

Differential calculus on the Faber polynomials

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Abstract

The Faber polynomials are presented as a coordinate system to study the geometry of the manifold of coefficients of univalent functions.

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Résumé

Les polynômes de Faber sont présentés comme un système de coordonnées pour étudier la géométrie de la variété des coefficients des fonctions univalentes.

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1. Introduction

We show how the methods introduced in [2] and [3] allow to do differential calculus on the manifold of coefficients of univalent functions. The Faber polynomials $(F_k)_{k \geq 1}$ are given by the identity [5,12]

$$1 + b_1 w + b_2 w^2 + \dots + b_k w^k + \dots = \exp\left(-\sum_{k=1}^{+\infty} \frac{F_k(b_1, b_2, \dots, b_k)}{k} w^k\right). \quad (1.1)$$

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The polynomials $(G_m)_{m \geq 1}$ and $(K_n^p)_{n \geq 1}$, $p \in \mathbb{Z}$ are given by

$$\frac{1}{1 + b_1 w + b_2 w^2 + \dots + b_k w^k + \dots} = 1 + \sum_{m=1}^{+\infty} G_m(b_1, b_2, \dots, b_m) w^m, \tag{1.2}$$

$$(1 + b_1 w + b_2 w^2 + \dots + b_k w^k + \dots)^p = 1 + \sum_{n \geq 1} K_n^p(b_1, b_2, \dots, b_n) w^n, \tag{1.3}$$

then $G_m = K_m^{-1}$ and $K_m^1 = b_m$. Important polynomials are also the $(P_n^k)_{n \geq 2}$, see [1, (A.1.7)]. If $f(z) = zh(z)$,

$$\left(\frac{zf'(z)}{f(z)} \right)^2 [h(z)]^k = \sum_{n \geq 2} P_n^{n+k} z^n \quad \text{for } k \in \mathbb{Z}. \tag{1.4}$$

The polynomials $(F_n)_{n \geq 0}$, $(G_n)_{n \geq 0}$, $(K_n^p)_{n \geq 0}$, $(P_n^k)_{n \geq 2}$ are homogeneous of degree n in the variables (b_1, b_2, \dots) where b_k has weight k . As in [1–3,7] let the function of the infinite number of variables

$$(b_1, b_2, \dots, b_k, \dots) \rightarrow h(z) = 1 + b_1 z + b_2 z^2 + \dots + b_k z^k + \dots.$$

On the infinite dimensional manifold of coefficients $\mathcal{M} = \{(b_1, b_2, \dots, b_k, \dots)\}$ of univalent functions, consider the operators [3],

$$W_j = -\frac{\partial}{\partial b_j} - b_1 \frac{\partial}{\partial b_{j+1}} - \dots - b_k \frac{\partial}{\partial b_{j+k}} - \dots \quad \text{for } j \geq 1. \tag{1.5}$$

For $j \geq 1$, it holds $\frac{\partial}{\partial b_j}[h(z)] = z^j$ and $W_j[h(z)] = -z^j h(z)$. We have (see [3])

$$W_j(F_m) = m \delta_{jm} \quad \text{and} \quad W_j(G_m) = G_{m-j} \times 1_{m \geq j}, \tag{1.6}$$

$$\frac{\partial}{\partial b_p} W_j - W_j \frac{\partial}{\partial b_p} = -\frac{\partial}{\partial b_{j+p}} \quad \text{and} \quad W_p W_q = W_q W_p, \tag{1.7}$$

$$W_p W_q + W_{p+q} = \sum_{k \geq 0} \sum_{m \geq 0} b_k b_m \frac{\partial^2}{\partial b_{q+m} \partial b_{k+p}}. \tag{1.8}$$

For $k \in \mathbb{Z}$, let

$$V_j^k = -\sum_{n \geq 0} K_n^{k+1} \frac{\partial}{\partial b_{n+j}}. \tag{1.9}$$

We consider for $j \geq 1$ and $a \in \mathbb{Z}$, the operators $(V_j^{aj})_{j \geq 1}$ and for $a = 1$, we put

$$V_j = -\sum_{n \geq 0} K_n^{j+1} \frac{\partial}{\partial b_{n+j}}. \tag{1.10}$$

Then $W_j = V_j^0$, $V_j^{-1} = -\frac{\partial}{\partial b_j}$, $V_j = V_j^j$, $j \geq 1$ and

$$V_j^k V_p^s - V_p^s V_j^k = (k - s) V_{j+p}^{k+s} \quad \text{for } p \geq 1, j \geq 1. \tag{1.11}$$

The differential operators $(V_j^k)_{j \geq 1}$, $k \in \mathbb{Z}$ form an algebra and for $a \in \mathbb{Z}$, the set of $(V_j^{aj})_{j \geq 1}$ is a subalgebra since

$$V_j^{aj} V_p^{ap} - V_p^{ap} V_j^{aj} = a(j - p) V_{j+p}^{a(j+p)}. \tag{1.12}$$

Let $f(z) = zh(z)$. For $j \geq 1$, the vector field V_j is the image through the map $f \rightarrow f^{-1}$ of the Kirillov operator

$$L_j = \frac{\partial}{\partial b_j} + \sum_{n \geq 1} (n+1)b_n \frac{\partial}{\partial b_{n+j}}. \tag{1.13}$$

Let x_1, x_2, \dots, x_n , be the roots of $\xi^n + b_1\xi^{n-1} + b_2\xi^{n-2} + \dots + b_{n-1}\xi + b_n = 0$ and consider Newton symmetric functions $\pi_k = x_1^k + x_2^k + \dots + x_n^k, k \geq 1$, it was proved in [3] that

$$\pi_k(b_1, b_2, \dots, b_n) = F_k(b_1, b_2, \dots, b_n) \quad \text{for } k \leq n, \tag{1.14}$$

where $(F_k)_{k \geq 1}$ are the Faber polynomials. This is a consequence of

$$\log(1 + b_1w + b_2w^2 + \dots + b_kw^k + \dots) = - \sum_{k=1}^{+\infty} \frac{F_k(b_1, b_2, \dots, b_k)}{k} w^k \tag{1.15}$$

or equivalently (1.1). With this identification, the exact coefficients of the polynomial $F_k(b_1, b_2, \dots, b_k), k \geq 1$, have been calculated in [3]. The polynomials $(F_k)_{k \geq 1}$ are completely determined as homogeneous polynomial solutions of the system of partial differential equations (1.6) involving $(W_j)_{j \geq 1}$ (See [3]). The exact coefficients of the polynomials (G_n) and of all the (K_n^p) have been given in [3].

