NEW OPERATIONAL FORMULAS AND GENERATING FUNCTIONS FOR THE GENERALIZED ZERNIKE POLYNOMIALS

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ABSTRACT. We establish new operational formulae of Burchnall type for the complex disk polynomials (generalized Zernike polynomials) on the hyperbolic unit disk of the complex plane and then apply them in order to derive related basic identities involving these polynomials. Mainely, we are interested in three terms recurrence formulas, like Nielsen's identity and Runge's addition formula. Various generating functions for these disk polynomials are also given.

1. INTRODUCTION

Operational formulae are powerful tools in the theory of orthogonal polynomials. They will enable proofs of new identities as well as alternative, more simpler, proofs of well-known ones. Particular examples of such identities are the so-called addition formulas, which can also be obtained from a group-theoretic point of view [35, 20, 36]. This approach was first employed by Burchnall [2], in a direct and simple way, to prove Nielsen's identity ([26]) for the classical real Hermite polynomials $H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m}(e^{-x^2})$. Since then many extension for specific one-variable real polynomials have been obtained [5, 1, 15, 6, 27, 7, 28, 18]. For the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined by their Rodrigues' formula ([29, 32]):

$$P_n^{(\alpha,\beta)}(x) := \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right),$$

R.P. Singh has developed in [31] the following operational relation

$$\prod_{j=1}^{n}\left\{(1-x^2)\frac{d}{dx} - (\alpha+\beta+2j)x + \beta - \alpha\right\} = \sum_{k=0}^{n} \frac{(-2)^{n-k} n!}{k!}(1-x^2)^k P_{n-k}^{(\alpha+k,\beta+k)}(x)\frac{d^k}{dx^k}$$

It is then used to derive some useful properties of these polynomials like the quadratic recurrence formula

$$P_{n+m}^{(\alpha,\beta)}(x) := \frac{n!m!}{(n+m)!}\sum_{k=0}^{n}\frac{(-1)^k(\alpha+\beta+2n+m+1)_k}{2^{2k}k!}(1-x^2)^k P_{n-k}^{(\alpha+k,\beta+k)}(x) P_{m-k}^{(\alpha+n+k,\beta+n+k)}(x).$$

In the present paper, we deal with the disk polynomials ([21, 37, 12, 9]) that we define here through:

$$\mathcal{Z}_{m,n}^{\gamma}(z,\bar z) = (-1)^{m+n}(1-|z|^2)^{-\gamma}\frac{\partial^{m+n}}{\partial z^m \partial \bar z^n}\left((1-|z|^2)^{\gamma+m+n}\right), \tag{1.1}$$

z and $\bar z$ being the variables in the unit disk D of the complex plane. They form a complete orthogonal system (basis) over the Hilbert space $L^2(D;(1-|z|^2)^\gamma d\lambda)$, where $\gamma > -1$ and $d\lambda$ being the Lebesgue measure, and are often referred to as generalized Zernike polynomials. Note that for $\gamma = 0$ and $m \le n$, the disk polynomials $P^0_{m,n}(z,\bar z)$ turn out to be related to the real Zernike polynomials $R_k^\nu(x)$ introduced by Zernike and Brinkman [40], being indeed

$$\mathcal{Z}_{m,n}^{0}(z,\bar z) = (m+n)! e^{i[(n-m)\arg z]} R_{m+n}^{n-m}(\sqrt{z\bar z}),$$

and used in the study of diffraction problems. For further applications in geometric and wave optics for systems with circular apertures and coherent state quantization on the disk, we refer the reader to [25, 23, 3, 38, 37, 34, 33] and the references therin. They were served in [39] by Y. Xu as a basic example to illustrate the connection between representations of orthogonal polynomials of two real variables and those of complex variables. Their structural relations is also explored there. Furthermore, the disk polynomials $\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})$ appear quite frequently when investigating spectral properties of some special differential operators of Laplacian type acting on $L^2(D;(1-|z|^2)^{\gamma}d\lambda)$ (see Section 2).

The motivation for considering the disk polynomials is the same as that for considering orthogonal polynomials in complex variable. This context is in general more convenient to obtain more elegant identities and formulae which is already done in the case of the complex Hermite polynomials ([11, 13, 17]):

$$H_{p,q}(z,\bar{z}) = (-1)^{p+q} e^{z\bar{z}} \frac{\partial^{p+q}}{\partial \bar{z}^p \partial z^q}\left(e^{-z\bar{z}}\right); \quad z \in \mathbb{C}. \tag{1.2}$$

The main object of this paper is to develop some operational formulae for the $\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})$, and next use them to derive some remarkably interesting identities, including Nielsen's identities and Runge's addition formula. New generating functions are derived from the obtained operational representation which are alternative to the Rodrigues-type representation. Furthermore, one can derive other generating functions for the product of disk polynomials.

Lastly we quote the important remark that an appropriate limiting procedure $\gamma \to +\infty$ leads to complex Hermite polynomials in (1.2). Moreover, it is a device for taking into account the hyperbolic geometry of the disk and the physical meaning of the parameter γ.

The remaining sections are organised as follows. In Section 2, we review the factorization method, *à la* Schrödinger, for the magnetic Laplacian on the hyperbolic unit disk and recall some properties of the suggested disk polynomials. In Section 3, we give operational formulae of Burchnall type involving these complex disk polynomials. Some of their applications are discussed in Sections 4, 5, 6 and 7. Indeed, Section 4 deals with recurrence quadratic formulas. In Section 5, we establish three term recurrence formulas. Section 6 is devoted to Runge's addition formula. In Section 7, new generating functions are obtained.

2. Generation of the complex disk polynomials: an algebraic approach

Let D be the hyperbolic unit disk equipped with its standard Bergman-Kähler structure described through the Hermitian metric $ds^2 := (1-|z|^2)^{-2} dz \otimes d\bar{z}$. The volume measure is then $d\mu = (1-|z|^2)^{-2} d\lambda$, where $d\lambda(z) = dxdy$ denotes the Lebesgue measure on D. Associated to the differential one-form $\theta := (\partial - \bar{\partial}) \log(1-|z|^2)$, there is the magnetic Schrödinger operator $\mathfrak{L}_\nu = (d+i\nu\theta)^*(d+i\nu\theta)$ acting on the L^2-Hilbert space $\mathcal{H} := L^2(D;d\mu)$. Its explicit expression, up to a multiplicative constant, is given by

$$\mathfrak{L}_\nu = -(1-|z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} - \nu(1-|z|^2)\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) + \nu^2|z|^2. \tag{2.1}$$

The twisted Laplacian $\mathfrak{L}_\nu$ is an elliptic self-adjoint second order differential operator whose concrete spectral theory is well known in the literature [41, 10, 14, 4]. In particular, its discrete L^2-spectrum is nontrivial only and only if $\nu > 1/2$. It is known to be given by the eigenvalues $\lambda_{\nu,m} := \nu(2m+1) - m(m+1)$ for varying positive integer m such that $0 \leq m < \nu - 1/2$. In the other hand, we know since Schrödinger (see [30, 16, 10]) that the factorization method allows one to construct L^2-eigenfunctions. Indeed, our Laplacian $\mathfrak{L}_\nu$ can be rewritten as

$$\mathfrak{L}_\nu = \nabla_{\nu-1}\nabla^*_{\nu-1} + \nu \quad \text{and} \quad \mathfrak{L}_{\nu-1} = \nabla^*_{\nu-1}\nabla_{\nu-1} - (\nu-1),$$

where the first order differential operator ∇_α and its formal adjoint ∇^*_α are given by

$$\nabla_\alpha = -(1-|z|^2)\frac{\partial}{\partial z} + \alpha\bar{z} \quad \text{and} \quad \nabla^*_\alpha = (1-|z|^2)\frac{\partial}{\partial \bar{z}} + (\alpha+1)z. \tag{2.2}$$

So that we have the following algebraic relationship

$$\mathfrak{L}_\nu\nabla_{\nu-1} = \left(\nabla_{\nu-1}\nabla^*_{\nu-1} + \nu\right)\nabla_{\nu-1} = \nabla_{\nu-1}\left(\mathfrak{L}_{\nu-1} + (2\nu-1)\right).$$

