx_1$.

Proof. We consider the inner-most integral in

$$\int_{\Delta_d} f\,\mathbf{dx} = \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\cdots-x_{d-1}} f(x_1, \ldots, x_d) dx_d \ldots dx_1 \tag{14}$$

and substitute x_d by $1 - x_1 - \cdots - x_d$. Then dx_d becomes $-dx_d$, and the limits of integration 0 and $1 - x_1 - \cdots - x_{d-1}$ get interchanged. Switching the limits of integration back will cancel the minus sign, which proves the lemma for $j = d$.

Next we note that because of symmetry of the object Δ_d we can take the iterated integral on the right of (14) in any order; in particular, x_j for any $j = 1, \ldots, d-1$ could be interchanged with x_d. The statement of the lemma is then obtained for j by the same easy substitution as above. This completes the proof. □

3. Proofs of the Theorems

Proof of Theorem 1. For a fixed $k \geq 2$ and $x_1, \ldots, x_k$ as in the theorem, consider the multiple sum

$$\widetilde{S}_n(x_1, \ldots, x_k) = \sum_{\substack{j_1+\ldots+j_k=n \\ j_1,\ldots,j_k \geq 1}} \frac{n!}{j_1! \ldots j_k!} x_1^{j_1} \ldots x_k^{j_k}. \tag{15}$$

Furthermore, for any integer r with $1 \leq r \leq k$, define

$$T_n(x_{i_1}, \ldots, x_{i_r}) := \sum_{\substack{j_1+\ldots+j_r=n \\ j_1,\ldots,j_r \geq 0}} \frac{n!}{j_1! \ldots j_r!} x_{i_1}^{j_1} \ldots x_{i_r}^{j_r}, \tag{16}$$

where $\{x_{i_1}, \ldots, x_{i_r}\}$ is a subset of the set of variables $\{x_1, \ldots, x_k\}$. Note that the summation indices in (16) start with 0, in contrast to the sum (15). By the multinomial theorem the sums (16) evaluate as

$$T_n(x_{i_1}, \ldots, x_{i_r}) = (x_{i_1} + \cdots + x_{i_r})^n, \qquad r = 1, \ldots, k. \tag{17}$$

In particular, we have

$$T_n(x_i) = x_i^n, \qquad T_n(x_{i_1}, x_{i_2}) = (x_{i_1} + x_{i_2})^n,$$

and, keeping in mind that $x_1 + \cdots + x_k = 1$, we get

$$T_n(x_1, \ldots, x_k) = 1. \tag{18}$$

Now we evaluate the sum $\widetilde{S}_n(x_1, \ldots, x_k)$ by using the sums (16) and the inclusion-exclusion principle. We do this by noting that $\widetilde{S}_n(x_1, \ldots, x_k)$ is obtained from the full sum $T_n(x_1, \ldots, x_k)$ by subtracting each of the $(k-1)$-fold sums that have $j_1 = 0$, respectively $j_2 = 0$, etc.; that is, we subtract the k sums $T_n(x_2, \ldots, x_k)$, $T_n(x_1, x_3, \ldots, x_k)$, up to $T_n(x_1, \ldots, x_{k-1})$. However, we subtracted too many terms and must therefore add the $\binom{k}{2}$ sums $T_n(x_{i_1}, \ldots, x_{i_{k-2}})$, and so on. Thus,

$$\begin{aligned}\widetilde{S}_n(x_1, \ldots, x_k) =& T_n(x_1, \ldots, x_k) - \sum_{1 \le i_1 < \ldots < i_{k-1} \le k} T_n(x_{i_1}, \ldots, x_{i_{k-1}}) \\ &+ \sum_{1 \le i_1 < \ldots < i_{k-2} \le k} T_n(x_{i_1}, \ldots, x_{i_{k-2}}) - \cdots + (-1)^{k-1} \sum_{i=1}^{k} T(x_i),\end{aligned}$$

with distinct summation indices i_j in each of the sums. Now, using (17) and (18), we see that in fact $\widetilde{S}_n(x_1, \ldots, x_k)$ is the same as $S_n(x_1, \ldots, x_k)$ in (5).