The object of this note is to prove that the polynomials (K_n^p) are all obtained as partial derivatives of the Faber polynomials and show how some of the recursion formulae on the polynomials are related to elementary differential calculus on \mathcal{M} . This is a step towards the classification of Faber type polynomials (see [8]). In the last section, we give the example of the conformal map from the exterior of the unit disk onto the exterior of $[-2, +2]$. This shows how to introduce non trivial second order differential operators on the manifold \mathcal{M} .

Main Theorem. *We have for $n \geq 1, k \geq 1$,*

$$\frac{\partial F_n}{\partial b_k} = -nG_{n-k} \times 1_{n \geq k}, \tag{1.16}$$

$$\frac{\partial}{\partial b_k} G_n = \frac{\partial}{\partial b_k} K_n^{-1} = -K_{n-k}^{-2} \times 1_{k \leq n}, \quad \frac{\partial}{\partial b_k} K_n^p = pK_{n-k}^{p-1} \times 1_{k \leq n},$$

$$\frac{\partial^2 F_j}{\partial b_k \partial b_p} = jK_{j-(p+k)}^{-2} \times 1_{j \geq k+p}, \quad \frac{\partial^3 F_j}{\partial b_r \partial b_k \partial b_p} = -2jK_{j-(p+k+r)}^{-3} \times 1_{j \geq k+p+r}.$$

For $j \geq k_1 + k_2 + \dots + k_s, k_1 \geq 1, \dots, k_s \geq 1$ and $s \geq 1$,

$$\frac{\partial^s F_j}{\partial b_{k_1} \partial b_{k_2} \dots \partial b_{k_s}} = (-1)^s (s-1)! j K_{j-(k_1+k_2+\dots+k_s)}^{-s}. \tag{T1}$$

Moreover for $n \geq 1, k \geq 1$,

$$\begin{aligned} &K_n^k(b_1, b_2, \dots, b_n) \\ &= K_n^{-k}(G_1(b_1), G_2(b_1, b_2), \dots, G_j(b_1, b_2, \dots, b_j), \dots, G_n(b_1, b_2, \dots, b_n)). \end{aligned} \tag{T2}$$

In the notation, all functions are functions of $(b_1, b_2, \dots, b_n, \dots)$.

The first (F_n) for $1 \leq n \leq 11$, as well as the first (G_n) are shown in [3]. The $(K_n^p), p \in \mathbb{Z}, p \neq 0$ and $n \leq 5$ are in [2]. In [3], an exact expression of the coefficients of all the polynomials (K_n^p) has been given. We explicit the first K_n^p ,