This implies that the operator $\nabla_{\nu-1}$ generates eigenfunctions of $\mathfrak{L}_\nu$ from those of $\mathfrak{L}_{\nu-1}$. Thus, if φ_0 is a nonzero L^2-eigenfuncion associated to the lowest eigenvalue (the lowest Landau level) of $\mathfrak{L}_{\nu-m}$, then $\nabla^\nu_m\varphi_0$, where we have set

$$\nabla^\nu_m := \nabla_{\nu-1}\circ\nabla_{\nu-2}\circ\cdots\circ\nabla_{\nu-m} = (-1)^m\prod_{j=1}^{m}\left((1-|z|^2)\frac{\partial}{\partial z} - (\nu-j)\bar{z}\right), \tag{2.3}$$

is an eigenfunction of $\mathfrak{L}_\nu$ belonging to

$$A^{2,\nu}_m(D) := \{\varphi \in L^2(D;d\mu);\ \ \mathfrak{L}_\nu\varphi = \lambda_{\nu,m}\varphi\}.$$

Conversely, this method describes completely the L^2-eigenspaces $A^{2,\nu}_m(D)$ (see [12]). More precisely, for fixed $\nu > 1/2$, the functions

$$\psi^\nu_{m,n}(z,\bar{z}) := \nabla^\nu_m\left(z^n(1-|z|^2)^{\nu-m}\right) \tag{2.4}$$

for varying $n = 0,1,2,\cdots$, are L^2-eigenfunctions of $\mathfrak{L}_\nu$. Furthermore, they constitute an orthogonal basis of the L^2-eigenspace $A^{2,\nu}_m(D)$. Whence, by observing that ∇_α can be rewritten as

$$\nabla_\alpha f = -(1-|z|^2)^{1-\alpha}\frac{\partial}{\partial z}[(1-|z|^2)^\alpha f],$$

the operator ∇^ν_m in (2.3) acts on sufficienty differentiable functions on D as

$$\nabla^\nu_m f = (-1)^m(1-|z|^2)^{-\nu}\left[(1-|z|^2)^2\frac{\partial}{\partial z}\right]^m\left((1-|z|^2)^{\nu-m}f\right). \tag{2.5}$$

which for the special case of $f(z) = \varphi^\nu_{n,m}(z) := z^n(1-|z|^2)^{\nu-m}$, belonging to the null space of $\nabla^*_{\nu-m-1}$, gives rise to

$$\psi^\nu_{m,n}(z,\bar{z}) = (-1)^m(1-|z|^2)^{-\nu}\left[(1-|z|^2)^2\frac{\partial}{\partial z}\right]^m\left(z^n(1-|z|^2)^{2(\nu-m)}\right). \tag{2.6}$$

The explicit expression of $\psi^\nu_{m,n}(z,\bar{z})$ involves the real Jacobi Polynomials $\mathrm{P}^{(\alpha,\beta)}_k(x)$. More precisely, we have

Proposition 2.1 ([12]). *Denote by $m \wedge n$ the minimum of the nonnegative integers m and n. The quantities $\psi^\nu_{m,n}(z,\bar{z})$ are given by*

$$\psi^\nu_{m,n}(z,\bar{z}) = (-1)^m(m\wedge n)!(1-z\bar{z})^{\nu-m}|z|^{|m-n|}e^{i[(n-m)\arg z]}\mathrm{P}^{(|m-n|,2(\nu-m)-1)}_{m\wedge n}(1-2|z|^2). \tag{2.7}$$

By means of (2.6) and (2.7), the suggested class of two variable polynomials

$$\begin{aligned} P^\gamma_{m,n}(z,\bar{z}) &:= (1-|z|^2)^{-\nu+m}\nabla^\nu_m\left(z^n(1-|z|^2)^{\nu-m}\right) \\ &= (1-|z|^2)^{-\nu+m}\psi^\nu_{m,n}(z,\bar{z}), \end{aligned} \tag{2.8}$$

where $\gamma = 2(\nu - m) - 1$, are given by

$$P^{\gamma}_{m,n}(z,\bar{z}) = (-1)^m (1-|z|^2)^{-(\gamma+m+1)} \left[(1-|z|^2)^2 \frac{\partial}{\partial z}\right]^m \left(z^n (1-|z|^2)^{\gamma+1}\right) \tag{2.9}$$

$$= (-1)^m (m\wedge n)! |z|^{|m-n|} e^{i[(n-m)\arg z]} \mathrm{P}^{(|m-n|,\gamma)}_{m\wedge n}(1-2|z|^2). \tag{2.10}$$

Up to a multiplicative constant, (2.10) leads to the so-called disk polynomials defined through (1.1). An accurate analysis of the basic properties of such polynomials, like the recurrence relations with respect to the indices m and n, the differential equations they obey, the generating functions and so on, from different point of views has been developed in many papers. For a very nice account on these polynomials, the interested can refer to [21, 22]. Other recent relevant references are [37, 34, 12, 24, 19, 39, 33].

In the next section we derive operational formulae for the disk polynomials. We follow in spirit [13] where analogous results are obtained for the complex Hermite polynomials $H_{p,q}(z,\overline{z})$ in (1.2).

3. Operational formulae for the disk polynomials

We start by noting that the intertwining invariant operator in the right hand side of (2.5), i.e.,

$$\mathfrak{D}^m = \mathfrak{D}\circ\mathfrak{D}\circ\cdots\circ\mathfrak{D}; \quad m\text{-times}, \quad \mathfrak{D} = h^2(z)\frac{\partial}{\partial z}; \quad h(z) := 1-|z|^2, \tag{3.1}$$

depends only on the geometry of the hyperbolic disk D. It is connected to the one in (2.3) through (2.5),

$$\mathfrak{D}^m f = (-1)^m (1-|z|^2)^{\nu} \nabla^{\nu}_m \Big((1-|z|^2)^{-\nu+m} f\Big). \tag{3.2}$$

Moreover, it can be realized in terms of the following one

$$\mathcal{A}_m(f) := h^{m+1}(z)\frac{\partial^m}{\partial z^m}\left(h^{m-1}(z) f(z)\right). \tag{3.3}$$

Namely, we assert

Proposition 3.1. *Let $\mathcal{A}$ and $\mathfrak{D}$ be as above. Then, we have $\mathfrak{D}^m f = \mathcal{A}_m(f)$. More explicitly*

$$\left[(1-|z|^2)^2\frac{\partial}{\partial z}\right]^m (f) = (1-|z|^2)^{m+1}\frac{\partial^m}{\partial z^m}\left((1-|z|^2)^{m-1} f\right). \tag{3.4}$$

In particular, we have $\mathcal{A}_{m+m'} = \mathcal{A}_m \circ \mathcal{A}_{m'}$.

Proof. The proof of (3.4) can be handled by induction. Obviously, for $m=0$ and $m=1$ the identity (3.4) holds good. Assume that (3.4) holds for given fixed positive integer m and note that the operators $\mathfrak{D}^m$ and $\partial h/\partial z = -\bar{z}$ commute. Hence, direct computation yields

$$\begin{aligned} h^{m+2}\frac{\partial^{m+1}}{\partial z^{m+1}}(h^m f) &= h\left\{h^{m+1}\frac{\partial^m}{\partial z^m}\left(\frac{\partial}{\partial z}(h^m f)\right)\right\} \\ &= h\left\{m\left(\frac{\partial h}{\partial z}\right)\mathfrak{D}^m(f) + \mathfrak{D}^m(h^{-1}\mathfrak{D}(f))\right\}. \end{aligned} \tag{3.5}$$

Note also that, for every given positive integer k such that $0\le k\le m$, we have

$$\mathfrak{D}^m(h^{-1}\mathfrak{D}(f)) = \mathfrak{D}^{m-k}(h^{-1}\mathfrak{D}^{k+1}(f)) - k\frac{\partial h}{\partial z}\mathfrak{D}^m(f) \tag{3.6}$$

which follows by repeated application of the fact $\mathfrak{D}^m(h^{-1}\mathfrak{D}(f)) = \mathfrak{D}^{m-1}(h^{-1}\mathfrak{D}^2(f)) - \frac{\partial h}{\partial z}\mathfrak{D}^m(f)$. Now by taking $k=m$ in (3.6) and substituting it in (3.5), we get the equality (3.3). The second assertion follows easily since $\mathcal{A}^{m+1}(f) = \mathfrak{D}^{m+1}(f) = \mathfrak{D}\circ\mathfrak{D}^m(f) = \mathcal{A}_1\circ\mathcal{A}_m(f)$. ∎

Remark 3.2. *According to* (2.9) *and Proposition 3.1, we have*

$$P^{\gamma}_{m,n}(z,\bar{z}) = (-1)^m h^{-\gamma}(z)\frac{\partial^m}{\partial z^m}(z^n h^{\gamma+m}(z)) \tag{3.7}$$

which gives rise to the Rodrigues' type formula [37, 12]*:*

$$C^{\gamma}_{m,n}P^{\gamma}_{m,n}(z,\bar{z}) = (-1)^{m+n}h^{-\gamma}(z)\frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n}\left(h^{\gamma+m+n}(z)\right), \tag{3.8}$$

where

$$C^{\gamma}_{m,n} := (\gamma+m+1)_n \tag{3.9}$$

$(a)_n$ *being the Pochhammer symbol* $(a)_n := a(a+1)\cdots(a+n-1)$.

