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Remarks on Faber Polynomials

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Abstract

With the differential calculus on the Faber polynomials, we calculate the Faber polynomials for powers of inverse functions. We apply the same methods to obtain majoration of the derivatives of the Faber polynomials of a univalent function of the class Σ.

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1 Introduction

Let $g(z) = z + b_1 + b_2 z^{-1} + \cdots + b_n z^{1-n} + \cdots$, the Faber polynomials $\psi_n(t)$ of $g(z)$ are given by

$$\frac{zg'(z)}{g(z) - t} = 1 + \sum_{m=1}^{\infty} \psi_m(t)\, z^{-m} \tag{1}$$

Let $F_k(b_1, b_2, .., b_k) = \psi_k(0)$, $k \geq 1$, then $\psi_n(t) = F_n(b_1 - t, b_2, .., b_n)$. Let

$$\frac{z}{g(z) - t} = 1 + \sum_{n=1}^{+\infty} G_n(b_1 - t, b_2, \cdots, b_n)\, z^{-n} \tag{2}$$

If $h(z) = zg(\frac{1}{z})$, then $\frac{1}{h(z)} = \sum_{n \geq 0} G_n(b_1, b_2, \cdots, b_n) z^n$. We have

$$G_n(b_1 - t, b_2, \cdots, b_n) = \frac{1}{n+1}\, \psi'_{n+1}(t) \;=\; -\frac{1}{k+1}\, \frac{\partial F_{k+1}}{\partial b_1}(b_1 - t, b_2, \cdots, b_k) \tag{3}$$

$$h(z)^p = \sum_{n \geq 0} K_n^p(b_1, b_2, \cdots, b_n)\, z^n \qquad \forall p \ real \tag{4}$$

$$(b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots)^m = \sum_{s \geq 0} D_{m+s}^m(b_1, b_2, \cdots) z^{s+m} \quad \forall m \in N \tag{5}$$

Then $D^0_n = 0$ if $n > 0$ and $D^0_0 = 1$, $G_n = K^{-1}_n$ and $K^0_n = 0$ if $n > 0$. We have

$$\lim_{p\to 0} \frac{K^p_n}{p} = -\frac{F_n}{n} \tag{6}$$

We denote K^p_n for $K^p_n(b_1, b_2, .., b_n)$ and $(G_1, G_2, ..)$ for $(G_1(b_1), G_2(b_1, b_2), ..)$. With $h(z)^p \times h(z) = h(z)^{p+1}$, we see that K^p_n satisfies the recurrence

$$K^p_n + b_1 K^p_{n-1} + b_2 K^p_{n-2} + \cdots + b_{n-1} K^p_1 + b_n = K^{p+1}_n \tag{7}$$

We have, see [3], $F_n(G_1, G_2, \cdots, G_n) = -F_n(b_1, b_2, \cdots, b_n)$,

$$K^{-p}_n(G_1, G_2, \cdots, G_n) = K^p_n(b_1, b_2, \cdots, b_n) \tag{8}$$

and see [9], [8] and [1],

$$\frac{1}{n} F_n(b_1, b_2, \cdots, b_n) = \sum_{m=1}^{n} \frac{(-1)^m}{m} D^m_n(b_1, b_2, \cdots, b_{n-m+1}) \tag{9}$$

The derivatives of $F_n(b_1, b_2, \cdots, b_n)$ with respect to any of the variables b_j are given in [3]. With respect to b_1,

$$\frac{1}{n} \frac{\partial^k}{\partial b_1^k} F_n(b_1, b_2, \cdots, b_n) = (-1)^k (k-1)!\, K^{-k}_{n-k}(b_1, \cdots, b_{n-k}) \tag{10}$$

$$= (-1)^k (k-1)!\, K^k_{n-k}(G_1, \cdots, G_{n-k}) = (-1)^k (k-1)!\, D^k_n(1, G_1, \cdots, G_{n-k})$$

Denote $C^p_m = \dfrac{p(p-1)\cdots(p-m+1)}{m!}$, then for $p \geq 1$, p, $q \in N$,

$$(-1)^q C^{-p}_q = \sum_{k=1}^{q} C^p_k \frac{(q-1)!}{(q-k)!(k-1)!} \tag{11}$$

Then $D^m_n(b_1 - t, b_2, .., b_{n-m+1}) = \sum_{k=0}^{m-1} C^m_k (-1)^k D^{m-k}_{n-k}(b_1, b_2, .., b_{n-m+1}) t^k$ for $1 \leq m < n$ and $D^n_n = b^n_1$.