$$\begin{aligned}
K_1^2 &= 2b_1, & K_5^2 &= 2b_5 + 2b_1b_4 + 2b_2b_3, \\
K_2^2 &= 2b_2 + b_1^2, & K_6^2 &= 2b_5b_1 + 2b_4b_2 + b_3^2 + 2b_6, \\
K_3^2 &= 2b_3 + 2b_1b_2, & K_7^2 &= 2b_5b_2 + 2b_7 + 2b_4b_3 + 2b_1b_6, \\
K_4^2 &= 2b_4 + 2b_1b_3 + b_2^2, & K_8^2 &= 2b_7b_1 + 2b_5b_3 + 2b_2b_6 + 2b_8 + b_4^2, \\
K_1^3 &= 3b_1, & K_3^3 &= 6b_1b_2 + b_1^3 + 3b_3, \\
K_2^3 &= 3b_1^2 + 3b_2, & K_4^3 &= 6b_1b_3 + 3b_1^2b_2 + 3b_4 + 3b_2^2, \\
K_5^3 &= 6b_1b_4 + 3b_1^2b_3 + 3b_1b_2^2 + 3b_5 + 6b_2b_3, \\
K_6^3 &= 6b_5b_1 + 6b_4b_2 + 3b_3^2 + 3b_6 + 3b_1^2b_4 + 6b_1b_2b_3 + b_2^3, \\
K_7^3 &= 6b_5b_2 + 3b_5b_1^2 + 3b_7 + 6b_4b_3 + 6b_1b_6 + 6b_1b_4b_2 + 3b_1b_3^2 + 3b_2^2b_3, \\
K_8^3 &= 6b_7b_1 + 6b_1b_5b_2 + 6b_1b_4b_3 + 3b_1^2b_6 + 6b_5b_3 + 3b_4b_2^2 \\
&\quad + 3b_2b_3^2 + 6b_2b_6 + 3b_8 + 3b_4^2, \\
K_1^4 &= 4b_1, & K_3^4 &= 12b_1b_2 + 4b_1^3 + 4b_3, \\
K_2^4 &= 6b_1^2 + 4b_2, & K_4^4 &= 12b_1b_3 + 12b_1^2b_2 + 6b_2^2 + b_1^4 + 4b_4, \\
K_5^4 &= 12b_1b_4 + 12b_3b_1^2 + 12b_1b_2^2 + 12b_2b_3 + 4b_2b_1^3 + 4b_5, \\
K_6^4 &= 12b_1b_5 + 12b_1^2b_4 + 24b_1b_2b_3 + 12b_2b_4 + 6b_3^2 + 4b_1^3b_3 + 4b_2^3 + 6b_1^2b_2^2 + 4b_6, \\
K_1^{-2} &= -2b_1, & K_3^{-2} &= -2b_3 + 6b_1b_2 - 4b_1^3, \\
K_2^{-2} &= 3b_1^2 - 2b_2, & K_4^{-2} &= 5b_1^4 + 6b_1b_3 + 3b_2^2 - 12b_1^2b_2 - 2b_4, \\
K_5^{-2} &= -12b_1b_2^2 - 4b_5 + 20b_2b_1^3 + 6b_1b_4 - 6b_1^5 - 12b_3b_1^2 + 6b_2b_3, \\
K_6^{-2} &= 3b_3^2 + 30b_2^2b_1^2 + 12b_5b_1 - 12b_1^2b_4 - 4b_2^3 - 24b_1b_2b_3 + 6b_2b_4 \\
&\quad + 7b_1^6 - 2b_6 + 20b_1^3b_3 - 30b_2b_1^4, \\
K_7^{-2} &= 12b_5b_2 + 6b_1b_6 - 12b_1b_3^2 - 12b_2^2b_3 - 24b_1b_4b_2 + 42b_2b_1^5 + 6b_4b_3 - 2b_7 \\
&\quad - 8b_1^7 + 60b_2b_3b_1^2 - 30b_1^4b_3 - 60b_2^2b_1^3 + 20b_2^3b_1 + 20b_1^3b_4 - 24b_5b_1^2, \\
K_8^{-2} &= -12b_4b_2^2 - 60b_2^3b_1^2 - 12b_2b_3^2 + 105b_2^2b_1^4 - 30b_1^4b_4 - 120b_2b_3b_1^3 - 12b_1^2b_6 \\
&\quad + 5b_2^4 + 60b_1^2b_4b_2 - 2b_8 + 6b_7b_1 + 12b_5b_3 + 6b_2b_6 + 60b_2^2b_3b_1 + 42b_3b_1^5 \\
&\quad - 48b_1b_5b_2 + 30b_1^2b_3^2 + 3b_2^4 + 9b_1^8 - 56b_2b_1^6 - 24b_1b_4b_3 + 40b_1^3b_5, \\
K_1^{-3} &= -3b_1, & K_3^{-3} &= -3b_3 - 10b_1^3 + 12b_1b_2, \\
K_2^{-3} &= 6b_1^2 - 3b_2, & K_4^{-3} &= 15b_1^4 + 6b_2^2 - 30b_2b_1^2 - 3b_4 + 12b_1b_3, \\
K_5^{-3} &= 12b_1b_4 + 12b_2b_3 + 60b_2b_1^3 - 30b_1b_2^2 - 30b_3b_1^2 - 21b_1^5 - 3b_5, \\
K_6^{-3} &= 12b_4b_2 + 60b_3^2 + 90b_2^2b_1^2 - 3b_6 - 30b_4b_1^2 + 28b_1^6 - 60b_3b_1b_2 + 60b_3b_1^3 \\
&\quad - 105b_1^4b_2 - 10b_3^3 + 12b_1b_5, \\
K_3^{-4} &= 4 \times (-5b_1^3 + 5b_1b_2 - b_3), \\
K_4^{-5} &= 5 \times (14b_1^4 - 21b_2b_1^2 + 6b_1b_3 - b_4 + 3b_2^2).
\end{aligned}$$

An expression of K_n^p can be obtained as follows, let

$$\phi_n(z) = b_1z + b_2z^2 + b_3z^3 + \dots + b_nz^n.$$

The line integral $D_n^k = \frac{1}{2i\pi} \int \frac{\phi_n(\xi)^k}{\xi^{n+1}} d\xi$ is equal to the coefficient of z^n in $\phi_n(z)^k$ and is of course independent of p . For any $p \in \mathbb{Z}$, we have,

$$K_n^p = pb_n + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + \frac{p!}{(p-4)!4!} D_n^4 + \dots + \frac{p!}{(p-n)!n!} D_n^n. \tag{T3}$$

D_n^k is the sum of terms having k factors in K_n^k and

$$C_n^p = \frac{p!}{n!(p-n)!} = \frac{p(p-1)\dots(p-n+1)}{n!}$$

is the binomial coefficient. If $b_1 \neq 0$,

$$D_n^k(b_1, b_2, \dots, b_n) = b_1^k K_{n-k}^k \left(\frac{b_2}{b_1}, \dots, \frac{b_{n-k+1}}{b_1} \right). \tag{T4}$$

Replacing in (T3) and iterating the procedure permits to obtain the exact expression on K_n^p , see [3] and Section 4.2 below.

We have relations between the partial derivatives of the Faber polynomials as

$$\frac{\partial G_n}{\partial b_k} = \frac{\partial G_{n+p}}{\partial b_{k+p}} = -K_{n-k}^{-2} \times 1_{n \geq k} \quad \text{for all } n \geq 1, k \geq 1, \text{ and } p \geq 0, \tag{1.17}$$

$$\frac{\partial^2 F_n}{\partial b_1^2} = -n \frac{\partial G_n}{\partial b_2}, \quad \frac{\partial^2 F_n}{\partial b_r \partial b_s} = -n \frac{\partial G_n}{\partial b_{r+s}}. \tag{1.18}$$

Let

$$X_0 = - \sum_{j \geq 1} b_j \frac{\partial}{\partial b_j} = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \dots - b_k \frac{\partial}{\partial b_k} - \dots = - \sum_{j \geq 1} G_j W_j, \tag{1.19}$$

then

$$X_0 F_n = -n G_n \quad \text{and} \quad \frac{\partial^2 F_n}{\partial b_r \partial b_s} = \frac{\partial}{\partial b_{r+s}} (X_0 F_n) \quad \forall r \geq 1, s \geq 1, \tag{1.20}$$

$$\frac{\partial}{\partial b_j} X_0 - X_0 \frac{\partial}{\partial b_j} = - \frac{\partial}{\partial b_j}. \tag{1.21}$$

This leads to the construction of differential operators on \mathcal{M} which transform one polynomial into the other. See Section 7.