The reader is advised that the normalization adopted here may not coincide with the ones adopted elsewhere. In the sequel, instead of $P^{\gamma}_{m,n}(z,\bar{z})$, we deal with the polynomials

$$\begin{aligned}\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) &:= C^{\gamma}_{m,n}P^{\gamma}_{m,n}(z,\bar{z})\\ &= (-1)^m (m\wedge n)! C^{\gamma}_{m,n}|z|^{|m-n|}e^{i[(n-m)\arg z]}\mathrm{P}^{(|m-n|,\gamma)}_{m\wedge n}(1-2|z|^2).\end{aligned} \tag{3.10}$$

so that

$$\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) \overset{(2.8)}{=} C^{\gamma}_{m,n}(1-|z|^2)^{-\frac{\gamma+1}{2}}\nabla_m^{\frac{\gamma+1}{2}+m}\left(z^n(1-|z|^2)^{\frac{\gamma+1}{2}}\right) \tag{3.11}$$

$$\overset{(3.7)}{=} (-1)^m C^{\gamma}_{m,n}h^{-\gamma}(z)\frac{\partial^m}{\partial z^m}(z^n h^{\gamma+m}(z)) \tag{3.12}$$

$$\overset{(3.8)}{=} (-1)^{m+n}h^{-\gamma}(z)\frac{\partial^{m+n}}{\partial z^m\partial\bar{z}^n}\left(h^{\gamma+m+n}(z)\right). \tag{3.13}$$

Associated to these formulas, we introduce the following differential operators

$$\mathcal{A}^{\gamma}_{m,n}(f) := (-1)^m C^{\gamma}_{m,n}h^{-\gamma}(z)\frac{\partial^m}{\partial z^m}\left(z^n h^{\gamma+m}(z)f\right) \tag{3.14}$$

$$\mathcal{Z}^{\gamma}_{m,n}(f) := (-1)^{m+n}h^{-\gamma}(z)\frac{\partial^{m+n}}{\partial z^m\partial\bar{z}^n}\left(h^{\gamma+m+n}(z)f\right), \tag{3.15}$$

$$\nabla^{\gamma}_{m,n}(f) = \nabla^{\gamma}_m \circ \overline{\nabla}^{\gamma}_n(f), \tag{3.16}$$

where $\nabla^{\alpha}_m = \nabla_{\alpha-1}\circ\nabla_{\alpha-2}\circ\cdots\circ\nabla_{\alpha-m}$ and $\overline{\nabla}^{\alpha}_n = \overline{\nabla}_{\alpha-1}\circ\overline{\nabla}_{\alpha-2}\circ\cdots\circ\overline{\nabla}_{\alpha-n}$ with $\nabla_\alpha = -h(z)\frac{\partial}{\partial z} + \alpha\bar{z}$ and $\overline{\nabla}_\alpha = -h(z)\frac{\partial}{\partial\bar{z}} + \alpha z$, so that

$$[\mathcal{A}^{\gamma}_{m,n}(1)](z) = [\mathcal{Z}^{\gamma}_{m,n}(1)](z) = \mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) \tag{3.17}$$

and $\mathcal{Z}^{\beta}_{0,s}(z,\bar{z}) = \overline{\mathcal{Z}^{\beta}_{s,0}(z,\bar{z})} = (\beta+1)_s z^s$. We should note also that the operator $\mathcal{A}^{\gamma}_{m,n}$ is connected to the one in (3.3) by

$$\mathcal{A}^{\gamma}_{m,n}(f) = (-1)^m(\gamma+m+1)_n h^{-(\gamma+m+1)}(z)\mathcal{A}_m(z^n h^{\gamma+1}(z)f). \tag{3.18}$$

The main results of this section are the following operational formulae of Burchnall type involving $\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})$.

Proposition 3.3. *For given positive integers m, n, and sufficiently differentiable function f, we have*

$$\mathcal{A}^{\gamma}_{m,n}(f) = m!n! \sum_{j=0}^{m} \frac{(-1)^j h^j(z)}{j!} \frac{\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})}{(m-j)!n!} \frac{\partial^j}{\partial z^j}(f). \tag{3.19}$$

$$\mathcal{Z}^{\gamma}_{m,n}(f) = m!n! \sum_{j=0}^{m}\sum_{k=0}^{n} \frac{(-1)^{j+k} h^{j+k}(z)}{j!k!} \frac{\mathcal{Z}^{\gamma+j+k}_{m-j,n-k}(z,\bar{z})}{(m-j)!(n-k)!} \frac{\partial^{j+k}}{\partial z^j \partial \bar{z}^k}(f) \tag{3.20}$$

$$\nabla^{\gamma}_{m,n}(f) = m!n! \sum_{j=0}^{m}\sum_{k=0}^{n} \frac{(\gamma+n-k-2)_{n-k}}{(\gamma+k)_{n-k}} \frac{(-1)^{j+k} h^{j+k}(z)}{j!k!} \frac{\mathcal{Z}^{(\gamma-1-m)+k+j}_{m-j,n-k}(z,\bar{z})}{(m-j)!(n-k)!} \frac{\partial^{k+j}}{\partial z^j \partial \bar{z}^k}(f). \tag{3.21}$$

Proof. Start from (3.14) and make use of the Leibnitz formula to get

$$\begin{aligned}\mathcal{A}^{\gamma}_{m,n}(f) &= (-1)^m C^{\gamma}_{m,n} h^{-\gamma}(z) \sum_{j=0}^{m} \binom{m}{j} \left(\frac{\partial^{m-j}}{\partial z^{m-j}}(z^n h^{\gamma+m}(z))\right)\left(\frac{\partial^j}{\partial z^j} f\right) \\ &= m! \sum_{j=0}^{m} \frac{(-1)^j h^j(z)}{j!(m-j)!} \left((-1)^{m-j} h^{-(\gamma+j)}(z) C^{\gamma}_{m,n} \frac{\partial^{m-j}}{\partial z^{m-j}}(z^n h^{(\gamma+j)+(m-j)}(z))\right) \frac{\partial^j}{\partial z^j} f.\end{aligned}$$

The statement follows from the fact (3.17) since $C^{\gamma}_{m,n} = C^{\gamma+j}_{m-j,n}$.