2 The Faber polynomials of the inverse function

Let $g(z)$ as in (1) and denote $g^{-1}(z)$ the inverse function, $(g\,o\,g^{-1} = Identity)$, then, see [2, (I.2.7)-(I.2.8)] and [3, proposition 6.2],

$$g^{-1}(z) = z - b_1 - \sum_{n\geq 1} \frac{1}{n} K^n_{n+1} \frac{1}{z^n} \tag{12}$$

$$\frac{1}{[g^{-1}(u)]^p} = \frac{1}{u^p} + \sum_{n\geq 1} \frac{p}{n+p} K_n^{n+p}(b_1, b_2, \cdots, b_n) \frac{1}{u^{n+p}} \tag{13}$$

This expansion is valid for any real number p with the convention that when $n+p=0$, we replace the coefficient $\dfrac{p}{n+p} K_n^{n+p}$ by F_n, this is in accordance with (6). This shows that for any $p \in R$,

$$K_n^{-p}(\,-b_1, -K_2^1, -\frac{1}{2}K_3^2, \cdots, -\frac{1}{n-1}K_n^{n-1}\,) \;=\; \frac{p}{n+p} K_n^{n+p}(b_1, b_2, \cdots) \tag{14}$$

where in the left hand side, K_n^p means $K_n^p(b_1, b_2, \cdots, b_n)$. Making $p \to 0$ as in (6), we deduce from (14),

$$F_n(\,-b_1, -K_2^1, -\frac{1}{2}K_3^2, \cdots, -\frac{1}{n-1}K_n^{n-1}\,) \;=\; K_n^n(b_1, b_2, \cdots, b_n) \tag{15}$$

The Faber polynomials $\phi_n(t)$ in terms of the (b_n), for the inverse function g^{-1} are well known, they are the Taylor part in the Laurent expansion of $g(z)^n$, see for example the proof of Lemma 1.3 in [8]. We give a different proof of this fact: In (15), we have the value $\phi_n(0)$ at $t=0$. We do a Taylor expansion of

$$\phi_n(t) = F_n(-b_1 - t,\, -K_2^1(b_1, b_2),\, -\frac{1}{2}K_3^2(b_1, b_2, b_3), \cdots) \tag{16}$$

Using (14) at the last step, we obtain

$$\frac{\phi_n^{(k)}(0)}{k!} = \frac{(-1)^k}{k!} \frac{\partial^k F_n}{\partial b_1^k}(-b_1,\, -K_2^1(b_1, b_2),\, -\frac{1}{2}K_3^2(b_1, b_2, b_3), \cdots) \tag{17}$$

$$= \frac{n}{k} K_{n-k}^{-k}\,(\,-b_1,\, -K_2^1(b_1, b_2),\, -\frac{1}{2}K_3^2(b_1, b_2, b_3), \cdots) \;=\; K_{n-k}^n(b_1, b_2, \cdots)$$

We obtain

$$\phi_n(t) = K_n^n(b_1, b_2, \cdots, b_n) \;+\; \sum_{k=1}^{n} K_{n-k}^n(b_1, b_2, \cdots, b_{n-k}) t^k \tag{18}$$

which is the Taylor part of $g(t)^n$. With the same method,

Theorem 2.1 *The Faber polynomials $\phi_n(t)$ of $\dfrac{z^{p+1}}{[g^{-1}(z)]^p}$ are given in terms of the (b_n) by*

$$\phi_n(t) = \,-p \sum_{k=0}^{n} \frac{n}{n-k(1+p)} K_{n-k}^{n-k(1+p)}(b_1, b_2, \cdots, b_{n-k})\, t^k \tag{19}$$

with the convention $\dfrac{n}{n-(p+1)k} K_{n-k}^{n-(p+1)k} \;=\; -\,\dfrac{(1+p)}{p}\; F_{kp}$ *if* $n = (p+1)k$.