On the other hand, from (T1), we obtain, see (3.19) and Section 4,

Main Corollary. *The coefficients of the Schwarzian derivative of $f(z) = z + b_1 z^2 + b_2 z^3 + \dots + b_n z^{n+1} + \dots$ are given in terms of Faber polynomials and their second derivatives as $z^2 S(f)(z) = z^2 [(\frac{f''}{f'})' - \frac{1}{2} (\frac{f''}{f'})^2] = \sum_{k \geq 2} \mathcal{P}_k z^k$ where*

$$\begin{aligned} \mathcal{P}_k = & -(k-2) F_k(2b_1, 3b_2, \dots, (j+1)b_j, \dots) \\ & - \frac{1}{2} K_k^2(F_1(2b_1), F_2(2b_1, 3b_2), \dots, F_k(2b_1, 3b_2, \dots, (k+1)b_k)) \end{aligned} \tag{C1}$$

and $K_k^2(F_1(2b_1), F_2(2b_1, 3b_2), \dots) = -\frac{1}{k+2} \frac{\partial^2 F_{k+2}}{\partial b_1^2}(c_1, c_2, \dots, c_k)$ is the second derivative F_{k+2} calculated at the point

$$(c_1, c_2, \dots, c_k) = (G_1(F_1(2b_1)), G_2(F_1(2b_1), F_2(2b_1, 3b_2)), \dots, G_k(F_1(2b_1), \dots, F_k(2b_1, 3b_2, \dots, (k + 1)b_k))).$$

The tool is the composition of maps on the manifold \mathcal{M} . We have

$$\frac{h'(w)}{h(w)} = - \sum_{k=1}^{+\infty} F_k(b_1, b_2, \dots, b_k) w^{k-1} = -(F_1 + F_2 w + F_3 w^2 + \dots). \tag{1.22}$$

The function $f(w) = wh(w) = w + b_1 w^2 + b_2 w^3 + \dots + b_n w^{n+1} + \dots$ satisfies

$$w \frac{f'(w)}{f(w)} = 1 + w \frac{h'(w)}{h(w)} = 1 - \sum_{k \geq 1} F_k(b_1, b_2, \dots, b_k) w^k \tag{1.23}$$

and $g(w) = \frac{w}{h(w)} = w + \sum_{n \geq 1} G_n(b_1, b_2, \dots, b_n) w^{n+1} + \dots$ satisfies

$$w \frac{g'(w)}{g(w)} = 1 - w \frac{h'(w)}{h(w)} = 1 + \sum_{k \geq 1} F_k(b_1, b_2, \dots, b_k) w^k. \tag{1.24}$$

From (1.23) and (1.24), we deduce

$$F_n(G_1(b_1), G_2(b_1, b_2), \dots, G_n(b_1, b_2, \dots, b_n)) + F_n(b_1, b_2, \dots, b_n) = 0. \tag{1.25}$$

We consider the following maps from $\mathcal{M} \rightarrow \mathcal{M}$,

$$\begin{aligned} F &: (b_1, b_2, \dots, b_n, \dots) \rightarrow (F_1(b_1), F_2(b_1, b_2), \dots, F_n(b_1, b_2, \dots, b_n), \dots) \\ F^{-1} &: (b_1, b_2, \dots, b_n, \dots) \rightarrow (c_1, c_2, \dots, c_n, \dots) \quad \text{such that} \\ F_1(c_1) &= b_1, \quad F_2(c_1, c_2) = (b_1, b_2), \quad \dots, \\ F_n(c_1, c_2, \dots, c_n) &= (b_1, b_2, \dots, b_n), \quad \dots, \\ G &: (b_1, b_2, \dots, b_n, \dots) \rightarrow (G_1(b_1), G_2(b_1, b_2), \dots, G_n(b_1, b_2, \dots, b_n), \dots), \\ S &: (b_1, b_2, \dots, b_n, \dots) \rightarrow (-b_1, -b_2, \dots, -b_n, \dots). \end{aligned}$$

The relation (1.25) means that G is obtained as the composition of maps

$$G = F^{-1} \circ S \circ F. \tag{1.26}$$

The first polynomials $(F_n^{-1})_{n \geq 1}$ defined by the map F^{-1} are given by

$$\begin{aligned} F_1^{-1}(b_1) &= -b_1 \quad \text{and} \quad F_2^{-1}(b_1, b_2) = \frac{1}{2}(b_1^2 - b_2), \\ F_3^{-1}(b_1, b_2, b_3) &= \frac{1}{6}(-b_1^3 + 3b_1 b_2 - 2b_3), \\ F_4^{-1}(b_1, b_2, b_3, b_4) &= \frac{1}{4!}(b_1^4 - 6b_1^2 b_2 + 3b_2^2 + 8b_1 b_3 - 6b_4), \\ F_5^{-1} &= \frac{1}{5!}(-b_1^5 - 15b_1 b_2^2 + 10b_1^3 b_2 - 20b_1^2 b_3 + 20b_2 b_3 + 30b_1 b_4 - 24b_5), \\ F_6^{-1} &= \frac{1}{6!}(b_1^6 + 144b_1 b_5 - 15b_2^3 + 45b_2^2 b_1^2 - 15b_2 b_1^4 - 120b_1 b_2 b_3 + 90b_2 b_4 \\ &\quad + 40b_1^3 b_3 - 120b_6 + 40b_3^2 - 90b_1^2 b_4), \end{aligned}$$

$$F_7^{-1} = \frac{1}{7!}(-b_1^7 - 504b_1^2b_5 + 504b_2b_5 + 840b_1b_6 + 21b_1^5b_2 + 420b_2b_1^2b_3 - 70b_1^4b_3 - 280b_1b_3^2 - 210b_2^2b_3 - 105b_2^2b_1^3 + 105b_1b_2^3 + 420b_3b_4 - 720b_7 + 210b_1^3b_4 - 630b_1b_2b_4),$$

$$F_8^{-1} = \frac{1}{8!}(b_1^8 - 4032b_1b_2b_5 + 1344b_1^3b_5 + 2688b_3b_5 - 3360b_1^2b_6 + 3360b_2b_6 + 5760b_1b_7 + 105b_2^4 - 420b_2^3b_1^2 + 1680b_1b_2^2b_3 - 1120b_1^3b_2b_3 - 420b_1^4b_4 + 210b_1^4b_2^2 - 1120b_2b_3^2 + 112b_1^5b_3 + 1120b_1^2b_3^2 - 28b_1^6b_2 - 5040b_8 + 1260b_4^2 + 2520b_2b_1^2b_4 - 1260b_2^2b_4 - 3360b_1b_3b_4).$$