The proof of (3.20) is quite similar. Indeed, repeated application of the Leibnitz formula to (3.15) yields

$$\begin{aligned}\mathcal{Z}^{\gamma}_{m,n}(f) &= (-1)^{m+n} h^{-\gamma}(z) \frac{\partial^m}{\partial z^m}\left\{\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\partial^{n-k}}{\partial \bar{z}^{n-k}}(h^{\gamma+m+n}(z))\right)\left(\frac{\partial^k}{\partial \bar{z}^k} f\right)\right\} \\ &= (-1)^{m+n} h^{-\gamma}(z) \sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j} \frac{\partial^{m-j}}{\partial z^{m-j}}\left(\frac{\partial^{n-k}}{\partial \bar{z}^{n-k}}(h^{\gamma+m+n}(z))\right)\frac{\partial^j}{\partial z^j}\left(\frac{\partial^k}{\partial \bar{z}^k} f\right) \\ &= (-1)^{m+n} h^{-\gamma}(z) \sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k}\left(\frac{\partial^{(m-j)+(n-k)}}{\partial z^{m-j}\partial \bar{z}^{n-k}}(h^{\gamma+m+n}(z))\right)\frac{\partial^{j+k}}{\partial z^j \partial \bar{z}^k} f.\end{aligned}$$

The desired identity follows then since

$$\frac{\partial^{(m-j)+(n-k)}}{\partial z^{m-j}\partial \bar{z}^{n-k}}(h^{\gamma+m+n}(z)) = (-1)^{(m-j)+(n-k)} h^{\gamma+j+k}(z) \mathcal{Z}^{\gamma+j+k}_{m-j,n-k}(z,\bar{z}).$$

To prove (3.21) let recall first that from (3.2), we have

$$\nabla^{\gamma}_m f = (-1)^m h^{-\gamma} \mathfrak{D}^m (h^{\gamma-m} f) \quad \text{and} \quad \overline{\nabla}^{\gamma}_n f = (-1)^n h^{-\gamma} \overline{\mathfrak{D}}^n (h^{\gamma-n} f),$$

so that

$$\nabla^{\gamma}_{m,n} f = \nabla^{\gamma}_m \circ \overline{\nabla}^{\gamma}_n f = (-1)^{m+n} h^{-\gamma} \mathfrak{D}^m (h^{-m} \overline{\mathfrak{D}}^n (h^{\gamma-n} f)).$$

By applying twice Proposition 3.1, we get

$$\nabla^{\gamma}_{m,n} f = (-1)^{m+n} h^{m+1-\gamma}(z) \frac{\partial^m}{\partial z^m}\left[h^n(z) \frac{\partial^n}{\partial \bar{z}^n}(h^{\gamma-1}(z) f)\right].$$

Now, making use of Leibnitz formula, combined with the fact that

$$\frac{\partial^k}{\partial z^k}(h^{\beta}(z)) = (-\beta)_k \bar{z}^k h^{\beta-k}(z), \tag{3.22}$$

leads to

$$\begin{aligned}\nabla^{\gamma}_{m,n} f &= (-1)^{m+n} h^{m+1-\gamma}(z) \sum_{j=0}^{m}\sum_{k=0}^{n} \binom{m}{j}\binom{n}{k} (1-\gamma)_{n-k} \frac{\partial^{m-j}}{\partial z^{m-j}} \left(z^{n-k} h^{\gamma-1+k}(z)\right) \frac{\partial^{k+j}}{\partial z^j \partial \bar{z}^k}(f) \\ &= \sum_{j=0}^{m}\sum_{k=0}^{n} (-1)^{n-j} \binom{m}{j}\binom{n}{k} \frac{(1-\gamma)_{n-k}}{(\gamma+k)_{n-k}} h^{j+k}(z) \mathcal{Z}^{(\gamma-1-m)+k+j}_{m-j,n-k}(z,\bar{z}) \frac{\partial^{k+j}}{\partial z^j \partial \bar{z}^k}(f).\end{aligned}$$

This proves (3.21) since $(-a)_k = (-1)^k (a-k+1)_k$. □

Remark 3.4. *The particular case of $f=1$ in* (3.21) *gives rise to*

$$\left(\nabla_{\gamma-1} \circ \nabla_{\gamma-2} \circ \cdots \nabla_{\gamma-m}\right) \circ \left(\overline{\nabla}_{\gamma-1} \circ \overline{\nabla}_{\gamma-2} \circ \cdots \overline{\nabla}_{\gamma-n}\right)(1) = \frac{(\gamma+n-2)_n}{(\gamma)_n} \mathcal{Z}^{(\gamma-1-m)}_{m,n}(z,\bar{z}).$$

Remark 3.5. *Using similar arguments as in the proof of* (3.21)*, we see that the operator $\nabla^{\alpha,\beta}_{m,n} f := \nabla^{\alpha}_{m} \circ \overline{\nabla}^{\beta}_{n} f$ coincides with the operator $\mathcal{Z}^{\gamma}_{m,n}(f)$ given through* (3.14) *for $\alpha = \gamma+m+1$ and $\beta = \alpha+n$.*

Corollary 3.6. *We have the following identities*

$$\sum_{j=0}^{m} (-1)^j (\gamma+m)_j \frac{\bar{z}^j}{j!} \frac{\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})}{(m-j)!} = 0 \tag{3.23}$$

whenever $m > n$, and

$$\sum_{j=0}^{m} (-1)^j (\gamma+m)_j \frac{\bar{z}^j}{j!} \frac{\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})}{(m-j)!n!} = (-1)^m (\gamma+1+m)_n \frac{z^{n-m}}{(n-m)!} \frac{(1-|z|^2)^m}{m!} \tag{3.24}$$

when $m \le n$.

Proof. The identities (3.23) when $m > n$ and (3.24) when $m \le n$ follow easily from (3.20) and (3.15), by taking there $f(z) = h^{-\gamma-m}(z)$, since

$$\mathcal{A}^{\gamma}_{m,n}(h^{-\gamma-m}(z)) = \begin{cases} 0 & \text{if } n < m \\ (-1)^m \dfrac{n!(\gamma+m+1)_n}{(n-m)!} z^{n-m} h^{-\gamma}(z) & \text{if } n \ge m \end{cases}.$$

□

Remark 3.7. *Taking $n=0$ in* (3.23)*, keeping in mind that $\mathcal{Z}^{\beta}_{s,0}(z,\bar{z}) = (\beta+1)_s \bar{z}^s$, leads to the following identity for Gamma function*

$$\sum_{j=0}^{m} (-1)^j \frac{\Gamma(\gamma+m+j)}{\Gamma(\gamma+1+j)} = 0. \tag{3.25}$$

Corollary 3.8. *The representation of the complex Hermite polynomials $H_{m,n}(z,\bar{z})$ restricted to the unit disk is given in terms of the disk polynomials by*

$$H_{m,n}(z,\bar{z}) = \frac{m!}{(\gamma+m+1)_n} h^{-m}(z) \sum_{j=0}^{m}\sum_{k=0}^{j} \frac{(-1)^k (\gamma+m)_k}{k!} \frac{\bar{z}^j h^{j-k}(z)}{(j-k)!} \frac{\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})}{(m-j)!}. \tag{3.26}$$

Proof. Note first that $\mathcal{A}^{\gamma}_{m,n}(h^{-\gamma-m}(z)e^{-|z|^2}) = (\gamma+m+1)_n h^{-\gamma}(z)e^{-|z|^2}H_{m,n}(z,\bar{z})$. Furthermore, the right hand side of (3.19), with $f(z) = h^{-\gamma-m}(z)e^{-|z|^2}$, yields

$$\mathcal{A}^{\gamma}_{m,n}(h^{-\gamma-m}(z)e^{-|z|^2}) = m!h^{-\gamma-m}(z)\sum_{j=0}^{m}\sum_{k=0}^{j}\frac{(-1)^k(\gamma+m)_k}{k!}\frac{\bar{z}^j h^{j-k}(z)}{(j-k)!}\frac{\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})}{(m-j)!}. \tag{3.27}$$

Thus (3.26) follows from (3.19) by straightforward computation making use of the Leibnitz rule combined with the fact (3.22). □

Some applications of the previous obtained results are discussed in the following sections.

4. Quadratic recurrence formula

For different specific functions f we deduce interesting identities. Indeed, for given positive integers m,n,s, we have

$$\mathcal{Z}^{\gamma}_{m,n+s}(z,\bar{z}) = m!n!s!C^{\gamma}_{m+n,s}\sum_{j=0}^{m\wedge s}\frac{(-1)^j z^{s-j}h^j(z)}{(s-j)!j!}\frac{\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})}{(m-j)!n!}, \tag{4.1}$$

which follows from (3.19) for $f = z^s$, together with the following useful fact

$$\mathcal{Z}^{\gamma+\beta}_{m,n+s}(z,\bar{z}) = \frac{C^{\gamma+\beta}_{m,n+s}}{C^{\gamma}_{m,n}}\mathcal{A}^{\gamma}_{m,n}(z^s h^{\beta}(z)) \tag{4.2}$$

for any positive integers m,n,s and real number γ,β. Also, when taking $f = h^s$, we obtain

$$\mathcal{Z}^{\gamma+s}_{m,n}(z,\bar{z}) = m!n!s!\frac{C^{\gamma+s}_{m,n}}{C^{\gamma}_{m,n}}\sum_{j=0}^{m\wedge s}\frac{\bar{z}^j}{(s-j)!j!}\frac{\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})}{(m-j)!n!}. \tag{4.3}$$

Moreover, we have the following identity (of Nielsen type) involving $\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})$.

