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Catalan-like Numbers and Determinants

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A class of numbers, called Catalan-like numbers, are introduced which unify many well-known counting coefficients, such as the Catalan numbers, the Motzkin numbers, the middle binomial coefficients, the hexagonal numbers, and many more. Generating functions, recursions and determinants of Hankel matrices are computed, and some interpretations are given as to what these numbers count.

1. INTRODUCTION

The starting point for this paper is the observation that the Catalan numbers C_n and the Motzkin numbers M_n enjoy several common properties, informally sketched in (A) and (B) below. Recall that the numbers C_n are defined by the recursion $C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$, $C_0 = 1$, and the numbers M_n by $M_n = M_{n-1} + \sum_{k=0}^{n-2} M_k M_{n-2-k}$, $M_0 = 1$, with generating functions $C(x) = (1 - \sqrt{1-4x})/2x$ and $M(x) = (1 - x - \sqrt{1-2x-3x^2})/2x^2$.

(A) A beautiful (and not so well-known) description of the Catalan numbers is the following (see [8]): The numbers C_n are the unique sequence of real numbers such that the Hankel matrices

$$\begin{pmatrix} C_0 & C_1 & \cdots & C_n \\ C_1 & C_2 & \cdots & C_{n+1} \\ \vdots & & & \\ C_n & C_{n+1} & \cdots & C_{2n} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C_1 & C_2 & \cdots & C_n \\ C_2 & C_3 & \cdots & C_{n+1} \\ \vdots & & & \\ C_n & C_{n+1} & \cdots & C_{2n-1} \end{pmatrix}$$

each have determinant 1, for all n. It was shown in [1] that, for the Motzkin numbers, the determinant of the first Hankel matrix is again 1 for all n, while the determinant of the second matrix is $1, 0, -1, -1, 0, 1$ for $n = 1, ..., 6$, repeating modulo 6 thereafter.

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(B) There are several classical formulae for the Catalan numbers involving binomial coefficients (see, e.g., [6]), such as

$$C_{n+1}=\sum_k \binom{n}{2k} 2^{n-2k} C_k, \qquad C_{n+1}=\sum_k (-1)^k \binom{n}{k} 4^{n-k} C_{k+1},$$

$$\binom{n}{\lfloor n/2 \rfloor}=\sum_k (-1)^k \binom{n}{k} 2^{n-k} C_k.$$

Similarly, we have, for example

$$M_n=\sum_k (-1)^k \binom{n}{k} 2^{n-k} M_k,$$

$$M_n=\sum_k \binom{n}{2k} C_k, \qquad C_{n+1}=\sum_k \binom{n}{k} M_k.$$

The purpose of the present paper is to provide a common framework for a series of coefficients (called Catalan-like numbers), with C_n and M_n as special cases, which show these common features. A major source of inspiration, especially for Section 3, was the paper [7] discussing the Riordan group. In Section 2 we introduce a class of infinite matrices which lead to the Catalan-like numbers. In Section 3 we look at the most interesting special cases, giving the generating functions and recursions. Section 4 is devoted to the Hankel matrices of Catalan-like numbers, and Section 5 briefly deals with some interpretations as to what these numbers count.

Apart from C_n and M_n we need the middle binomial coefficients, which we denote by $W_n=\binom{n}{\lfloor n/2 \rfloor}$ and the central binomial coefficients $B_n=\binom{2n}{n}$. Their generating functions are $W(x)=(1-2x-\sqrt{1-4x^2})/(-2x+4x^2)$ and $B(x)=1/\sqrt{1-4x}$. For all terms not defined the reader may consult any of the standard texts such as [3, 6, 8].

2. A CLASS OF MATRICES

We consider infinite matrices $A=(a_{n,k})$, indexed by $\{0, 1, 2, ...\}$, and denote by $r_m=(a_{m,0}, a_{m,1}, ...)$ the mth row.

DEFINITION. $A=(a_{n,k})$ is called *admissible* if

(i) $a_{n,k}=0$ for $n<k$, $a_{n,n}=1$ for all n (that is, A is lower triangular with main diagonal equal to 1),

(ii) $r_m \cdot r_n=a_{m+n,0}$ for all m, n, where $r_m \cdot r_n=\sum_k a_{m,k}a_{n,k}$ is the usual inner product.

The numbers $a_{n,0}$ of the first column will be our Catalan-like numbers. Our first result describes these matrices.

LEMMA 1. *An admissible matrix* $A = (a_{n,k})$ *is uniquely determined by the sequence* $(a_{1,0}, a_{2,1}, ...a_{n+1,n}, ...)$. *Conversely, to every sequence* $(b_0, b_1, ..., b_n, ...)$ *of real numbers there exists an (and therefore precisely one) admissible matrix* $(a_{n,k})$ *with* $a_{n+1,n} = b_n$ *for all n.*

Proof. It suffices to show that the sequence $(b_0, b_1, ...)$ uniquely determines an admissible matrix $(a_{n,k})$ with $a_{n+1,n} = b_n$ for all n.

The rows r_0 and r_1 are given. Then $a_{2,0} = r_1 \cdot r_1$, and so row r_2 has been found. Now $a_{3,0} = r_1 \cdot r_2$, $a_{4,0} = r_2 \cdot r_2$, and then the equation $r_1 \cdot r_3 = a_{4,0}$ determines $a_{3,1}$, and so row r_3 has been found. Assume inductively that we know rows $r_0, r_1, ..., r_n$ and the entries $a_{0,0}, a_{1,0}, ..., a_{2n-2,0}$. Then we find

$$a_{2n-1,0} = r_{n-1} \cdot r_n, \qquad a_{2n,0} = r_n \cdot r_n,$$

and then the equations

$$r_k \cdot r_{n+1} = a_{k+n+1,0} \qquad (k = 1, ..., n-1)$$

determine $a_{n+1,k}$, $k = 1, ..., n-1$, that is, the row r_{n+1}. ∎

PROPOSITION 1. *Let* $A = (a_{n,k})$ *be an admissible matrix with* $a_{n+1,n} = b_n$ *for all n. Set* $s_0 = b_0$, $s_1 = b_1 - b_0$, ..., $s_n = b_n - b_{n-1}$, ... *Then we have*

$$\begin{aligned} &a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + a_{n-1,k+1} \qquad (n \geqslant 1) \\ &a_{0,0} = 1, \qquad a_{0,k} = 0 \qquad \textit{for} \quad k > 0. \end{aligned} \tag{1}$$

Conversely, if $a_{n,k}$ *is given by the recursion* (1), *then* $(a_{n,k})$ *is an admissible matrix with* $a_{n+1,n} = s_0 + \cdots + s_n$.