Proof. We give the proof for $p = 1$. Then $\phi_n(t) = -\sum_{k=0}^{n} \frac{n}{n-2k} K_{n-k}^{n-2k} t^k$ with the convention $\frac{n}{n-2k} K_{n-k}^{n-2k} = -2F_k$ if $n = 2k$. With (13),

$$K_n^p(\, b_1,\, \frac{1}{3}K_2^3(b_1, b_2), \cdots, \frac{1}{n+1}K_n^{n+1}(b_1, b_2, \cdots, b_n)\,) = \frac{p}{n+p}K_n^{n+p} \tag{20}$$

With (6), dividing by p and making $p \to 0$,

$$\phi_n(0) = F_n(b_1, \frac{1}{3}K_2^3(b_1, b_2), .., \frac{1}{n+1}K_n^{n+1}(b_1, b_2, .., b_n)) = -K_n^n(b_1, b_2, .., b_n)$$

Then with (10), we apply Taylor formula to,

$$\phi_n(t) = F_n(\, b_1 - t,\, \frac{1}{3}K_2^3(b_1, b_2), \cdots, \frac{1}{n+1}K_n^{n+1}(b_1, b_2, \cdots, b_n)\,) \quad \bullet$$

Similarly, let $f(z) = z + \sum_{n\geq 1} b_n z^{n+1}$ and the inverse $f^{-1}(z)$, $(fof^{-1} = Id)$, then for any real p, see [2, (I.2.6)-(I.2.9)],

$$[f^{-1}(z)]^p = z^p[1 + \sum_{n\geq 1} \frac{p}{n+p}K_n^{-(n+p)}z^n] \tag{21}$$

with the convention that we replace $\frac{p}{p-n}K_n^{p-n}$ in (21) by $-F_n$ if $p = n$. In particular

$$\frac{z}{f^{-1}(z)} = 1 + b_1 z - \sum_{n\geq 2} \frac{1}{n-1}K_n^{1-n}z^n \tag{22}$$

Theorem 2.2 *Let $f(z) = z + \sum_{n\geq 1} b_n z^{n+1}$, the Faber polynomials $\phi_n(t)$ of $z^{1+p}\,[f^{-1}(\frac{1}{z})]^p$ are given in terms of the (b_n) by*

$$\phi_n(t) = -p \sum_{k=0}^{n} \frac{n}{n-k(1+p)}\, K_{n-k}^{-n+k(1+p)}(b_1, b_2, \cdots, b_{n-k})t^k \tag{23}$$

Proof. With (21), $K_n^q(-pb_1, .., \frac{p}{n+p}K_n^{-(n+p)}) = \frac{pq}{pq+n}K_n^{-pq-n}(b_1, b_2, .., b_n)$ if $n + pq \neq n$, otherwise we use (6). As in (6), we divide by q and make $q \to 0$, $F_n(-pb_1, \cdots, \frac{p}{n+p}K_n^{-(n+p)}) = -pK_n^{-n}(b_1, b_2, .., b_n)$. We have $\phi_n(t) = F_n(-pb_1 - t, .., \frac{p}{n+p}K_n^{-(n+p)})$. We calculate its coefficients $\frac{\phi_n^k(0)}{k!}$ with (10) $\bullet$
For $p = -1$, $\phi_n(t) = \sum_{k=0}^{n} K_{n-k}^{-n}(b_1, b_2, \cdots, b_{n-k})t^k$ and similarly to (18), this case can be obtained directly since the inverse of $g(z) = \frac{1}{f^{-1}(\frac{1}{z})}$ is $g^{-1}(u) = \frac{1}{f(\frac{1}{u})}$. A different proof of (19) and (23) would consist in making a change of variables in Cauchy integral formula. A succession of manipulations on $f(z)$ and $g(z)$ as raising to the power and taking the inverse as in Theorems 2.1 and 2.2 brings always series with coefficients in the class of polynomials K_n^p.