We put $F_0^{-1} = 1$. We have $\exp(-\sum_{j \geq 1} \frac{b_j}{j} z^j) = 1 + \sum_{k \geq 1} F_k^{-1}(b_1, b_2, \dots, b_k) z^k$ and $\frac{\partial}{\partial b_1} F_j^{-1} = -F_{j-1}^{-1}, \forall j \geq 2$,

$$\frac{\partial}{\partial b_k} F_p^{-1} = 0 \quad \text{if } k \geq p + 1, \tag{1.27}$$

$$\frac{\partial}{\partial b_k} F_k^{-1} = -\frac{1}{k} \quad \text{and} \quad \frac{\partial}{\partial b_k} F_p^{-1} = -\frac{1}{k} F_{p-k}^{-1} \quad \text{if } k \leq p. \tag{1.28}$$

Differentiating (1.25), we obtain systems of partial differential equations satisfied by the $(F_n)_{n \geq 1}$ and the (F_n^{-1}) . If $p \geq 1$ is an integer, we denote $p \times S$ the map

$$p \times S : (b_1, b_2, \dots, b_n, \dots) \rightarrow (-pb_1, -pb_2, \dots, -pb_n, \dots) \tag{1.29}$$

and $p \times I$ the map

$$p \times I : (b_1, b_2, \dots, b_n, \dots) \rightarrow (pb_1, pb_2, \dots, pb_n, \dots). \tag{1.30}$$

Consider the maps, for $p \in \mathbb{Z}, p \neq 0$,

$$K^p : (b_1, b_2, \dots, b_n, \dots) \rightarrow (K_1^p(b_1), K_2^p(b_1, b_2), \dots, K_j^p(b_1, b_2, \dots, b_j), \dots).$$

We obtain for $p \geq 1$,

$$K^{-p} = F^{-1} \circ (p \times S) \circ F, \tag{1.31}$$

$$K^p = F^{-1} \circ (p \times I) \circ F. \tag{1.32}$$

This last relation shows that $K^p \circ K^q = K^q \circ K^p = K^{pq}$ for $p \neq 0, q \neq 0, p, q \in \mathbb{Z}$. In particular, it is enough to know the (K^p) when p are prime numbers, to obtain all the other $K^p, p \in \mathbb{Z}$, by composition of maps. From (1.31)–(1.32), we see that

$$F_n(K_1^{-p}(b_1), K_2^{-p}(b_1, b_2), \dots, K_n^{-p}(b_1, b_2, \dots, b_n)) + F_n(K_1^p(b_1), K_2^p(b_1, b_2), \dots, K_n^p(b_1, b_2, \dots, b_n)) = 0$$

which extends (1.25).

The coefficients of $f^{-1}(z)$ the inverse map of $f(z) = z + b_1z^2 + b_2z^3 + \dots$ are given by

$$f^{-1}(z) = z + \sum_{n \geq 1} \frac{1}{n+1} K_n^{-(n+1)} z^{n+1} \tag{1.33}$$

(compare with [13]) and the coefficients of $g^{-1}(z)$ the inverse map of $g(z) = z + b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \dots + \frac{b_{n+1}}{z^n} + \dots$ are given by

$$g^{-1}(z) = z - b_1 - \sum_{n \geq 1} \frac{1}{n} K_{n+1}^n \frac{1}{z^n}. \tag{1.34}$$

Thus the two maps ϕ_f and ϕ_g defined by

$$\begin{aligned} \phi_f : (b_1, b_2, b_3, b_4, \dots) &\rightarrow \left(\frac{1}{2} K_1^{-2}, \frac{1}{3} K_2^{-3}, \frac{1}{4} K_3^{-4}, \frac{1}{5} K_4^{-5}, \dots, \frac{1}{n+1} K_n^{-(n+1)}, \dots \right), \\ \phi_g : (b_1, b_2, b_3, \dots) &\rightarrow \left(-b_1, -K_2^1, -\frac{1}{2} K_3^2, -\frac{1}{3} K_4^3, -\frac{1}{4} K_5^4, \dots, -\frac{1}{n} K_{n+1}^n, \dots \right) \end{aligned}$$

satisfy $\phi_f \circ \phi_f = \text{Id}_{\mathcal{M}}$ and $\phi_g \circ \phi_g = \text{Id}_{\mathcal{M}}$.

2. Identities between the polynomials

First, we recall the basic facts relative to the polynomials $(F_j)_{j \geq 0}$, $(G_j)_{j \geq 0}$, (K_n^p) , (P_n^p) , $n \geq 1$, $p \in \mathbb{Z}$ and the differential operators $(W_j)_{j \geq 1}$.

2.1. Zeroes and particular values of the polynomials

We have

$$\begin{aligned} G_1(1) &= -1, & G_n(1, 1, 1, \dots, 1) &= 0 \quad \text{for } n \geq 2, \\ G_1(2) &= -2, & G_2(2, 3) &= 1, \\ G_1(3) &= -3, & G_n(2, 3, 4, \dots, k, \dots, n+1) &= 0 \quad \text{for } n \geq 2, \\ G_2(3, 6) &= 3, & G_3(3, 6, 10) &= -1, \\ G_4(3, 6, 10, 15) &= 0, \\ G_n\left(3, 6, 10, 15, 21, \dots, \frac{(n+1)(n+2)}{2}\right) &= 0 \quad \text{for } n \geq 5, \\ G_n\left(1, \frac{1}{2!}, \frac{1}{3!}, \dots, \frac{1}{n!}\right) &= (-1)^n \frac{1}{n!} \quad \text{for } n \geq 1, \end{aligned} \tag{2.1}$$

$$\begin{aligned} K_1^{-2}(1) &= -2, & K_2^{-2}(1, 1) &= 1, \\ K_2^{-3}(1, 1) &= 3, & K_3^{-3}(1, 1, 1) &= -1, \\ K_n^{-2}(1, 1, 1, \dots, 1) &= 0 \quad \text{for } n \geq 3, \\ K_n^{-3}(1, 1, \dots, 1) &= 0 \quad \text{for } n \geq 4, \\ K_n^2(-4, -2, -4, -2, -4, -2, \dots) &= \begin{cases} 8(n-2) & \text{if } n \text{ is odd,} \\ 2(5n-4) & \text{if } n \text{ is even.} \end{cases} \end{aligned} \tag{2.2}$$