Proof. By the uniqueness (Lemma 1) it suffices to verify the second part. That is, we have to show that if $a_{n,k}$ is defined according to (1), then

(i) $a_{n,k} = 0$ for $n < k$, $a_{n,n} = 1$

(ii) $a_{n+1,n} = s_0 + \cdots + s_n$

(iii) $r_n \cdot r_\ell = a_{n+\ell,0}$.

(i) Suppose (i) is true up to $n-1$. Then

$$\begin{aligned} a_{n,k} &= a_{n-1,k-1} + s_k a_{n-1,k} + a_{n-1,k+1} \\ &= \begin{cases} 0, & n < k \\ 1, & n = k. \end{cases} \end{aligned}$$

(ii) We have $a_{1,0} = s_0 a_{0,0} = s_0$, and hence by induction $a_{n+1,n} = a_{n,n-1} + s_n a_{n,n} + a_{n,n+1} = (s_0 + \cdots + s_{n-1}) + s_n$.

(iii) By (i) we have $r_0 \cdot r_\ell = a_{0,0} a_{\ell,0} = a_{\ell,0}$. Suppose the assertion is true for all $m \leqslant n-1$ and all ℓ. Then

$$\begin{aligned} r_n \cdot r_\ell &= \sum_{j=0}^{n} a_{n,j} a_{\ell,j} = \sum_{j=0}^{n} (a_{n-1,j-1} + s_j a_{n-1,j} + a_{n-1,j+1}) a_{\ell,j} \\ &= \sum_{j=0}^{n-1} (a_{\ell,j+1} + s_j a_{\ell,j} + a_{\ell,j-1}) a_{n-1,j} \\ &= \sum_{j=0}^{n-1} a_{\ell+1,j} a_{n-1,j} = r_{n-1} \cdot r_{\ell+1} = a_{n+\ell,0}. \quad \blacksquare \end{aligned}$$

According to the proposition we may (and will do so from now on) consider an admissible matrix $(a_{n,k})$ as given by the sequence $\sigma = (s_0, s_1, s_2, \ldots)$ via the recursion (1). We will then write shortly $A = A^{(\sigma)}$ and call $C_n^{(\sigma)} = a_{n,0}$ the *Catalan-like numbers* of type σ.

EXAMPLE 1. Consider the following three matrices determined by the sequences written on top:

$$\begin{array}{cccccc} \multicolumn{6}{c}{(1, 0, 0, \ldots)} \\ 1 \\ 1 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 1 & 1 \\ 6 & 4 & 4 & 1 & 1 \\ 10 & 10 & 5 & 5 & 1 & 1 \\ & & \cdots \end{array} \qquad \begin{array}{cccccc} \multicolumn{6}{c}{(1, 1, 1, \ldots)} \\ 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 4 & 5 & 3 & 1 \\ 9 & 12 & 9 & 4 & 1 \\ 21 & 30 & 25 & 14 & 5 & 1 \\ & & \cdots \end{array} \qquad \begin{array}{cccccc} \multicolumn{6}{c}{(1, 2, 2, \ldots)} \\ 1 \\ 1 & 1 \\ 2 & 3 & 1 \\ 5 & 9 & 5 & 1 \\ 14 & 28 & 20 & 7 & 1 \\ 42 & 90 & 75 & 35 & 9 & 1 \\ & & \cdots \end{array}$$

We will see that the Catalan-like numbers are W_n, M_n and C_n, respectively. (It is, of course, well known that these numbers can be generated in this way via the recursion (1).)

Our next result deals with two basic properties of the admissible matrices $A^{(\sigma)}$.

PROPOSITION 2. *Let $A^{(\sigma)} = (a_{n,k})$ be determined by $\sigma = (s_0, s_1, \ldots)$.*

(i) *Set $b_{n,k} = (-1)^{n+k} a_{n,k}$, then $(b_{n,k}) = A^{(-\sigma)}$, where $-\sigma = (-s_0, -s_1, -s_2, \ldots)$.*

(ii) *Let $P = (\binom{n}{k})$ be the binomial matrix with $\binom{n}{k}$ as its (n,k)-entry. Then $A^{(\sigma+1)} = PA^{(\sigma)}$, where $\sigma + 1 = (s_0 + 1, s_1 + 1, s_2 + 1, \ldots)$.*

Proof. (i) We have to verify the recursion (1) for the coefficients $b_{n,k}$ with sequence $-\sigma$. With $b_{0,0}=a_{0,0}=1$, we find by induction

$$b_{n,k}=(-1)^{n+k}\,a_{n,k}=(-1)^{n+k}\,(a_{n-1,k-1}+s_k a_{n-1,k}+a_{n-1,k+1})$$
$$=b_{n-1,k-1}-s_k b_{n-1,k}+b_{n-1,k+1}.$$