3 Majoration of the derivatives of the Faber polynomials for $g(z) = \dfrac{1}{f(\frac{1}{z})}$

Theorem 3.1 *Let $f(z) = z + \sum_{n\geq 1} b_n z^{n+1}$. The n^{th} Faber polynomial of $g(z) = z^{1+p}[f(\frac{1}{z})]^p$ is $\phi_n(t) = pF_n(b_1, b_2, .., b_n) + \sum_{k=1}^{n} \frac{n}{k} K_{n-k}^{-pk}(b_1, b_2, .., b_{n-k})t^k$. In particular for $p = -1$, see [9, (30)], let $\phi_n(t)$ be the n^{th} Faber polynomial of $g(z) = [f(\frac{1}{z})]^{-1}$, then*

$$\frac{1}{n}\phi_n(t) = -\frac{1}{n}F_n(b_1, b_2, \cdots, b_n) + \sum_{k=1}^{n} \frac{t^k}{k} D_n^k(1, b_1, b_2, \cdots, b_{n-k}) \tag{24}$$

Proof. We have $g(z) = z + pb_1 + \dfrac{K_2^p}{z} + \cdots + \dfrac{K_{n+1}^p}{z^n} + \cdots$. Because of (1),

$$\phi_n(t) = F_n(pb_1 - t, K_2^p, \cdots, K_n^p) \tag{25}$$

By Taylor's formula, $\phi_n(t) = \phi_n(0) + \sum_{k=1}^{n} \dfrac{\phi^{(k)}(0)}{k!} t^k$. For $k \geq 1$, the k^{th} derivative is

$$\phi_n^{(k)}(t) = (-1)^k(\frac{\partial^k}{\partial b_1^k}F_n)(pb_1 - t, K_2^p, .., K_n^p) = (k-1)! n K_{n-k}^{-k}(pb_1 - t, K_2^p, .., K_{n-k}^p)$$

Thus $\dfrac{\phi_n^{(k)}(0)}{k!} = \frac{n}{k} K_{n-k}^{-pk}(b_1, b_2, .., b_{n-k})$. When $p = -1$, we obtain (24) •
Our proof uses differential calculus whereas the proofs in [9, 10] are combinatory. Remark that (24) is also an immediate consequence of (21) since (24) is the Taylor part of $\frac{1}{n}[f^{-1}(\frac{1}{z})]^{-n}$. See (18). When $p = -1$, the derivatives $\phi_n^{(k)}$ are given by

$$\frac{1}{n}\phi_n^{(k)}(t) = (k-1)!\, K_{n-k}^{-k}(G_1 - t, G_2, \cdots, G_{n-k}) = \sum_{s=0}^{n-k} \frac{k!}{s+k} D_n^{s+k} t^s \tag{26}$$

In [9], when $g(z) = \dfrac{1}{f(\frac{1}{z})}$ and $f(z) = z + b_1 z^2 + b_2 z^3 + ..$ is univalent in $|z| \leq 1$, the positivity of the coefficients of the polynomials D_n^k and De Branges theorem $|b_n| \leq n + 1$ permit to obtain a majoration of the first derivative $\phi_n'(t)$ for $|t| \leq 1$, the Koebe function is extremal. Because of (26), we can apply the same method, see Theorem 3.3, to all the derivatives $\phi_n^{(k)}(t)$ and the Koebe function is again extremal.
Let $f(z) = \dfrac{z}{(1-z)^2}$ be the Koebe function and $g(z) = \dfrac{1}{f(\frac{1}{z})} = z - 2 + \frac{1}{z}$.
We denote $x_1(t)$ and $x_2(t)$ the two roots of

$$x^2 - (2 + t)\, x + 1 = 0 \tag{27}$$

As in section **4**, the Faber polynomials of $g(z)$ are the Tchebicheff polynomials, see for example [11]. For $n \geq 1$,