For $n \geq 1$,

- $F_n(1, 1, 1, \dots, 1) = -1,$ (i)
- $F_n(-1, 1, \dots, (-1)^n) = (-1)^{n+1},$ (ii)
- $F_n(4, 9, \dots, (n+1)^2) = -3 + (-1)^n,$ (iii)

$$F_n(2, 3, 4, 5, \dots, n + 1) = -2, \tag{iv}$$

$$F_n\left(1, \frac{1}{2!}, \frac{1}{3!}, \dots, \frac{1}{n!}\right) = 0 \quad \text{for } n \geq 2. \tag{v}$$

For any $p \in \mathbb{Z}$, $p \neq 0$, $n \geq 1$

$$F_n^{-1}(p, p, p, \dots, p) = (-1)^n C_n^p \quad \text{with } C_n^p = \frac{p(p-1)(p-2)\dots(p-n+1)}{n!}, \tag{vi}$$

$$F_n^{-1}(b_1 + 1, b_2 + 1, \dots, b_n + 1) = F_n^{-1}(b_1, b_2, \dots, b_n) - F_{n-1}^{-1}(b_1, b_2, \dots, b_{n-1}), \tag{vii}$$

$$F_n^{-1}(-p, -p, \dots, -p) = C_n^{n+p-1}, \tag{viii}$$

$$K_n^p(2, 3, 4, 5, \dots, n + 1) = C_n^{2p+n-1} = (-1)^n C_n^{-2p}, \tag{ix}$$

$$K_n^p(-1, 1, -1, \dots, (-1)^n) = C_n^{-p}. \tag{x}$$

Proof. We take the particular case of the function

$$h(z) = 1 + z + z^2 + \dots + z^n + \dots = \frac{1}{1 - z} \quad \text{for } |z| < 1.$$

Since $\frac{1}{h(z)} = 1 - z$, we find $G_n(1, 1, 1, \dots, 1)$ for all $n \geq 1$. In the same way, for $|z| < 1$, consider $h'(z) = 1 + 2z + 3z^2 + \dots + (n + 1)z^n + \dots$. Since

$$\frac{1}{h'(z)} = (1 - z)^2 = 1 - 2z + z^2$$

we find $G_n(2, 3, 4, \dots, n + 1)$. We continue with

$$\frac{1}{(1 - z)^3} = \frac{h''(z)}{2} = 1 + 3z + 6z^2 + 10z^3 + 15z^4 + 21z^5 + \dots + \frac{(n + 1)(n + 2)}{2} z^n + \dots$$

For K_n^{-2} , since $\frac{1}{h(z)^2} = (1 - z)^2$, we obtain $K_n^{-2}(1, 1, 1, \dots, 1)$. For $K_n^{-3}(1, 1, 1, \dots)$, we use $\frac{1}{h(z)^3} = 1 - 3z + 3z^2 - z^3$. In this way, we find particular values of $(b_1, b_2, \dots, b_n, \dots)$ such that the functions G_n and K_n^{-p} , $p \geq 1$ are zero. We obtain the zeros of $(K_n^p)_{n \geq 1}$, $p \geq 1$, using the identity (T2) in the Main theorem. To find zeros of $(F_n)_{n \geq 1}$, we take $h(z) = \exp(z)$. Moreover since for a homogeneous polynomial P_n of degree n ,

$$P_n(r b_1, r^2 b_2, \dots, r^n b_n) = r^n P_n(b_1, b_2, \dots, b_n) \quad \forall r \in \mathbb{C}$$

we obtain for the polynomials $(G_n)_{n \geq 1}$, $(K_n^{-p})_{n \geq 1}$ and $(F_n)_{n \geq 1}$, curves of zeros in the manifold \mathcal{M} . Of course for these polynomials, there are many other manifolds of zeros. See [4]. To find the special values for $(F_n)_{n \geq 1}$, for (i), we consider $h(z) = \frac{1}{1-z}$ which gives $\frac{h'}{h} = \frac{1}{1-z}$. For (ii), we take $h(z) = \frac{1}{1+z}$. For (iii), (iv), (v) and (2.1), we take the Koebe function

$$f(z) = \frac{z}{(1 - z)^2}, \quad \text{then } \frac{zf'}{f} = 1 + \frac{2z}{1 - z} = \frac{1 + z}{1 - z},$$

$$h(z) = f'(z) = \frac{1 + z}{(1 - z)^3} = 1 + 4z + 9z^2 + \dots + (n + 1)^2 z^n + \dots,$$

$$\frac{h'}{h} = \frac{f''}{f'} = \frac{1}{1 + z} + \frac{3}{1 - z} = 4 + 2z + 4z^2 + 2z^3 + 4z^4 + \dots, \tag{iii}'$$

$$\begin{aligned}
 z^2 S_f(z) &= z^2 \left[\left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2 \right] = -\frac{6z^2}{(1-z^2)^2} = \sum_{k \geq 2} \mathcal{P}_k z^k \\
 &= z^2 (2 + 8z + 6z^2 + 16z^3 + 10z^4 + \dots) \\
 &\quad - \frac{z^2}{2} (16 + 16z + 36z^2 + 32z^3 + 4 \times 14z^4 + \dots). \tag{iii}''
 \end{aligned}$$

We calculate (iv) with $\frac{zf'}{f}$ and (iii) with (iii)'. To calculate (2.2), we use

$$\sum_{k \geq 0} K_k^2 (-4, -2, -4, -2, -4, -2, \dots) z^k = \left(1 - z \frac{f''}{f'} \right)^2 = \left(\frac{1}{1+z} - \frac{3z}{1-z} \right)^2.$$