(ii) The (n,ℓ)-entry of $PA^{(\sigma)}$ is $c_{n,\ell}=\sum_k \binom{n}{k}\,a_{k,\ell}$. Hence we have to verify the recursion (1) for $c_{n,\ell}$ with sequence $\sigma+1$. By induction on n,

$$c_{n,\ell}=\sum_k \binom{n}{k}\,a_{k,\ell}=\sum_k \binom{n-1}{k-1}\,a_{k,\ell}+\sum_k \binom{n-1}{k}\,a_{k,\ell}$$
$$=\sum_k \binom{n-1}{k}\,a_{k+1,\ell}+\sum_k \binom{n-1}{k}\,a_{k,\ell}$$
$$=\sum_k \binom{n-1}{k}\,(a_{k,\ell-1}+s_\ell a_{k,\ell}+a_{k,\ell+1})+\sum_k \binom{n-1}{k}\,a_{k,\ell}$$
$$=\sum_k \binom{n-1}{k}\,(a_{k,\ell-1}+(s_\ell+1)\,a_{k,\ell}+a_{k,\ell+1})$$
$$=c_{n-1,\ell-1}+(s_\ell+1)\,c_{n-1,\ell}+c_{n-1,\ell+1}. \quad \blacksquare$$

It is well known (and immediately seen) that the power P^t $(t\in\mathbb{Z})$ of the binomial matrix P has $\binom{n}{k}\,t^{n-k}$ as its (n,k)-entry. Hence we obtain from Proposition 2:

COROLLARY 1. *Let $A^{(\sigma)}=(a_{n,k})$, $t\in\mathbb{Z}$. Then the (n,ℓ)-entry $b_{n,\ell}$ of $A^{(\sigma+t)}$ is given by $b_{n,\ell}=\sum_k \binom{n}{k}\,t^{n-k}a_{k,\ell}$.*

3. THE SPECIAL CASE $\sigma=(a,s,s,...)$

The most important special cases arise when $\sigma=(s_0,s_1,...)$ has constant $s_1=s_2=\cdots=s$ with $s_0=a$. We then write shortly $\sigma=(a,s)$ and $A^{(\sigma)}=A^{(a,s)}$. Let $C_k(x)=\sum_{n\geq 0} a_{n,k}x^n$ be the generating function of the kth column of $A^{(a,s)}$. The recursion (1) translates to

$$\begin{aligned} C_k(x)&=x(C_{k-1}(x)+sC_k(x)+C_{k+1}(x)) \qquad (k\geq 1)\\ C_0(x)&=x(aC_0(x)+C_1(x))+1. \end{aligned} \tag{2}$$

In this case it is easy to compute the generating function $C_0(x)$. Let $f(x)$ satisfy

$$f(x) = x(1 + sf(x) + f(x)^2), \tag{3}$$

then we claim that $C_k(x) = f(x)^k C_0(x)$. Indeed, by (3),

$$f(x)^k C_0(x) = x(f(x)^{k-1} C_0(x) + sf(x)^k C_0(x) + f(x)^{k+1} C_0(x))$$

which means that the functions $f(x)^k C_0(x)$ satisfy precisely the recursion (2) for $k \geqslant 1$, implying $C_k(x) = f(x)^k C_0(x)$.

Matrices of this type were introduced in [7] via the so-called Riordan group. In the framework of Riordan groups we have $A^{(a,s)} = (C_0(x), f(x))$.

In the sequel we concentrate on the generating function $C_0(x)$ of the Catalan-like numbers associated with the sequence $\sigma = (a, s)$, and write $C^{(a,s)}(x) = C_0(x) = \sum_{n \geqslant 0} C_n^{(a,s)} x^n$. To determine $C^{(a,s)}(x)$, we find from (3) by an easy computation

$$f(x) = \frac{1 - sx - \sqrt{1 - 2sx + (s^2 - 4)\, x^2}}{2x}.$$

Substituting $f(x)$ into Eq. (2)

$$C^{(a,s)}(x) = x(aC^{(a,s)}(x) + f(x)\, C^{(a,s)}(x)) + 1$$

yields the following result:

Proposition 3. *The generating function $C^{(a,s)}(x)$ of the Catalan-like numbers $C_n^{(a,s)}$ is given by*

$$C^{(a,s)}(x) = \frac{1 - (2a - s)x - \sqrt{1 - 2sx + (s^2 - 4)\, x^2}}{2(s - a)\, x + 2(a^2 - as + 1)x^2}. \tag{5}$$

Now let us look at some examples. It turns out that the cases $a = s$, $a = s + 1$ and $a = s - 1$ yield the most interesting sequences.

3.1. *Examples*

(1) $a = s$,

$$C^{(s,s)}(x) = \frac{1 - sx - \sqrt{1 - 2sx + (s^2 - 4)\, x^2}}{2x^2}.$$

For $s = 0$ we obtain $C^{(0,0)}(x) = (1 - \sqrt{1 - 4x^2})/2x^2 = C(x^2)$ with sequence $(C_0, 0, C_1, 0, C_2, 0, C_3, \ldots)$.

For $s=1$, $C^{(1,1)}(x)=(1-x-\sqrt{1-2x-3x^2})/2x^2=M(x)$, the Motzkin series with sequence $(M_0, M_1, \ldots)$.

For $s=2$, $C^{(2,2)}(x)=(1-2x-\sqrt{1-4x})/2x^2=(1/x)(C(x)-1)$ with sequence $(C_1, C_2, C_3, \ldots)$.

For $s=3$, $C^{(3,3)}(x)=(1-3x-\sqrt{1-6x+5x^2})/2x^2$ which is the generating function of the restricted hexagonal numbers H_n described in [4]. The first terms are (1, 3, 10, 36, 137, 543, ...).

(2) $a=s+1$,

$$C^{(s+1,s)}(x)=\frac{1-(s+2)x-\sqrt{1-2sx+(s^2-4)x^2}}{-2x+2(s+2)x^2}.$$

For $s=0$, this gives $C^{(1,0)}(x)=(1-2x-\sqrt{1-4x^2})/(-2x+4x^2)=W(x)$ as announced in Example 1.

For $s=1$, $C^{(2,1)}(x)=(1-3x-\sqrt{1-2x-3x^2})/(-2x+6x^2)$ with the first terms (1, 2, 5, 13, 35, 96, ...). We will return to this sequence in 3.4.

For $s=2$, $C^{(3,2)}=(1-4x-\sqrt{1-4x})/(-2x+8x^2)=(1/2x)(B(x)-1)$. Hence $C_n^{(3,2)}=\frac{1}{2}\binom{2n+2}{n+1}=\binom{2n+1}{n}$.