Proposition 4.1. *For every fixed positive integers m,n,r and s, we have the following quadratic recurrence formulae*

$$\frac{\mathcal{Z}^{\gamma}_{m+r,n+s}(z,\bar{z})}{m!n!r!s!} = \sum_{j=0}^{m\wedge r}\sum_{k=0}^{n\wedge s}(\alpha+r+1)_j(\alpha+s+1)_k\frac{(-1)^{j+k}h^{j+k}(z)}{j!k!}\frac{\mathcal{Z}^{\gamma+j+k}_{m-j,n-k}(z,\bar{z})}{(m-j)!(n-k)!}\frac{\mathcal{Z}^{\gamma+m+n+j+k}_{r-k,s-j}(z,\bar{z})}{(s-j)!(r-k)!} \tag{4.4}$$

and

$$\frac{\mathcal{Z}^{\gamma}_{m+r,n}(z,\bar{z})}{m!n!} = \sum_{j=0}^{m\wedge n}(\gamma+m+r+1)_j(\gamma+j+1)_{m-j}\frac{(-1)^j\bar{z}^{m-j}h^j(z)}{(m-j)!j!}\frac{\mathcal{Z}^{\gamma+m+j}_{r,n-j}(z,\bar{z})}{(n-j)!}. \tag{4.5}$$

Proof. Using the representation of the operator $\mathcal{Z}^{\gamma}_{m,n}(f)$, given through (3.15), as well as the fact $\mathcal{Z}^{\beta}_{r,s}(z,\bar{z}) = \mathcal{Z}^{\beta}_{r,s}(1)$, we check easily that

$$\mathcal{Z}^{\gamma}_{m+r,n+s}(z,\bar{z}) := \mathcal{Z}^{\gamma}_{m,n}(\mathcal{Z}^{\gamma+m+n}_{r,s}(z,\bar{z})). \tag{4.6}$$

Thus, the operational formula (3.20) gives rise to

$$\mathcal{Z}^{\gamma}_{m+r,n+s}(z,\bar{z}) = m!n!\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{(-1)^{j+k}h^{j+k}(z)}{j!k!}\frac{\mathcal{Z}^{\gamma+j+k}_{m-j,n-k}(z,\bar{z})}{(m-j)!(n-k)!}\frac{\partial^{j+k}}{\partial z^j\partial\bar{z}^k}\left(\mathcal{Z}^{\gamma+m+n}_{r,s}(z,\bar{z})\right)$$

which reduces further to

$$m!n!r!s!\sum_{j=0}^{m\wedge r}\sum_{k=0}^{n\wedge s}(\alpha+r+1)_j(\alpha+s+1)_k\frac{(-1)^{j+k}h^{j+k}(z)}{j!k!}\frac{\mathcal{Z}^{\gamma+j+k}_{m-j,n-k}(z,\bar{z})}{(m-j)!(n-k)!}\frac{\mathcal{Z}^{\gamma+m+n+j+k}_{r-k,s-j}(z,\bar{z})}{(s-j)!(r-k)!}.$$

The last equality is due to the fact that

$$\frac{\partial^{j+k}}{\partial z^j\partial\bar{z}^k}\left(\mathcal{Z}^{\alpha}_{r,s}(z,\bar{z})\right)=\frac{r!s!(\alpha+r+1)_j(\alpha+s+1)_k}{(r-k)!(s-j)!}\mathcal{Z}^{\alpha+j+k}_{r-k,s-j}(z,\bar{z}) \tag{4.7}$$

which follows by induction since

$$\frac{\partial}{\partial z}\left(\mathcal{Z}^{\alpha}_{r,s}(z,\bar{z})\right)=s(\alpha+r+1)\mathcal{Z}^{\alpha+1}_{r,s-1}(z,\bar{z})\quad\text{and}\quad\frac{\partial}{\partial\bar{z}}\left(\mathcal{Z}^{\alpha}_{r,s}(z,\bar{z})\right)=r(\alpha+s+1)\mathcal{Z}^{\alpha}_{r-1,s}(z,\bar{z}).$$

The proof of (4.5) lies essentially on the fact that

$$\mathcal{Z}^{\gamma}_{m+r,n}(z,\bar{z}):=\mathcal{A}^{\gamma}_{m,0}(\mathcal{Z}^{\gamma+m}_{r,n}(z,\bar{z})), \tag{4.8}$$

which can be handled easily from the representation (3.14) of the operator $\mathcal{A}^{\gamma}_{m,0}(f)$ together with the fact that $\mathcal{Z}^{\beta}_{r,n}(z,\bar{z})=\mathcal{A}^{\beta}_{r,n}(1)$. Next, the operational formula (3.20) infers

$$\begin{aligned}\mathcal{Z}^{\gamma}_{m+r,n}(z,\bar{z})&=m!\sum_{j=0}^{m\wedge n}\frac{(-1)^jh^j(z)}{j!}\frac{\mathcal{Z}^{\gamma+j}_{m-j,0}(z,\bar{z})}{(m-j)!}\frac{\partial^j}{\partial z^j}(\mathcal{Z}^{\gamma+m}_{r,n}(z,\bar{z}))\\&=m!n!\sum_{j=0}^{m\wedge n}(\gamma+m+r+1)_j\frac{(-1)^jh^j(z)}{j!}\frac{\mathcal{Z}^{\gamma+j}_{m-j,0}(z,\bar{z})}{(m-j)!}\frac{\mathcal{Z}^{\gamma+m+j}_{r,n-j}(z,\bar{z})}{(n-j)!}.\end{aligned}$$

This leads to (4.5) according to $\mathcal{Z}^{\alpha}_{0,s}(z,\bar{z})=(\alpha+1)_sz^s$. □

Remark 4.2. *We recover* (4.1) *from* (4.4) *by taking $s=0$. For specific choices of the parameters, $r=0$ in* (4.5) *or $n=0$ in* (4.1)*, we get the explicit expression of the polynomials $\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})$:*

$$\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})=m!n!\sum_{j=0}^{m\wedge n}\frac{\Gamma(\gamma+m+n+1)}{\Gamma(\gamma+j+1)}\frac{(-1)^jh^j(z)}{j!}\frac{\bar{z}^{m-j}}{(m-j)!}\frac{z^{n-j}}{(n-j)!}$$

which can be rewritten in terms of the Gauss hypergeometric function ${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|x\right)$ as

$$\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})=(\gamma+1)_{m+n}\bar{z}^mz^n{}_2F_1\left(\begin{matrix}-m,-n\\\gamma+1\end{matrix}\middle|1-\frac{1}{|z|^2}\right).$$

5. Three terms recurrence formulas

Proposition 5.1. *We have the following three terms recurrence formulas*

$$\mathcal{Z}^{\gamma}_{m,n+1}(z,\bar{z})=(\gamma+m+n+1)\left\{z\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})-mh(z)\mathcal{Z}^{\gamma+1}_{m-1,n}(z,\bar{z})\right\} \tag{5.1}$$

$$\mathcal{Z}^{\gamma+1}_{m,n}(z,\bar{z})=\left(\frac{\gamma+m+n+1}{\gamma+m+1}\right)\left\{\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})+m\bar{z}\mathcal{Z}^{\gamma+1}_{m-1,n}(z,\bar{z})\right\} \tag{5.2}$$

$$h(z)\frac{\partial}{\partial z}(\mathcal{Z}^{\gamma+1}_{m,n}(z,\bar{z}))=(\gamma+1)\bar{z}\mathcal{Z}^{\gamma+1}_{m,n}(z,\bar{z})-\mathcal{Z}^{\gamma}_{m+1,n}(z,\bar{z}) \tag{5.3}$$

as well as their conjugate

$$\mathcal{Z}^{\gamma}_{m+1,n}(z,\bar{z}) = (\gamma+m+n+1)\left\{\bar{z}\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) - nh(z)\mathcal{Z}^{\gamma+1}_{m,n-1}(z,\bar{z})\right\} \tag{5.4}$$

$$\mathcal{Z}^{\gamma+1}_{m,n}(z,\bar{z}) = \left(\frac{\gamma+m+n+1}{\gamma+n+1}\right)\left\{\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) + nz\mathcal{Z}^{\gamma+1}_{m,n-1}(z,\bar{z})\right\} \tag{5.5}$$

$$h(z)\frac{\partial}{\partial\bar{z}}(\mathcal{Z}^{\gamma+1}_{m,n}(z,\bar{z})) = (\gamma+1)z\mathcal{Z}^{\gamma+1}_{m,n}(z,\bar{z}) - \mathcal{Z}^{\gamma}_{m,n+1}(z,\bar{z}). \tag{5.6}$$