To prove (vi),

$$\exp \left(-p \sum_{j \geq 1} \frac{z^j}{j} \right) = \exp [p \log(1-z)] = (1-z)^p = 1 + \sum_{n \geq 1} F_n^{-1}(p, p, \dots, p) z^n.$$

To prove (vii),

$$\begin{aligned}
 \exp \left(- \sum_{j \geq 1} (b_j + 1) \frac{z^j}{j} \right) &= 1 + \sum_{n \geq 1} F_n^{-1}(b_1 + 1, b_2 + 1, \dots, b_n + 1) z^n \\
 &= \exp \left(- \sum_{j \geq 1} b_j \frac{z^j}{j} \right) \times \exp \left(- \sum_{j \geq 1} \frac{z^j}{j} \right) = (1-z) \times \left(1 + \sum_{n \geq 1} F_n^{-1}(b_1, b_2, \dots, b_n) z^n \right)
 \end{aligned}$$

and we identify equal powers of z . The identity (vii) generalizes the classical identity $C_n^{p+1} = C_{n-1}^p + C_n^p$ for the binomial coefficients. See for example [6, vol. 1, II-12].

To prove (viii), we have to calculate $F^{-1} \circ S$, it comes from

$$\exp \left(\sum_{j \geq 1} \frac{b_j}{j} z^j \right) = 1 + \sum_{k \geq 1} F_k^{-1}(-b_1, -b_2, \dots, -b_k) z^k.$$

Taking $b_j = n$ for all $j \geq 1$, $\exp(n \sum_{j \geq 1} \frac{1}{j} z^j) = \frac{1}{(1-z)^n} = \sum_{j \geq 0} C_j^{n+j-1} z^j$.

To prove (ix), we take the Koebe function $f(z)$, then $h(z) = \frac{f(z)}{z} = \frac{1}{(1-z)^2}$ and $[h(z)]^p = \frac{1}{(1-z)^{2p}} = 1 + \sum_{n \geq 1} C_n^{2p+n-1} z^n$. We can also deduce (ix) using the composition of maps: $K^p(2b_1, 3b_2, \dots, (n+1)b_n, \dots) = F^{-1} \circ pI \circ F(2b_1, 3b_2, \dots)$. For $b_1 = b_2 = \dots = 1$, we replace $F(2, 3, \dots)$ using (iv), then we use (viii). To prove (x), we take $h(z) = \frac{1}{1+z}$. \square

Remark 2.1. We verify the main corollary (C1) when $f(z)$ is the Koebe function. From (C1), for the Koebe function,

$$\mathcal{P}_k = -(k-2)F_k(4, 9, 16, \dots, (k+1)^2) - \frac{1}{2}K_k^2(-4, -2, -4, -2, -4, \dots).$$

If k is odd, from (iii), we have $F_k(4, 9, 16, \dots, (k+1)^2) = -4$, thus from (2.2), we find that $\mathcal{P}_k = 0$. If k is even, from (iii), $F_k(4, 9, 16, \dots, (k+1)^2) = -2$ and using (2.2), we find $\mathcal{P}_k = -(k-2) \times (-2) - (5k-4) = -3k$. Thus $\mathcal{P}_{2p} = -6p$. Compare with (iii)'.

We obtain values of F_n and G_n when the $(b_j)_{j \geq 1}$ are binomial coefficients,

Proposition 2.1. *We have*

$$F_n \left(-\binom{n}{1}, \binom{n}{2}, -\binom{n}{3}, \dots, (-1)^n \binom{n}{n} \right) = n,$$

$$G_n \left(-\binom{n}{1}, \binom{n}{2}, -\binom{n}{3}, \dots, (-1)^n \binom{n}{n} \right) = \binom{2n-1}{n},$$

where $\binom{n}{k} = C_k^n$ is the binomial coefficient. More generally, let $q \in C$, and let

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1}) \dots (1-q^{n-k+1})}{(1-q)(1-q^2) \dots (1-q^k)}$$

be the Gaussian polynomial, then

$$G_n \left(-\begin{bmatrix} n \\ 1 \end{bmatrix}, q \begin{bmatrix} n \\ 2 \end{bmatrix}, -q^3 \begin{bmatrix} n \\ 3 \end{bmatrix}, \dots, (-1)^n q^{\frac{n(n-1)}{2}} \begin{bmatrix} n \\ n \end{bmatrix} \right) = \begin{bmatrix} 2n-1 \\ n \end{bmatrix}.$$

Proof. We obtain F_n with

$$-n \sum_{j \geq 1} \frac{w^j}{j} = \log \left(1 - \binom{n}{1} w + \binom{n}{2} w^2 - \binom{n}{3} w^3 + \dots + (-1)^n \binom{n}{n} w^n \right)$$

$$= n \log(1-w).$$

We can also deduce the identity for F_n from (2.2)–(vi) and the composition of maps $F \circ F^{-1} = \text{Id}$. We obtain G_n with the relation

$$\left[1 - \binom{n}{1} w + \binom{n}{2} w^2 - \binom{n}{3} w^3 + \dots + (-1)^n \binom{n}{n} w^n \right]^{-1}$$

$$= \frac{1}{(1-w)^n} = \sum_{j \geq 0} \binom{n+j-1}{j} w^j$$

when $\binom{n}{k}$ is the binomial coefficient, and in the case of the Gaussian polynomial,

$$\left[1 - \begin{bmatrix} n \\ 1 \end{bmatrix} w + q \begin{bmatrix} n \\ 2 \end{bmatrix} w^2 - q^3 \begin{bmatrix} n \\ 3 \end{bmatrix} w^3 + \dots + (-1)^n q^{\frac{n(n-1)}{2}} \begin{bmatrix} n \\ n \end{bmatrix} w^n \right]^{-1}$$

$$= \frac{1}{\prod_{k=0}^{n-1} (1-q^k w)} = \sum_{k \geq 0} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} w^k.$$