(3) $a=s-1$,

$$C^{(s-1,s)}(x)=\frac{1-(s-2)x-\sqrt{1-2sx+(s^2-4)\,x^2}}{2x-2(s-2)\,x^2}.$$

For $s=1$, $C^{(0,1)}(x)=(1+x-\sqrt{1-2x-3x^2})/(2x+2x^2)$ with coefficients (1, 0, 1, 1, 3, 6, 15, 36, ...). We will consider these numbers in 3.4.

For $s=2$, $C^{(1,2)}(x)=(1-\sqrt{1-4x})/2x=C(x)$, the Catalan function.

For $s=3$, $C^{(2,3)}(x)=(1-x-\sqrt{1-6x+5x^2})/(2x-2x^2)$ with first terms (1, 2, 5, 15, 51, 188, ...). We will return to this sequence in Section 5.

3.2. *Binomial Formulae*

To illustrate our approach we will apply Corollary 1 to some examples. If we set $\ell=0$ in Corollary 1, then we obtain

$$C_n^{(\sigma+t)}=\sum_k \binom{n}{k} t^{n-k} C_k^{(\sigma)}. \tag{6}$$

Thus, we obtain as examples the binomial formulae mentioned in the Introduction by taking $\sigma=(0,0)$, $t=1$, $\sigma=(1,1)$, $t=1$; $\sigma=(0,0)$, $t=2$, respectively.

$$M_n=\sum_k \binom{n}{2k} C_k, \qquad C_{n+1}=\sum_k \binom{n}{k} M_k, \qquad C_{n+1}=\sum_k \binom{n}{2k} 2^{n-2k} C_k.$$

Furthermore, the values $\sigma=(0,0)$, $t=3$; $\sigma=(1,1)$, $t=2$; $\sigma=(2,2)$, $t=1$ yield for the hexagonal numbers

$$H_n=\sum_k \binom{n}{2k} 3^{n-2k} C_k=\sum_k \binom{n}{k} 2^{n-k} M_k=\sum_k \binom{n}{k} C_{k+1}.$$

The values $\sigma=(-1,-1)$, $t=2$; $\sigma=(-1,-1)$, $t=3$; $\sigma=(-2,-2)$, $t=4$ produce the formulae

$$M_n=\sum_k (-1)^k \binom{n}{k} 2^{n-k} M_k, \qquad C_{n+1}=\sum_k (-1)^k \binom{n}{k} 3^{n-k} M_k,$$

$$C_{n+1}=\sum_k (-1)^k \binom{n}{k} 4^{n-k} C_{k+1}.$$

The value $\sigma=(-1,0)$, $t=2$ in (6) gives

$$C_n=\sum_k (-1)^k \binom{n}{k} 2^{n-k} \binom{k}{\lfloor k/2 \rfloor}$$

and $\sigma=(-1,-2)$, $t=4$ yields

$$\binom{2n+1}{n}=\sum_k (-1)^k \binom{n}{k} 4^{n-k} C_k.$$

3.3. *Recursions*

Formula (5) leads by an easy computation to the equation

$$C^{(a,s)}(x)=(2a-s)\,xC^{(a,s)}(x)+((s-a)x+(a^2-as+1)\,x^2)\,C^{(a,s)}(x)^2+1.$$

In particular, this yields the following recursions for the Catalan-like numbers of our main examples:

$$C_n^{(s,s)}=sC_{n-1}^{(s,s)}+\sum_{k=0}^{n-2} C_k^{(s,s)} C_{n-2-k}^{(s,s)} \qquad (n\geqslant 1) \tag{7}$$

$$C_n^{(s+1,s)}=(s+2)^n-\sum_{k=0}^{n-1} C_k^{(s+1,s)} C_{n-1-k}^{(s+1,s)} \qquad (n\geqslant 1) \tag{8}$$

$$C_n^{(s-1,s)}=(s-2)^n+\sum_{k=0}^{n-1} C_k^{(s-1,s)} C_{n-1-k}^{(s-1,s)} \qquad (n\geqslant 1). \tag{9}$$

For $a=s=1$ and $a=1$, $s=2$ this gives the recursions for the Motzkin number and Catalan numbers, respectively. For the hexagonal numbers H_n we obtain $H_n=3H_{n-1}+\sum_{k=0}^{n-2} H_k H_{n-2-k}$ described in [4].

3.4. *Connecting $C^{(s+1,s)}(x)$ and $C^{(s-1,s)}(x)$ to $C^{(s,s)}(x)$*

Looking at (5) again, the following formulae are easily derived:

$$C^{(s+1,s)}(x) = -\frac{xC^{(s,s)}(x)}{1-(s+2)x} + \frac{1}{1-(s+2)x}$$

$$C^{(s-1,s)}(x) = \frac{xC^{(s,s)}(x)}{1-(s-2)x} + \frac{1}{1-(s-2)x}.$$

This yields, for example,

$$C_n^{(1,0)} = \binom{n}{\lfloor n/2 \rfloor} = 2^n - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} 2^{n-1-2k} C_k \tag{10}$$

$$C_n^{(2,1)} = 3^n - \sum_{k=0}^{n-1} 3^{n-1-k} M_k \tag{11}$$

$$C_n^{(0,1)} = \sum_{k=0}^{n-1} (-1)^{n-1-k} M_k + (-1)^n = M_{n-1} - M_{n-2} \pm \cdots + (-1)^n M_1. \tag{12}$$

Considering the instance $\sigma = (-1, -2)$, $t = 3$ in (6) we therefore obtain

$$3^n - \sum_{k=0}^{n-1} 3^{n-1-k} M_k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k} C_k$$

and therefore another equation relating the Catalan and Motzkin numbers:

$$\sum_{k=1}^{n} \left((-1)^k \binom{n}{k} C_k + M_{k-1} \right) 3^{n-k} = 0 \qquad (n \geqslant 1).$$