To conclude the proof, we divide (15) by the monomial $x_1 \ldots x_k$, which leaves the resulting quotient still as a symmetric polynomial. With the notation as used in Lemma 3, and keeping in mind that $x_k = 1 - x_1 - \cdots - x_{k-1}$, (3.1) and (2.3) immediately give

$$\begin{aligned}\int_{\Delta_{k-1}} \frac{S_n(x_1, \ldots, x_n)}{x_1 \ldots x_k} \mathbf{dx} &= \sum_{(*)} \frac{n!}{j_1! \ldots j_k!} \int_{\Delta_{k-1}} x_1^{j_1 - 1} \ldots x_k^{j_k - 1} \mathbf{dx} \\ &= \sum_{(*)} \frac{n!}{j_1! \ldots j_k!} \cdot \frac{(j_1 - 1)! \ldots (j_k - 1)!}{(j_1 + \cdots + j_k - 1)!} \\ &= n \sum_{(*)} \frac{1}{j_1 \ldots j_k},\end{aligned} \tag{19}$$

where $(*)$ indicates that the sum is taken over all $j_1, \ldots, j_k \ge 1$ with $j_1 + \cdots + j_k = n$. Finally, combining (10) with (19) we immediately get (6), which completes the proof of Theorem 1. $\square$

Proof of Theorem 2. For fixed integers $k \ge 2$ and $m \ge 0$ define the function

$$g_m(x_1, \ldots, x_{k-1}) := \frac{1}{x_1 \ldots x_{k-1}} \sum_{r=1}^{k-1} (-1)^r \sum_{1 \le i_1 < \ldots < i_r \le k-1} \left(x_{i_1} + \cdots + x_{i_r}\right)^m, \tag{20}$$

and for greater ease of notation we set $g_m = g_m(x_1, \ldots, x_{k-1})$. Here and in what follows the summation indices i_j are once again assumed to be distinct in each sum.

For $1 \le j \le k-1$ let σ_j be the linear operator acting on g_m (or any rational function in $x_1, \ldots, x_k$) that changes x_j to x_k, and let σ_k be the identity operator. Then we have

$$\begin{aligned}\sum_{j=1}^{k} \sigma_j(g_m) &= \frac{1}{x_1 \ldots x_k} \sum_{j=1}^{k} x_j \sum_{r=1}^{k-1} (-1)^r \sigma_j \sum_{1 \le i_1 < \ldots < i_r \le k-1} (x_{i_1} + \cdots + x_{i_r})^m \\ &= \frac{1}{x_1 \ldots x_k} \sum_{r=1}^{k-1} (-1)^r \left[\sum_{j=1}^{k} x_j \sigma_j \sum_{1 \le i_1 < \ldots < i_r \le k-1} (x_{i_1} + \cdots + x_{i_r})^m \right].\end{aligned} \tag{21}$$

Let A_r be the expression in large brackets in this last line. For each j, σ_j applied to the r-fold sum in A_r contains no x_j at all. Therefore, if we change the order of summation, we get

$$A_r = \sum_{1 \le i_1, \ldots, i_r \le k} (x_{i_1} + \cdots + x_{i_r})^m \sum_{\substack{j=1 \\ j \notin \{i_1, \ldots, i_r\}}}^{k} x_j \tag{22}$$

(note that the summation indices i_j range from 1 to k, in contrast to the sums in (21)). Since $x_1 + \cdots + x_k = 1$, the second summation in (22) is $1 - (x_{i_1} + \cdots + x_{i_r})$, and we get with (22) and (21),

$$\begin{aligned}\sum_{j=1}^{k} \sigma_j(g_m) &= \frac{-1}{x_1 \ldots x_k} \sum_{r=1}^{k-1} (-1)^r \sum_{1 \le i_1 < \ldots < i_r \le k} (x_{i_1} + \cdots + x_{i_r})^{m+1} \\ &\quad + \frac{1}{x_1 \ldots x_k} \sum_{r=1}^{k-1} (-1)^r \sum_{1 \le i_1 < \ldots < i_r \le k} (x_{i_1} + \cdots + x_{i_r})^m .\end{aligned} \tag{23}$$

Now we add both sides of (23) for $m = 0, 1, \ldots, n-1$. Then the sum on the right telescopes, and the final term (for $m = 0$) is

$$\sum_{r=1}^{k-1} (-1)^r \sum_{1 \le i_1 < \ldots < i_r \le k} 1 = \sum_{r=1}^{k-1} (-1)^r \binom{k}{r} = -1 - (-1)^k. \tag{24}$$