$$\pi_n(t) = x_1(t)^n + x_2(t)^n = F_n(\,-(2+t),\, 1,\, 0, \cdots, 0,\,) \tag{28}$$

$$\pi_n(t) = \frac{1}{2^{2n-1}} \sum_{k=0}^{n} C_{2k}^{2n}\, t^k\, (4+t)^{n-k} = (\frac{2+t+\sqrt{4t+t^2}}{2})^n + (\frac{2+t-\sqrt{4t+t^2}}{2})^n$$

$$\pi_1(t) = 2+t,\ \pi_2(t) = 2+4t+t^2 \ =\ \pi_1(v),\ \pi_3(t) = (2+t)(1+v),\ \cdots \text{ with }$$
$$v = 4t+t^2$$

If $x(t)$ is a root of (27), then $x(t)^4 \,-\, (2+v)x(t)^2+1 = 0$; for $y(v) = x(t)^2$, it gives $\pi_n(v) = y_1(v)^n + y_2(v)^n = x_1(t)^{2n} + x_2(t)^{2n} = \pi_{2n}(t)$. Thus

$$\pi_{2n}(t) = 2^{1-2n} \sum_{k=0}^{n} C_{2k}^{2n}(2+t)^{2k}(4t+t^2)^{n-k} \ =\ 2^{1-2n} \sum_{k=0}^{n} C_{2k}^{2n}(4+v)^k v^{n-k}$$

This proves the validity of the expression of $\pi_n(t)$ following (28). We also have

$$\pi_n(t) =\ 2\,T_n(\frac{t+2}{2}) \tag{29}$$

where T_n is the n^{th} Tchebicheff polynomial, see section **4**. From (27),

$$\forall n \geq 3, \quad \pi_n(t) = (2+t)\pi_{n-1}(t) - \pi_{n-2}(t) \tag{30}$$

Since the polynomial $\pi_n(t)$ has positive coefficients, its maximum value on $|t| \leq 1$ is at $t = 1$. Since $\pi_1(1) = 3$, $\pi_2(1) = 7$, we obtain $\pi_n(1)$ from (30). For the derivatives $\pi_n^{(k)}(t)$, we have $\pi_1'(t) = 1$, $\pi_2'(t) =\ 2(2+t)$,

$$\frac{1}{n}\pi_n'(t) =\ \frac{1}{\sqrt{4t+t^2}}[\,(\frac{2+t+\sqrt{4t+t^2}}{2})^n \,-\, (\frac{2+t-\sqrt{4t+t^2}}{2})^n\,] \tag{31}$$

$$\frac{1}{n}\pi_n''(t) = \frac{n}{4t+t^2}\,\pi_n(t) \,-\, \frac{2+t}{4t+t^2}\,\frac{\pi_n'(t)}{n} \tag{32}$$

$$\frac{1}{2p+1}\pi_{2p+1}'(t) = \frac{1}{2^{2p}} \sum_{j=0}^{p} C_{2j+1}^{2p+1}(4t+t^2)^j(2+t)^{2(p-j)} \tag{33}$$

and a similar expression for $\frac{1}{2p}\pi_{2p}'(t)$. Remark that (32) is a particular case of Theorem 9.1 in [3]. On the other hand, (28)-(33) give interesting relations if we put $2+t = \cos(\theta)$ as in section 4, or $2+t = \cosh\theta$.

Theorem 3.2 $\max_{|t|\leq 1} |\pi_n^{(k)}(t)| \ =\ \pi_n^{(k)}(1)$ *for* $k \geq 1$. *Moreover* $a_n(k) = \frac{1}{n}\,\pi_n^{(k)}(1) \ =\ (k-1)!K_{n-k}^{-k}(-3,1,0,\cdots,0)$ *are the coefficients of the series*

$$\frac{z^k}{(1-3z+z^2)^k} = z^k \,+\, \frac{1}{(k-1)!} \sum_{n\geq k+1} a_n(k)z^n \tag{34}$$

Theorem 3.3 *Let* $f(z) = z + b_1 z^2 + \cdots$ *be univalent in the disc* $|z| \leq 1$ *and* $g(z) = \frac{1}{f(\frac{1}{z})}$ *as in Theorem 3.1. For* $|t| \leq 1$, $k \geq 1$, *we have* $|\phi_n^{(k)}(t)| \leq \pi_n^{(k)}(1)$.