3.5. *The Sum Coefficients*

Several further interesting coefficients arise by considering the row sums of admissible matrices. Consider an arbitrary admissible matrix $A^{(\sigma)} = (a_{n,k})$ with sequence $\sigma = (s_0, s_1, s_2, \ldots)$. The *sum matrix* $B^{(\sigma)} = (b_{n,k})$ is defined by

$$b_{n,k} = \sum_{i \geqslant k} a_{n,i} \qquad \text{for all} \quad n \text{ and } k.$$

By introducing the infinite lower triangular matrix E of 1's, we can thus succinctly write

$$B^{(\sigma)} = A^{(\sigma)} E.$$

EXAMPLE. The sequence $\sigma=(0, 1)$ gives rise to

$$A^{(0,1)} \qquad\qquad B^{(0,1)}$$

$$\begin{array}{llllll} 1 \\ 0 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ 3 & 6 & 6 & 3 & 1 \\ 6 & 15 & 15 & 10 & 4 & 1 \\ & & \cdots \end{array} \qquad \begin{array}{llllll} 1 \\ 1 & 1 \\ 3 & 2 & 1 \\ 7 & 6 & 3 & 1 \\ 19 & 16 & 10 & 4 & 1 \\ 51 & 45 & 30 & 15 & 5 & 1 \\ & & \cdots \end{array}$$

Since by Proposition 2, $B^{(\sigma+1)}=A^{(\sigma+1)}E=PA^{(\sigma)}E=PB^{(\sigma)}$ we obtain:

PROPOSITION 4. *Let $A^{(\sigma)}$ be an admissible matrix, $B^{(\sigma)}$ its sum matrix and P the binomial matrix. Then*

$$B^{(\sigma+1)}=PB^{(\sigma)}.$$

Now let us look again at the special case $\sigma=(a, s)$.

LEMMA 2. *Let $\sigma=(a, s)$, $A^{(a,s)}=(a_{n,k})$, and $B^{(a,s)}=(b_{n,k})$ the sum matrix. Then*

$$\begin{aligned} b_{n,k}&=b_{n-1,k-1}+sb_{n-1,k}+b_{n-1,k+1} \qquad (k\geqslant 1)\\ b_{n,0}&=(a+1)\,b_{n-1,0}+(s-a+1)\,b_{n-1,1}. \end{aligned} \tag{13}$$

Proof. We have for $k\geqslant 1$

$$\begin{aligned} b_{n,k}&=\sum_{i\geqslant k} a_{n,i}=\sum_{i\geqslant k}(a_{n-1,i-1}+sa_{n-1,i}+a_{n-1,i+1})\\ &=b_{n-1,k-1}+sb_{n-1,k}+b_{n-1,k+1}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} b_{n,0}&=\sum_{i\geqslant 0} a_{n,i}\\ &=\sum_{i\geqslant 1}(a_{n-1,i-1}+sa_{n-1,i}+a_{n-1,i+1})+(aa_{n-1,0}+a_{n-1,1})\\ &=b_{n-1,0}+sb_{n-1,1}+b_{n-1,2}+aa_{n-1,0}+a_{n-1,1}\\ &=(a+1)\,b_{n-1,0}+(s-a)\,b_{n-1,1}+b_{n-1,1}. \quad \blacksquare \end{aligned}$$

Let $S_k(x)$ be the generating function of the kth column of $B^{(a,s)}$. Then we find from (13) with $f(x)$ as in (4)

$$S_k(x) = f(x)^k\, S_0(x)$$
$$S_0(x) = x((a+1)\, S_0(x) + (s-a+1)\, f(x)\, S_0(x)) + 1,$$

and by substituting $f(x)$ we arrive at the following expression for the generating function $S^{(a,s)}(x) = S_0(x) = \sum_{n \geqslant 0} S_n^{(a,s)} x^n$ of the sum coefficients $S_n^{(a,s)} = b_{n,0}$.

Proposition 5. *Let $S^{(a,s)}(x)$ be the generating function of the sum coefficients $S_n^{(a,s)}$ associated with $\sigma = (a, s)$. Then*

$$S^{(a,s)}(x) = \frac{2}{(1-s+a)(1-(s+2)x) + (1+s-a)\sqrt{1-2sx+(s^2-4)\,x^2}}. \tag{14}$$

Examples. For $a = s$, we obtain by an easy computation

$$S^{(s,s)}(x) = C^{(s+1,s)}(x),$$

hence $S_n^{(s,s)} = C_n^{(s+1,s)}$.

For $a = s+1$, Eq. (14) gives

$$S^{(s+1,s)}(x) = \frac{1}{1-(s+2)x},$$

hence $S_n^{(s+1,s)} = (s+2)^n$.

For $a = s-1$, we obtain

$$S^{(s-1,s)}(x) = \frac{1}{\sqrt{1-2sx+(s^2-4)\,x^2}}.$$

This last expression leads to some interesting coefficients. For $s = 0$, we find $S^{(-1,0)}(x) = 1/\sqrt{1-4x^2} = B(x^2)$, thus

$$S_n^{(-1,0)} = \begin{cases} \dbinom{n}{n/2}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

With Proposition 4 we thus obtain

$$S_n^{(0,\,1)}=\sum_k\binom{n}{2k}\binom{2k}{k}=\sum_k\binom{n}{k}\binom{n-k}{k}.$$

For $s=2$, we find $S^{(1,\,2)}(x)=1/\sqrt{1-4x}=B(x)$, hence

$$S_n^{(1,\,2)}=\binom{2n}{n},$$

and this yields with Proposition 4 again

$$S_n^{(2,\,3)}=\sum_k\binom{n}{k}\binom{2k}{k}.$$

Applying Proposition 4 once again we arrive at some well-known binomial formulae such as, with $(-1,0)\rightarrow(2,3)$

$$\sum_k\binom{n}{k}\binom{2k}{k}=\sum_k\binom{n}{2k}\binom{2k}{k}3^{n-2k}.$$

There is no general relationship for the sum matrix $B^{(\sigma)}$ replacing σ by $-\sigma$ as in Proposition 2. However, there is one interesting instance which is immediately derived from (14):

$$S^{(-s-1,\,-s)}(x)=S^{(s-1,\,s)}(-x). \tag{15}$$

Applying Proposition 4 we thus find via $\sigma=(-s-1,\,-s)$, $t=2s$,

$$S_n^{(s-1,\,s)}=\sum_k(-1)^k\binom{n}{k}(2s)^{n-k}\,S_k^{(s-1,\,s)}. \tag{16}$$