Proof. (5.1) is a special case of (4.1) by taking $s=1$. While (5.2) is an immediate consequence of (4.3) with $s=1$. (5.3) is checked by writing $\mathcal{Z}^{\gamma}_{m+1,n}(z,\bar{z}) = \mathcal{A}^{\gamma}_{m+1,n}(1)$ as

$$\mathcal{Z}^{\gamma}_{m+1,n}(z,\bar{z}) = -h^{-\gamma}(z)\frac{\partial}{\partial z}\left(h^{\gamma+1}(z)\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})\right)$$

and next applying the derivative rule to the involved product. Finally, since

$$\overline{\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})} = \mathcal{Z}^{\gamma}_{m,n}(\bar{z},z) = \mathcal{Z}^{\gamma}_{n,m}(z,\bar{z}),$$

the recurrence formulae (5.4), (5.5) and (5.6) are the conjugate counterparts of (5.1), (5.2) and (5.3) respectively. Note that (5.1) can also be derived directly from (3.17) by applying again the derivative rule to

$$\mathcal{Z}^{\gamma}_{m+1,n}(z,\bar{z}) = (-1)^{m+n}C^{\gamma}_{m,n}h^{-\gamma}(z)\frac{\partial^m}{\partial z^m}\left(\frac{\partial}{\partial z}\left(z^n h^{\gamma+m+1}(z)\right)\right).$$

□

6. Runge's addition formula

Proposition 6.1. *We have de following addition formula*

$$h^{\gamma}\left(\frac{z+w}{\sqrt{2}}\right)Z^{\gamma}_{m,n}\left(\frac{z+w}{\sqrt{2}},\overline{\frac{z+w}{\sqrt{2}}}\right) = \left(\frac{1}{2}\right)^{\gamma+m+\frac{m+n}{2}} m!n!(\gamma+m+n)!\times \tag{6.1}$$
$$\sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{|\mathbf{s}|=\gamma+m}\frac{(-1)^{s_3+s_4}\bar{z}^{s_4}\bar{w}^{s_3}h^{s_1-j}(z)h^{s_2-m+j}(w)}{\mathbf{s}!j!k!(m-j)!(n-k)!}\frac{\mathcal{Z}^{s_1-j}_{j,s_3+k}(z,\bar{z})\mathcal{Z}^{s_2-m+j}_{m-j,s_4+n-k}(w,\bar{w})}{(s_1+1)_{s_3+k}(s_2+1)_{s_4+n-k}}$$

with $\binom{m+\gamma}{\mathbf{s}} := (\gamma+m)!/\mathbf{s}!$, $\mathbf{s}! = s_1!s_2!s_3!s_4!$ *and* $|\mathbf{s}| = s_1+s_2+s_3+s_4$ *for* $s=(s_1,s_2,s_3,s_4)$; $s_j\in\mathbb{Z}^+$, *and where* $\gamma+m$ *is assumed to be a positive integer.*

Proof. From $\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) = (-1)^m(\gamma+m+1)_n h^{-\gamma}(z)\frac{\partial^m}{\partial z^m}(z^n h^{\gamma+m}(z))$, we can write

$$h^{\gamma}\left(\frac{z+w}{\sqrt{2}}\right)Z^{\gamma}_{m,n}\left(\frac{z+w}{\sqrt{2}},\overline{\frac{z+w}{\sqrt{2}}}\right) = (-1)^m(\gamma+m+1)_n\frac{\partial^m}{\partial\left(\frac{z+w}{\sqrt{2}}\right)^m}\left(\left(\frac{z+w}{\sqrt{2}}\right)^n h^{\gamma+m}\left(\frac{z+w}{\sqrt{2}}\right)\right).$$

Making use of the facts that $\frac{\partial}{\partial\left(\frac{z+w}{\sqrt{2}}\right)} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial w}\right)$ and $h\left(\frac{z+w}{\sqrt{2}}\right) = \frac{1}{2}\left(h(z)+h(w)-z\bar{w}-\bar{z}w\right)$ as well as the binomial formulas, including $(X_1+X_2+X_3+X_4)^m = \sum_{|\mathbf{s}|=m}\binom{m}{\mathbf{s}}X_1^{s_1}X_2^{s_2}X_3^{s_3}X_4^{s_4}$, one obtains

$$h^{\gamma}\left(\frac{z+w}{\sqrt{2}}\right) Z^{\gamma}_{m,n}\left(\frac{z+w}{\sqrt{2}}, \overline{\frac{z+w}{\sqrt{2}}}\right) = (-1)^m \left(\frac{1}{2}\right)^{\gamma+2m+\frac{m+}{2}} (\gamma+m+1)_n \times$$
$$\sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{|\mathbf{s}|=\gamma+m} (-1)^{s_3+s_4}\binom{m}{j}\binom{n}{k}\binom{m+\gamma}{\mathbf{s}} \bar{z}^{s_4}\bar{w}^{s_3}\frac{\partial^j}{\partial z^j}\left(z^{s_3+k}h^{s_1}(z)\right)\frac{\partial^{m-j}}{\partial w^{m-j}}\left(w^{s_4+n-k}h^{s_2}(w)\right)$$

that we can rewrite in terms of $\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})$ as

$$h^{\gamma}\left(\frac{z+w}{\sqrt{2}}\right) Z^{\gamma}_{m,n}\left(\frac{z+w}{\sqrt{2}}, \overline{\frac{z+w}{\sqrt{2}}}\right) = \left(\frac{1}{2}\right)^{\gamma+m+\frac{m+n}{2}} (\gamma+m+1)_n \sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{|\mathbf{s}|=\gamma+m} (-1)^{s_3+s_4}\times$$
$$\binom{m}{j}\binom{n}{k}\binom{m+\gamma}{\mathbf{s}} \bar{z}^{s_4}\bar{w}^{s_3} h^{s_1-j}(z) h^{s_2-m+j}(w) \frac{\mathcal{Z}^{s_1-j}_{j,s_3+k}(z,\bar{z})\mathcal{Z}^{s_2-m+j}_{m-j,s_4+n-k}(w,\bar{w})}{(s_1+1)_{s_3+k}(s_2+1)_{s_4+n-k}}.$$

This yields the desired result (6.1). □

Remark 6.2. *Under the assumption $\gamma+m$ is a positive integer and by writing down* (6.1) *for the particular case of $z+w=0$, we get*

$$\sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{|\mathbf{s}|=\gamma+m} \frac{(-1)^k \bar{z}^{s_3+s_4} h^{s_1+s_2}(z)}{\mathbf{s}!k!(n-k)!} \frac{\mathcal{Z}^{s_1-j}_{j,s_3+k}(z,\bar{z})\mathcal{Z}^{s_2-m+j}_{m-j,s_4+n-k}(z,\bar{z})}{j!(m-j)!(s_1+1)_{s_3+k}(s_2+1)_{s_4+n-k}} = 0,$$

when $m \neq n$ as well as

$$\sum_{j=0}^{m}\sum_{k=0}^{m}\sum_{|\mathbf{s}|=\gamma+m} \frac{(-1)^{j+k} \bar{z}^{s_3+s_4} h^{s_1+s_2}(z)}{\mathbf{s}!k!(m-k)!} \frac{\mathcal{Z}^{s_1-j}_{j,s_3+k}(z,\bar{z})\mathcal{Z}^{s_2-m+j}_{m-j,s_4+m-k}(z,\bar{z})}{j!(m-j)!(s_1+1)_{s_3+k}(s_2+1)_{s_4+m-k}} = \frac{(-1)^m 2^{\gamma+2m}}{m!(\gamma+m)!} h^m(z)$$

when $m=n$. The particular case of $m=0$ infers the identity

$$\sum_{s_1+s_2+s_3+s_4=\gamma} \frac{|z|^{2(s_3+s_4)}(1-|z|^2)^{s_1+s_2}}{s_1!s_2!s_3!s_4!} = \frac{2^{\gamma}}{\gamma!}.$$