4 Faber polynomials as symmetric functions of the roots and the Tchebicheff polynomials

The Faber polynomials $\psi_n(t)$ introduced in [5] for expansions of analytic functions and studied by P. Montel [7] can be obtained by Schiffer's elimination procedure [6]. A recent point of view, see [4], [8], is to consider the $\psi_n(t)$ as symmetric functions of the roots of an algebraic equation. Let $g(z)$ as in (1) and

$$Q_m(\xi,t) = \xi^m + (b_1 - t)\,\xi^{m-1} + \cdots + b_m = \prod_{k=1}^{m}(\xi - x_k(t)) \tag{35}$$

where $x_1(t)$, $x_2(t)$, $\cdots$, $x_m(t)$ are the roots of $Q_m(\xi,t)$ and let $\pi_j(t) = x_1(t)^j + x_2(t)^j + \cdots + x_m^j(t)$ be the symmetric polynomial of the roots of Q_m, it satisfies

$$\pi_m + (b_1 - t)\pi_{m-1} + \cdots + b_{m-1}\pi_1 + m b_m = 0 \tag{36}$$

From (1), the relation (36) is also valid for ψ_m. Compare (36) with (6)-(7).

Theorem 4.1 *We have* $\psi_n(t) = \pi_n(t)$

Using De Moivre formula, we interpret Theorem 4.1 with the Tchebicheff polynomials $T_n(x)$ and $U_n(x)$ where

$$T_n(\cos\theta) = \cos(n\theta) \quad \textit{and} \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

It is well known that $(x^2-1)U_n'(x) = (n+1)T_{n+1}(x) - xU_n(x)$,

$$T_n(x) = U_n(x) - xU_{n-1}(x) \quad \textit{and} \quad (1-x^2)U_{n-1}(x) = xT_n(x) - T_{n+1}(x)$$

$$T_n'(x) = nU_{n-1}(x) \tag{37}$$

Theorem 4.2 *Let* $g(z) = z + b_1 + \frac{1}{z}$ *and* $(\psi_m(t))_{m\geq 1}$ *the Faber polynomials associated to* $g(z)$. *Then*

$$\psi_m(t) = 2T_m\left(\frac{t-b_1}{2}\right) = 2\cos(m\theta) \quad \textit{with} \quad 2\cos(\theta) = t - b_1 \tag{38}$$

Proof. Let $Q_m(\xi,t) = \xi^m + (b_1-t)\xi^{m-1} + \xi^{m-2}$. It has $m-2$ zero roots and the two others non zero roots are $x_1(t) = e^{i\theta}$ and $x_2(t) = e^{-i\theta}$ with $2\cos\theta = t - b_1$. From Theorem 4.1, $\psi_m(t) = x_1(t)^m + x_2(t)^m = e^{im\theta} + e^{-im\theta} = 2T_m(\cos\theta)$ •

Theorem 4.3 *Let* $g(z) = z + b_1 + \frac{1}{z}$ *and* $G_n(t) = G_n(b_1 - t, 1, 0, \cdots, 0)$ *defined by* $\frac{z}{g(z)-t} = 1 + \sum_{m=1}^{+\infty} G_m(t) z^{-m}$, *then*

$$G_m(t) = \frac{\sin((m+1)\theta)}{\sin\theta} \quad \textit{with} \quad 2\cos\theta = t - b_1 \tag{39}$$

Proof. From (38), $G_m(t) = \dfrac{1}{m+1}\psi'_{m+1}(t) = \dfrac{2}{n+1}\, T'_{m+1}(\dfrac{t-b_1}{2}) \times \dfrac{1}{2}$. Using (37), we get $G_m(t) = U_m(\dfrac{t-b_1}{2})$, thus (39).

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