For our examples, this yields

$$\sum_k\binom{n}{2k}\binom{2k}{k}=\sum_k(-1)^k\binom{n}{k}2^{n-k}\sum_i\binom{k}{2i}\binom{2i}{i}$$

$$\binom{2n}{n}=\sum_k(-1)^k\binom{n}{k}4^{n-k}\binom{2k}{k}$$

$$\sum_k\binom{n}{k}\binom{2k}{k}=\sum_k(-1)^k\binom{n}{k}6^{n-k}\sum_i\binom{k}{i}\binom{2i}{i}.$$

4. DETERMINANTS OF HANKEL MATRICES

Let us return to general admissible matrices $A^{(\sigma)} = (a_{n,k})$ associated with the sequence $\sigma = (s_0, s_1, s_2, ...)$, and let $C_n^{(\sigma)}$ be the Catalan-like numbers appearing in the 0th column. Our goal is to compute the determinants of the matrices

$$\tilde{A}_n = \begin{pmatrix} C_0^{(\sigma)} & C_1^{(\sigma)} & \cdots & C_n^{(\sigma)} \\ C_1^{(\sigma)} & C_2^{(\sigma)} & \cdots & C_{n+1}^{(\sigma)} \\ \vdots & & & \\ C_n^{(\sigma)} & C_{n+1}^{(\sigma)} & \cdots & C_{2n}^{(\sigma)} \end{pmatrix}, \qquad \tilde{B}_n = \begin{pmatrix} C_1^{(\sigma)} & C_2^{(\sigma)} & \cdots & C_n^{(\sigma)} \\ C_2^{(\sigma)} & C_3^{(\sigma)} & \cdots & C_{n+1}^{(\sigma)} \\ \vdots & & & \\ C_n^{(\sigma)} & C_{n+1}^{(\sigma)} & \cdots & C_{2n-1}^{(\sigma)} \end{pmatrix}.$$

PROPOSITION 6. *Let $C_n^{(\sigma)}$ be the Catalan-like numbers associated with σ. Then* det $\tilde{A}_n = 1$ *for all n.*

Proof. Let $A^{(\sigma)} = (a_{n,k})$ and A_n be the submatrix of $A^{(\sigma)}$ consisting of the rows and columns with index 0, 1, ..., n. Since $\tilde{A}_n$ has as (k, ℓ)-entry $C_{k+\ell}^{(\sigma)} = a_{k+\ell,0} = r_k \cdot r_\ell$, we find $\tilde{A}_n = A_n A_n^T$, and thus det $\tilde{A}_n = 1$, since A_n is lower triangular with diagonal 1. ∎

Of course, this anticipated result was the motivation for the definition of admissible matrices to begin with.

To compute det $\tilde{B}_n$ we have to do a little more work. The following approach which is nicer than the original proof was suggested by the referee. Let P_n, Q_n, J_n be the following $n \times n$-matrices:

$$P_n = \begin{pmatrix} a_{0,0} & 0 & & \cdots & 0 \\ a_{1,0} & a_{1,1} & 0 & \cdots & 0 \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots & 0 \\ & & \cdots & & \\ a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix},$$

$$Q_n = \begin{pmatrix} a_{1,0} & a_{1,1} & 0 & \cdots\cdots & 0 \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots\cdots & 0 \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} \quad \cdots & 0 \\ & & & \cdots & \\ a_{n,0} & a_{n,1} & a_{n,3} & \cdots & a_{n,n-1} \end{pmatrix},$$

$$J_n = \begin{pmatrix} s_0 & 1 & 0 & \cdots & 0 \\ 1 & s_1 & 1 & \cdots & 0 \\ 0 & 1 & s_2 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & s_{n-1} \end{pmatrix}.$$

Then, by $C_{k+\ell}^{(\sigma)}=r_k\cdot r_\ell$, we have

$$\widetilde{B}_n=Q_nP_n^T. \tag{17}$$

Note that (17) holds in spite of the fact that the last row of Q_n does not contain the last entry $a_{n,n}$ of r_n, since the missing $a_{n,n}$ does not affect the products $r_n\cdot r_j$ for $j=0, ..., n-1$.

Recursion (1) of Proposition 1 implies $Q_n=P_nJ_n$, whence we obtain

$$\widetilde{B}_n=P_nJ_nP_n^T,$$

and thus

$$\det \widetilde{B}_n=\det J_n \tag{18}$$

since $\det P_n=1$.

By expanding the determinant of J_n with respect to the last column we therefore find the following result on $\widetilde{B}_n$.

PROPOSITION 7. *Let $C_n^{(\sigma)}$ be the Catalan-like numbers associated with $\sigma=(\sigma_i)$. Then* $\det \widetilde{B}_n=d_n$ $(n\geqslant 1)$, *where d_n satisfies the recursion*

$$d_n=s_{n-1}d_{n-1}-d_{n-2}, \qquad d_0=1. \tag{19}$$

Taking all things together we can therefore state: The Catalan-like numbers $C_n^{(\sigma)}$ are the unique sequence of real numbers with $\det \widetilde{A}_n=1$ and $\det \widetilde{B}_n=d_n$ for all n, whenever $d_n\neq 0$ for all n.

EXAMPLES. Let us look at some of our main examples $\sigma=(a, s)$ discussed in the previous sections. First, all matrices $\widetilde{A}_n$ have determinant 1. For $a=1$, $s=2$, and $a=s=2$, respectively, we thus obtain our initial example concerning the Catalan numbers. Now let us consider the sequences $d_n=\det \widetilde{B}_n$ defined according to (19).