7. Generating functions

Proposition 7.1. *We have*

$$\sum_{n=0}^{\infty} \frac{v^n}{n!}\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) = m!\left(\frac{1}{1-vz}\right)^{\gamma+1}\left(\frac{\bar{z}}{1-vz}\right)^m P^{(\gamma,0)}_m\left(1-2\frac{v(1-z\bar{z})}{\bar{z}(1-vz)}\right) \tag{7.1}$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{u^m}{m!}\frac{v^n}{n!}\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) = \left(\frac{1}{1-vz-u\bar{z}}\right)^{\gamma+1} {}_2F_1\left(\begin{matrix}\frac{\gamma+1}{2},\frac{\gamma+2}{2}\\ \gamma+1\end{matrix}\,\middle|\, -\frac{4uv(1-z\bar{z})}{(1-vz-u\bar{z})^2}\right). \tag{7.2}$$

Proof. Starting from (3.14) and using the fact that $\sum\limits_{n=0}^{+\infty}\frac{(a)_n}{n!}x^n = (1-x)^{-a}$, we get

$$\sum_{n=0}^{\infty} \frac{v^n}{n!}\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) = (-1)^m h^{-\gamma}(z)\frac{\partial^m}{\partial z^m}\left((1-vz)^{-(\gamma+1+m)}h^{\gamma+m}(z)\right).$$

From the well established fact that

$$\frac{\partial^m}{\partial t^m}\left((1-xt)^{\alpha}(1-yt)^{\beta}\right) = (-1)^m m! y^m (1-xt)^{\alpha}(1-yt)^{\beta-m} P_m^{(\beta-m,-\alpha-\beta-1)}\left(1-2\frac{x(1-yt)}{y(1-xt)}\right),$$

with $\alpha=-(\gamma+1+m)$, $\beta=\gamma+m$, $x=v$, $y=\bar{z}$ and $t=z$, we have

$$\sum_{n=0}^{\infty}\frac{v^n}{n!}\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) = m!\left(\frac{1}{1-vz}\right)^{\gamma+1}\left(\frac{\bar{z}}{1-vz}\right)^{m} P_m^{(\gamma,0)}\left(1-2\frac{v(1-z\bar{z})}{\bar{z}(1-vz)}\right).$$

Moreover,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{u^m}{m!}\frac{v^n}{n!}\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) = \left(\frac{1}{1-vz}\right)^{\gamma+1}\sum_{m=0}^{\infty}\left(\frac{u\bar{z}}{1-vz}\right)^{m} P_m^{(\gamma,0)}\left(1-2\frac{v(1-z\bar{z})}{\bar{z}(1-vz)}\right).$$

By means of the generating function [29, p.256], to wit

$$\sum_{m=0}^{\infty}\frac{(\alpha+\beta+1)_m}{(\alpha+1)_m}t^m P_m^{(\alpha,\beta)}(x) = (1-t)^{-(\alpha+\beta+1)}{}_2F_1\left(\begin{matrix}\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta+2}{2}\\ \alpha+1\end{matrix}\middle| 2\frac{t(x-1)}{(1-t)^2}\right),$$

with

$$t=\frac{u\bar{z}}{1-vz},\quad \alpha=\gamma,\quad \beta=0 \quad\text{and}\quad x=1-2\frac{v(1-z\bar{z})}{\bar{z}(1-vz)},$$

it follows

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{u^m}{m!}\frac{v^n}{n!}\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) = \left(\frac{1}{1-vz-u\bar{z}}\right)^{\gamma+1}{}_2F_1\left(\begin{matrix}\frac{\gamma+1}{2},\frac{\gamma+2}{2}\\ \gamma+1\end{matrix}\middle| -\frac{4uv(1-z\bar{z})}{(1-vz-u\bar{z})^2}\right).$$

□

Corollary 7.2. *The followings generating functions hold*

$$\sum_{n=0}^{+\infty}\frac{\bar{z}^n}{n!}\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) = m!P_m^{(\gamma,0)}(-1)\bar{z}^m h^{-(\gamma+m+1)}(z) \tag{7.3}$$

$$\sum_{m,n=0}^{+\infty}\frac{z^m\bar{z}^n}{m!n!}\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) = \left(1-2|z|^2\right)^{-(\gamma+1)}{}_2F_1\left(\begin{matrix}\frac{\gamma+1}{2},\frac{\gamma+2}{2}\\ \gamma+1\end{matrix}\middle| -\frac{4|z|^2(1-|z|^2)}{(1-2|z|^2)^2}\right) \tag{7.4}$$

$$\sum_{m,n=0}^{+\infty}\frac{h^m(z)\bar{z}^n}{m!n!}\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) = \left(\frac{1}{(1-\bar{z})h(z)}\right)^{\gamma+1}{}_2F_1\left(\begin{matrix}\frac{\gamma+1}{2},\frac{\gamma+2}{2}\\ \gamma+1\end{matrix}\middle| -\frac{4\bar{z}}{(1-\bar{z})^2}\right) \tag{7.5}$$

$$\sum_{m,n=0}^{+\infty}\frac{h^m(z)\bar{z}^n}{m!n!}\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) = \frac{h^{-(\gamma+1)}(z)}{1+\bar{z}} \tag{7.6}$$

Proof. (7.3) and (7.4) follow from (7.1) and (7.2), respectively by setting there $v=\bar{z}$ and $u=z$. While (7.5) follows also from (7.4) by taking $u=h(z)$ and $v=\bar{z}$. The last one, i.e. (7.6), is checked easily making use of (7.3) or directly from $\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z}) = (-1)^m(\gamma+m+1)_n h^{-\gamma}(z)\frac{\partial^m}{\partial z^m}(z^n h^{\gamma+m}(z))$ combined with the fact that $\frac{\partial^m}{\partial z^m}(h^{-\beta}(z)) = (\beta)_m\bar{z}^m h^{-\beta-m}(z)$. □

Remark 7.3. *From* (7.5) *and* (7.6) *we get the following identity*

$${}_2F_1\left(\begin{matrix}\frac{\gamma+1}{2},\frac{\gamma+2}{2}\\ \gamma+1\end{matrix}\middle| -\frac{4\bar{z}}{(1-\bar{z})^2}\right) = \frac{(1-\bar{z})^{\gamma+1}}{1+\bar{z}}. \tag{7.7}$$

An other generating function for the $\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})$ is the consequence of the following

Proposition 7.4. *For every fixed positive integers m and r, we have the following*

$$\sum_{k=0}^{\infty}\frac{(-1)^k\bar{z}^k}{k!}\frac{\mathcal{Z}^{\gamma}_{m,n+r+k}(z,\bar{z})}{(\gamma+m+n+1)_{r+k}} = e^{-|z|^2}\sum_{j=0}^{m}\frac{(-1)^j(-m)_jh^j(z)}{j!}\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})H_{r,j}(z,\bar{z}). \tag{7.8}$$

Proof. Taking $f(z)=z^re^{-|z|^2}$ in the operational formula (3.19) yields

$$\mathcal{A}^{\gamma}_{m,n}(z^re^{-|z|^2}) = \sum_{j=0}^{m}\frac{m!}{(m-j)!}\frac{(-1)^jh^j(z)}{j!}\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})\frac{\partial^j}{\partial z^j}\left(z^re^{-|z|^2}\right). \tag{7.9}$$

The j^{th} derivative of $z^re^{-|z|^2}$ in the right hand side of the last equality (7.9) is connected to the complex Hermite polynomials by

$$\frac{\partial^j}{\partial z^j}\left(z^re^{-|z|^2}\right) = (-1)^je^{-|z|^2}H_{r,j}(z,\bar{z})$$

so that (7.9) reduces further to

$$\mathcal{A}^{\gamma}_{m,n}(z^re^{-|z|^2}) = e^{-|z|^2}\sum_{j=0}^{m}\frac{m!}{(m-j)!}\frac{h^j(z)}{j!}\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})H_{r,j}(z,\bar{z}). \tag{7.10}$$