Hence we have

$$\begin{aligned}\sum_{m=0}^{n-1} \sum_{j=1}^{k} \sigma_j(g_m) &= \frac{-1}{x_1 \ldots x_k} \\ &\quad - \frac{(-1)^k}{x_1 \ldots x_k} \left(1 + \sum_{r=1}^{k-1} (-1)^{k-r} \sum_{1 \le i_1 < \ldots < i_r \le k} (x_{i_1} + \cdots + x_{i_r})^n \right).\end{aligned} \tag{25}$$

Next, from (20) we get as a result of finite geometric sums,

$$\sum_{m=0}^{n-1} g_m = \frac{1}{x_1 \dots x_{k-1}} \sum_{r=1}^{k-1} (-1)^r \sum_{1 \le i_1 < \dots < i_r \le k-1} \frac{(x_{i_1} + \dots + x_{i_r})^n - 1}{x_{i_1} + \dots + x_{i_r} + 1}. \tag{26}$$

Also, since $x_1 + \dots + x_k = 1$, we have

$$\frac{1}{x_1 \dots x_k} = \frac{1}{x_2 \dots x_k} + \frac{1}{x_1 x_3 \dots x_k} + \dots + \frac{1}{x_1 \dots x_{k-1}} = \sum_{j=1}^{k} \sigma_j \left(\frac{1}{x_1 \dots x_{k-1}} \right).$$

By the linearity of the operator σ_j, this with (26) and (25) gives the following identity, where we use the notations introduced in (5) and (7):

$$\sum_{j=1}^{k} \sigma_j \left(\frac{R_n(x_1, \dots, x_{k-1})}{x_1 \dots x_{k-1}} \right) = (-1)^{k-1} \frac{S_n(x_1, \dots, x_k)}{x_1 \dots x_k}. \tag{27}$$

As our final step we take the $(k-1)$-fold integral of both sides of (27) over Δ_{k-1}, and note the crucial fact that by Lemma 3 the integral of each of the k summands on the left of (27) is the same for each j. We may therefore evaluate it for the case $j = k$ (corresponding to the identity operator σ_k), and we finally get

$$\begin{aligned} k \int_{\Delta_{k-1}} \frac{R_n(x_1, \dots, x_{k-1})}{x_1 \dots x_{k-1}} \mathbf{dx} &= (-1)^{k-1} \int_{\Delta_{k-1}} \frac{S_n(x_1, \dots, x_k)}{x_1 \dots x_k} \mathbf{dx} \\ &= (-1)^{n-1} \frac{k!}{(n-1)!} s(n,k), \end{aligned}$$

where the last identity comes from Theorem 1. The proof of Theorem 2 is now complete. □

4. Further Remarks

1. In his recent paper [9], Qi derived three distinct integral representations for the Stirling numbers. However, they are all very different from our results and involve higher derivatives and limits.

It should also be mentioned here that easier integrals, related to the polygamma and other special functions, were obtained independently by Butzer and Hauss [3] and Adamchik [1] in the course of their work on extending the Stirling numbers $s(n,k)$ to real or complex parameters. These integrals, however, do not apply to the case of integers $1 \le k \le n$.

2. The harmonic numbers $H_n := 1 + \frac{1}{2} + \dots + \frac{1}{n}$, which were briefly discussed in the introduction, have been generalized in several different ways. In fact, five different generalizations are listed and studied in [5], among them

$$H_{n,r} := \sum_{1 \le j_1 < \cdots < j_r \le n} \frac{1}{j_1 j_2 \ldots j_r} \quad (n, r \ge 1), \quad H_{n,0} = 1$$

and

$$H(n,r) := \sum_{\substack{j_0,\ldots,j_r \ge 1 \\ 1 \le j_0 + j_1 + \cdots + j_r \le n}} \frac{1}{j_0 j_1 \ldots j_r} \quad (n \ge 1, r \ge 0).$$

We already noted in (9) that the identity

$$H_{n,r} = (-1)^{n-r} \frac{1}{n!} s(n+1, r+1), \tag{28}$$

was obtained in [5] and [7]; in the same papers it was also shown that

$$H(n,r) = (-1)^{n-r+1} (r+1)! s(n+1, r+2). \tag{29}$$

The identities (28) and (29) therefore show that our theorems can also be considered integral representations for these two related types of generalized harmonic numbers.

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