For $a=s=2$, we obtain $d_n=2d_{n-1}-d_{n-2}$, $d_0=1$, which yields $d_n=n+1$ by induction. Hence we find for the Catalan numbers

$$\det\begin{pmatrix} C_2 & C_3 & \cdots & C_{n+1} \\ C_3 & C_4 & \cdots & C_{n+2} \\ \vdots & & & \\ C_{n+1} & C_{n+2} & \cdots & C_{2n} \end{pmatrix}=n+1.$$

For $a=s=3$, we have $d_n=3d_{n-1}-d_{n-2}$, $d_0=1$, which yields $d_n=F_{2n+2}$ (the Fibonacci number), since $d_0=F_2=1$, $d_1=F_4=3$ and $F_{2n+2}=3F_{2n}-F_{2n-2}$. The hexagonal numbers H_n are thus the unique sequence with det $\tilde{A}_n=1$, det $\tilde{B}_n=F_{2n+2}$ for all n.

For $a=3$, $s=2$, this gives $d_1=3$, $d_2=5$, $d_n=2d_{n-1}-d_{n-2}$ from which $d_n=2n+1$ results. Hence the binomial numbers $\binom{2n+1}{n}$ are the unique sequence with det $\tilde{A}_n=1$, det $\tilde{B}_n=2n+1$.

Finally, for $a=2$, $s=3$, we compute $d_0=1$, $d_1=2$, $d_n=3d_{n-1}-d_{n-2}$, from which $d_n=F_{2n+1}$ results.

Remark. Consider the case when $\sigma=(s,s)$ is constant. It is known and easy to prove by induction that

$$\det\begin{pmatrix} 2\cos\vartheta & 1 & 0 & \cdots & 0 \\ 1 & 2\cos\vartheta & 1 & & \\ 0 & 1 & 2\cos\vartheta & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 1 & & 2\cos\vartheta \end{pmatrix}=\frac{\sin(n+1)\vartheta}{\sin\vartheta},$$

where n is the size of the determinant. Setting $s=2\cos\vartheta$, we therefore obtain

$$d_n=U_n\left(\frac{s}{2}\right), \tag{20}$$

where $U_n(x)=\sin(n+1)\vartheta/\sin\vartheta$, $\cos\vartheta=x$, is the Tchebychev polynomial of the second kind. Since $U_n(x/2)=\sum_k(-1)^k\binom{n-k}{k}x^{n-2k}$ (see [6]), we infer from (20)

$$d_n=\sum_k(-1)^k\binom{n-k}{k}s^{n-2k}. \tag{21}$$

Applying (21) to our examples, we obtain for $s=2$

$$n+1=\sum_k(-1)^k\binom{n-k}{k}2^{n-2k}$$

(a result of Coxeter, see [6, p. 76]), and for $s=3$ the curious formula

$$F_{2n+2}=\sum_k(-1)^k\binom{n-k}{k}3^{n-2k}.$$

In analogy we have

$$\det\begin{pmatrix} \cos\vartheta & 1 & 0 & \cdots & 0 \\ 1 & 2\cos\vartheta & 1 & \cdots & 0 \\ 0 & 1 & 2\cos\vartheta & \cdots & 0 \\ & & \cdots & \cdots & \\ 0 & 0 & & 1 & 2\cos\vartheta \end{pmatrix} = \cos n\vartheta$$

Setting $\cos\vartheta = a$ we thus obtain for the sequence $\sigma = (a, 2a)$

$$d_n = T_n(a),$$

where $T_n(x)$ is the Tchebychev polynomial of the first kind. In particular, for $a = 1$ (the Catalan numbers), we obtain again $d_n = 1$ for all n.

Let us, finally, take a look at the Hankel matrices $\tilde{A}_n$ for the sum coefficients. Let $\sigma = (a, s)$ and let as before $S_n^{(a,s)}$ be the coefficients of $S^{(a,s)}(x)$. The matrix $\tilde{A}_n$ is again defined as (suppressing (a, s))

$$\tilde{A}_n = \begin{pmatrix} S_0 & S_1 & \cdots & S_n \\ S_1 & S_2 & \cdots & S_{n+1} \\ \vdots & & & \\ S_n & S_{n+1} & \cdots & S_{2n} \end{pmatrix}.$$

By an analogous argument as the one leading up to Proposition 7, the following result can be shown.

PROPOSITION 8. *Let $S_n^{(a,s)}$ be the sum coefficients associated with the sequence $\sigma = (a, s)$. Then for all n,*

$$\det \tilde{A}_n = (s - a + 1)^n.$$

As an example, we obtain $\det \tilde{A}_n = 2^n$ whenever $a = s - 1$. Looking up our list in Subsection 3.5, we thus find, for example $(s = 2)$,

$$\det\begin{pmatrix} \binom{0}{0} & \binom{2}{1} & \cdots & \binom{2n}{n} \\ \vdots & & & \\ \binom{2n}{n} & & \cdots & \binom{4n}{2n} \end{pmatrix} = 2^n.$$

5. INTERPRETATIONS

In his book [8] R. Stanley lists a wealth of combinatorial settings which are counted by the Catalan numbers. In addition, he gives several instances counted by the Motzkin numbers, drawing mostly from the survey [2]. In this final section we make a few remarks as to what the Catalan-like numbers $C_n^{(\sigma)}$ and the sum numbers $S_n^{(\sigma)}$ count. There are three sources: The matrices $A^{(\sigma)}$ via the recursion (1), the binomial formulae in Propositions 2 and 3, and the recursions in Subsection 3.3, in particular (7), (8), (9).