In the other hand, by expanding $e^{-|z|^2}$ as series and inserting it in the expression of $\mathcal{A}^{\gamma}_{m,n}(z^re^{-|z|^2})$ given through (3.14), we get

$$\begin{aligned}\mathcal{A}^{\gamma}_{m,n}(z^re^{-|z|^2}) &= \sum_{k=0}^{\infty}(-1)^mC^{\gamma}_{m,n}h^{-\gamma}(z)\frac{\partial^m}{\partial z^m}\left(z^{n+r+k}h^{\gamma+m}(z)\right)\frac{(-1)^k\bar{z}^k}{k!}\\ &= \sum_{k=0}^{\infty}\left(\frac{C^{\gamma}_{m,n}}{C^{\gamma}_{m,n+r+k}}\right)\frac{(-1)^k\bar{z}^k}{k!}\mathcal{Z}^{\gamma}_{m,n+r+k}(z,\bar{z})\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{C^{\gamma}_{m+n,r+k}}\frac{\bar{z}^k}{k!}\mathcal{Z}^{\gamma}_{m,n+r+k}(z,\bar{z}).\end{aligned} \tag{7.11}$$

Equating the two right hand sides of (7.10) and (7.11) infers

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{C^{\gamma}_{m+n,r+k}}\frac{\bar{z}^k}{k!}\mathcal{Z}^{\gamma}_{m,n+r+k}(z,\bar{z}) = e^{-|z|^2}\sum_{j=0}^{m}\frac{m!}{(m-j)!}\frac{h^j(z)}{j!}\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})H_{r,j}(z,\bar{z})$$

that we can rewrite as

$$\sum_{k=0}^{\infty}\frac{(-1)^k\bar{z}^k}{k!}\frac{\mathcal{Z}^{\gamma}_{m,n+r+k}(z,\bar{z})}{(\gamma+m+n+1)_{r+k}} = e^{-|z|^2}\sum_{j=0}^{m}\frac{(-1)^j(-m)_jh^j(z)}{j!}\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})H_{r,j}(z,\bar{z}).$$

□

Corollary 7.5. *Let ${}_1F_1(a;c;x)$ denotes the confluent hypergeometric function. We have*

$$\sum_{k=0}^{\infty}\frac{(-1)^k\bar{z}^k}{k!}\frac{\mathcal{Z}^{\gamma}_{m,k}(z,\bar{z})}{(\gamma+m+1)_k} = (\gamma+1)_m\bar{z}^me^{-|z|^2}{}_1F_1(-m;\gamma+1;|z|^2-1). \tag{7.12}$$

Proof. This is in fact a particular case of (7.8). Indeed, by taking $r = 0$ and $n = 0$ keeping in mind that $H_{0,j}(z,\bar{z}) = \bar{z}^j$ and $\mathcal{Z}^{\gamma+j}_{m-j,0}(z,\bar{z}) = (\gamma+j+1)_{m-j}\bar{z}^{m-j}$, it follows

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k \bar{z}^k}{k!} \frac{\mathcal{Z}^{\gamma}_{m,k}(z,\bar{z})}{(\gamma+m+1)_k} &= (\gamma+1)_m \bar{z}^m e^{-|z|^2} \sum_{j=0}^{m} \frac{(-m)_j}{(\gamma+1)_j} \frac{(-1)^j h^j(z)}{j!} \\ &= (\gamma+1)_m \bar{z}^m e^{-|z|^2} {}_1F_1(-m;\gamma+1;|z|^2-1).\end{aligned}$$

□

The following result show that the monomial z^m restricted to the unit disk has an expansion in terms of the polynomials $\mathcal{Z}^{\alpha}_{j,k}(z,\bar{z})$.

Proposition 7.6. *For every fixed positive integer m, we have the following*

$$\bar{z}^m = m! e^{-|z|^2} \sum_{n=0}^{\infty} \sum_{j=0}^{m} \frac{(-1)^j \bar{z}^{n+j} h^j(z)}{j!(\gamma+1)_{m+n}} \frac{\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})}{(m-j)!n!}. \tag{7.13}$$

Proof. We begin noting that the action of the operator $\mathcal{A}^{\gamma}_{m,n}$ in (3.14) on $f(z) = \bar{z}^n e^{-|z|^2}$ reads

$$\mathcal{A}^{\gamma}_{m,n}(\bar{z}^n e^{-|z|^2}) = (-1)^m C^{\gamma}_{m,n} h^{-\gamma}(z) \frac{\partial^m}{\partial z^m}\left(|z|^{2n} h^{\gamma+m}(z) e^{-|z|^2}\right).$$

Therefore, from (3.22) and $(-a)_k = (-1)^k (a-k+1)_k$, it is easy to see that

$$\sum_{n=0}^{\infty} \frac{1}{n! C^{\gamma}_{m,n}} \mathcal{A}^{\gamma}_{m,n}(\bar{z}^n e^{-|z|^2}) = (\gamma+1)_m \bar{z}^m. \tag{7.14}$$

In the other hand, making appeal of the operational formula (3.19), the left hand side of (7.14) can be rewritten as

$$\sum_{n=0}^{\infty} \frac{1}{n! C^{\gamma}_{m,n}} \mathcal{A}^{\gamma}_{m,n}(\bar{z}^n e^{-|z|^2}) = m! e^{-|z|^2} \sum_{n=0}^{\infty} \sum_{j=0}^{m} \frac{(-1)^j \bar{z}^{n+j} h^j(z)}{j! C^{\gamma}_{m,n}} \frac{\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})}{(m-j)!n!}. \tag{7.15}$$

Hence from (7.14) and (7.15), it follows

$$\bar{z}^m = e^{-|z|^2} \frac{m!}{(\gamma+1)_m} \sum_{n=0}^{\infty} \sum_{j=0}^{m} \frac{(-1)^j \bar{z}^{n+j} h^j(z)}{j!(\gamma+m+1)_n} \frac{\mathcal{Z}^{\gamma+j}_{m-j,n}(z,\bar{z})}{(m-j)!n!}. \tag{7.16}$$

which gives rise to (7.13). □

Corollary 7.7. *We have*

$$e^{\bar{z}(1+z)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \bar{z}^{n+j} h^j(z)}{j!(\gamma+1)_{m+j+n}} \frac{\mathcal{Z}^{\gamma+j}_{m,n}(z,\bar{z})}{m!n!}. \tag{7.17}$$

Proof. The result follows easily from (7.13) since $\sum\limits_{m=0}^{\infty} \sum\limits_{j=0}^{m} A(m,j) = \sum\limits_{m=0}^{\infty} \sum\limits_{j=0}^{\infty} A(m+j,j)$. □

Further fascinating identities including those involving the product of disk polynomials may be obtained, from their analogous for the Jacobi polynomials, by means of (3.10). For example, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2m+1)_n}{(2m+n+1)_n} \frac{\mathcal{Z}^{m}_{m+n,n}(z,\bar{z})}{(m+n)!n!} = \frac{(-1)^m}{m!} \bar{z}^m \left[1 - 2(1-2|z|^2)w + w^2\right]^{-(m+\frac{1}{2})} \tag{7.18}$$

which can be handled making use of the generating relation ([29, Eq.3, p.276]):

$$\sum_{n=0}^{\infty} \frac{(2\alpha+1)_n}{(\alpha+1)_n} t^n P_n^{(\alpha,\beta)}(x) = [1-2xt+t^2]^{-\alpha-\frac{1}{2}}.$$

Also, from []

$$\sum_{n=0}^{\infty} \frac{n!(\alpha+\beta+1)_n}{(\alpha+1)_n(1+\beta)_n} t^n P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y)$$
$$= (1+t)^{\alpha-\beta-1} F_4\left(\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2}; \alpha+1, \beta+1; \frac{(1-x)(1-y)t}{(1+t)^2}, \frac{(1+x)(1+y)t}{(1+t)^2}\right)$$

where $F_4(a,b;c,c';x,y)$ stands for the fourth Appell's function ([29, p.265]), one deduces the following generating function for the product of disk polynomials with equal arguments and indices but with different variables:

$$\sum_{n=0}^{\infty} \frac{t^n \mathcal{Z}_{n,n}^{\gamma-1}(z,\bar{z}) \mathcal{Z}_{n,n}^{\gamma-1}(w,\bar{w})}{[n!(\gamma+n)_n]^2} = \frac{1}{(1+t)^{\gamma}} F_4\left(\frac{\gamma}{2}, \frac{\gamma+1}{2}; \gamma-1, \gamma; \frac{4|z|^2|w|^2 t}{(1+t)^2}, \frac{4(1-|z|^2)(1-|w|^2)t}{(1+t)^2}\right).$$

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