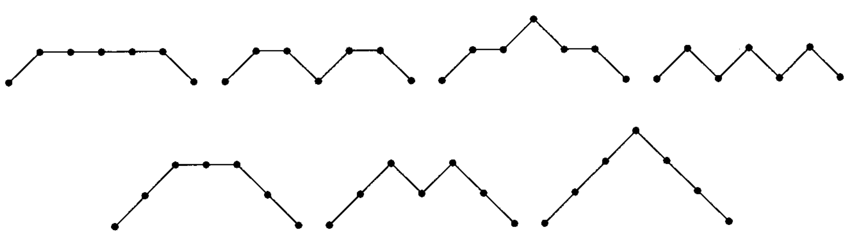
Lattice Paths

Consider $A^{(\sigma)}=(a_{n,k})$ with $\sigma=(s_0, s_1, s_2, ...)$ consisting of non-negative integers. By rotating $A^{(\sigma)}$ counterclockwise by 90° we place $a_{n,k}$ at the lattice point (n, k). Recursion (1) immediately implies the following result: Consider all paths starting from the origin $(0, 0)$ never falling below the x-axis with diagonal steps $(1, 1)$, $(1, -1)$ and s_i types of horizontal steps $(1, 0)$ on the line $y=i$. Then $a_{n,k}$ counts the number of these paths ending at (n, k).

For $\sigma=(0, 0)$ we therefore obtain the classical result that $C_{2n}^{(0,0)}=C_n$ counts the number of paths with n steps $(1, 1)$ and n steps $(1, -1)$ (so-called Dyck paths) starting at $(0, 0)$ and ending at $(2n, 0)$. Furthermore, $S_n^{(0,0)}=\binom{n}{\lfloor n/2 \rfloor}=\sum_{k=0}^{n} a_{n,k}$ implies that $\binom{n}{\lfloor n/2 \rfloor}$ is the number of paths with n diagonal steps. By extending these paths symmetrically (that is, if step i is $(1, 1)$, then step $2n+1-i$ is $(1, -1)$ and vice versa), we can express this result also in the following form: $\binom{n}{\lfloor n/2 \rfloor}$ counts the number of symmetric Dyck paths with $2n$ diagonal steps.

For $\sigma=(1, 1)$ we similarly obtain that the Motzkin number M_n counts the number of paths with n steps $(1, 1)$, $(1, -1)$ or $(1, 0)$ starting at $(0, 0)$ and ending at $(n, 0)$. As before, $S_n^{(1,1)}=C_n^{(2,1)}$ counts the number of symmetric Motzkin paths with $2n$ steps.

A final example: $C_n^{(0,1)}$ counts the number of paths with n steps $(1, 1)$, $(1, -1)$, $(1, 0)$ without horizontal edges at the x-axis, and $S_n^{(0,1)}$ counts the number of symmetric paths with $2n$ steps. For example, $S_3^{(0,1)}=7$ gives the paths:



B. *Paths along the Integers*

Consider paths on the non-negative integers. We call a path *admissible* if it starts and ends at 0 and never enters the negative integers. By interpreting a diagonal step $(1, 1)$ as 1 and $(1, -1)$ as -1, we find from (A) that $C_n^{(0,0)}$ counts the admissible paths with n steps 1 or -1. Using (A) or Proposition 2 this yields the well–known result (see [2]) that M_n counts the admissible paths with n steps 1, -1 or 0 ($=$loop). Similarly, $C_n^{(2,2)} = C_{n+1}$ counts these paths with n steps 1, -1 and two kinds of loops, $C_n^{(3,3)}$ the paths with n steps 1, -1 and three kinds of loops, and so on. There are analogous interpretations for general $C_n^{(\sigma)}$.

Let us now look at the sum coefficients $S_n^{(s-1,s)}$. We know

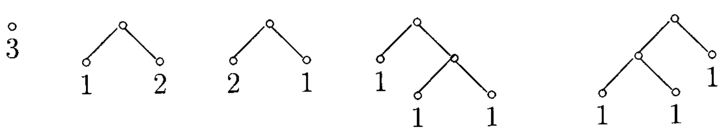
$$S_n^{(-1,0)} = \begin{cases} \dbinom{n}{n/2}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

Hence $S_n^{(-1,0)}$ counts the paths starting at 0, ending at n, with n steps 2 or 0. Applying Proposition 3, we find that $S_n^{(0,1)} = \sum_k \binom{n}{2k}\binom{2k}{k}$ counts the paths starting at 0, ending at n, with n steps 2, 1, or 0 (a result which goes back to Euler according to [7]). $S_n^{(1,2)} = \binom{2n}{n}$ is then the number of these paths with n steps 2, 1, 0, where the 1-steps may be carried out above or below the line.

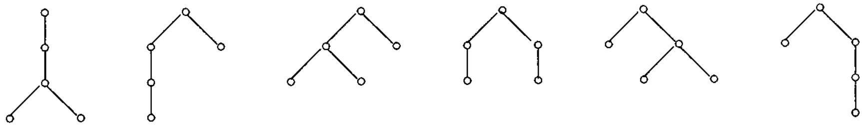
C. *Trees*

The recursion (7) or the lattice paths considered in (A) lead immediately to the counting of rooted binary trees. We have s possible directions when the out-degree is 1, and left/right for out–degree 2. Hence $M_n = C_n^{(1,1)}$ counts the rooted binary trees on n edges where we draw a single edge (out-degree 1) vertically (called Motzkin trees), $C_{n+1} = C_n^{(2,2)}$ counts all rooted binary trees on n edges. The hexagonal number $H_{n-1} = C_{n-1}^{(3,3)}$ count the binary trees on n edges with the single root edge at angles 30°, 90°, or 150°, the double root edges at 120°, and all other edges at angles 120° or 180°. This interpretation leads directly to the hexagonal animals as counted in [4] by regarding the vertices as the centers of the hexagons. A similar analysis leads to the enumeration of certain octagonal animals, and so on.

There is also a nice interpretation for $C_n^{(2,3)}$ which can be immediately derived from (9) or Proposition 2. $C_n^{(2,3)}$ counts the complete binary trees with positive labels at the leaves summing to $n+1$. Thus for $n=2$ we obtain the following $C_2^{(2,3)} = 5$ trees:



Finally, using (12) we find that $C_n^{(0,1)}$ counts the number of Motzkin trees on n vertices with an even number of vertices of out-degree 1 or 0 at the beginning. For example, we obtain for $n=5$ the $C_5^{(0,1)}=6$ trees:



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