From (1.26), we know that $G = F^{-1} \circ S \circ F$, then $G \circ F^{-1} = F^{-1} \circ S$, we can deduce the first identity for G_n from (2.2)–(viii). \square

2.2. Basic identities

The function $h(z) = 1 + b_1 z + b_2 z^2 + \dots + b_k z^k + \dots$ satisfies

$$h(z) - zh'(z) = 1 - \sum_{n \geq 2} (n-1) b_n z^n,$$

$$h(z) - zh'(z) + \frac{z^2 h''(z)}{2} = 1 + \sum_{n \geq 3} \frac{(n-1)(n-2)}{2} b_n z^n,$$

$$h(z) - zh'(z) + \frac{z^2 h''(z)}{2} - \frac{z^3 h'''(z)}{3!} = 1 - \sum_{n \geq 4} \frac{(n-1)(n-2)(n-3)}{3!} b_n z^n,$$

...

2.3. Relations between the polynomials

We differentiate (1.1) with respect to w ,

$$b_1 + 2b_2 w + \dots + kb_k w^{k-1} + \dots = (1 + b_1 w + b_2 w^2 + \dots + b_p w^p + \dots) \times \left(- \sum_{j \geq 1} F_j w^{j-1} \right).$$

We equal coefficients of same powers of w , it gives the recurrence for the polynomials $(F_k)_{k \geq 0}$, $F_0 = 1$,

$$-kb_k = \sum_{1 \leq j \leq k} F_j b_{k-j}. \tag{2.3}$$

With the same approach, one find other relations between the polynomials as

Proposition 2.2.

$$F_{j+1} = - \sum_{0 \leq r \leq j} (r+1)b_{r+1} G_{j-r}, \tag{2.4}$$

$$nG_n = \sum_{1 \leq j \leq n} F_j G_{n-j}, \tag{2.5}$$

$$\frac{n}{p-1} K_n^{1-p} = \frac{1}{r-1} \sum_{1 \leq j \leq n} j K_j^{1-r} K_{n-j}^{r-p} \text{ for } 2 \leq r < p, \tag{2.6}$$

$$\frac{n}{p-1} K_n^{1-p} = \sum_{1 \leq j \leq n} F_j K_{n-j}^{1-p} \quad \forall p \neq 1, p \in \mathbb{Z}, \tag{2.7}$$

$$K_n^p = \sum_{0 \leq j \leq n} K_j^r K_{n-j}^{p-r}. \tag{2.8}$$

Proof of the identities (2.4)–(2.8). To find (2.4), we consider $h(w) = 1 + b_1 w + b_2 w^2 + b_3 w^3 + \dots + b_k w^k + \dots$,

$$h'(w) = b_1 + 2b_2 w + 3b_3 w^2 + \dots + nb_n w^{n-1} + \dots.$$

Since $\frac{1}{h(w)} = \sum_{n \geq 0} G_n w^n$, multiplying by $h'(w)$ gives

$$\frac{h'(w)}{h(w)} = \sum_{r \geq 0} \left[\sum_{0 \leq r \leq j} (r+1)b_{r+1} G_{j-r} \right] w^r.$$

To obtain (2.4), we compare with (1.22). For (2.5), we identify the two following expansions

$$\frac{h'(w)}{h(w)^2} = \frac{h'(w)}{h(w)} \times \frac{1}{h(w)} = \left(\sum_{j \geq 0} -F_{j+1} w^j \right) \times \left(\sum_{p \geq 0} G_p w^p \right),$$

$$\frac{h'(w)}{h(w)^2} = -\frac{d}{dw} \frac{1}{h(w)} = -\sum_{n \geq 0} (n+1)G_{n+1}w^n.$$

To find (2.6) and (2.7), we use that for $0 \leq r \leq p$,

$$\frac{h'(w)}{h(w)^p} = \frac{h'(w)}{h(w)^r} \times \frac{1}{h(w)^{p-r}}. \tag{i}$$

If $p \neq 1$ and $r \neq 1$, (2.6) comes from

$$\frac{1}{(p-1)} \frac{d}{dw} \frac{1}{h(w)^{p-1}} = \frac{1}{(r-1)} \left(\frac{d}{dw} \frac{1}{h(w)^{r-1}} \right) \times \sum_{j \geq 0} K_j^{r-p} w^j.$$

In (i), we take $r = 1$ and we obtain (2.7) with

$$\frac{1}{p-1} \frac{d}{dw} \frac{1}{h(w)^{p-1}} = \left(\sum_{j \geq 0} F_{j+1} w^j \right) \times \frac{1}{h(w)^{p-1}}. \quad \square$$

Remark 2.1. For $k \geq 1$, (1.22) yields

$$\begin{aligned} \frac{w^{1-k}h'(w)}{h(w)} + F_1w^{1-k} + F_2w^{2-k} + \dots + F_mw^{m-k} + \dots + F_{k-1}w^{-1} + F_k \\ = -(F_{k+1}w + \dots + F_{k+r}w^r + \dots) \end{aligned}$$

and

$$\begin{aligned} (F_{k+1}w + \dots + F_{k+r}w^r + \dots) \times h(w) \\ = \sum_{j \geq 1} (F_{k+1}b_{j-1} + F_{k+2}b_{j-2} + \dots + F_{j+k}b_0)w^j. \end{aligned}$$

The relations (2.4), (2.5), (2.6) and (2.7) involve the first derivative of h , we can find other relations by multiplying powers of h . For example (2.8) comes from $h(w)^p = h(w)^r \times h(w)^{p-r}$.

Below, we give further identities between the polynomials. We consider the polynomials $(P_n^k)_{n \geq 1}, k \in Z$ defined by (1.4). We define B_n^k by

$$\left(\frac{zh'(z)}{h(z)} \right)^2 [h(z)]^k = \sum_{n \geq 2} B_n^{n+k} z^n. \tag{2.9}$$

Since $\frac{zf'}{f} = 1 + \frac{zh'}{h}$, we have

$$P_n^{k+n} = B_n^{n+k} \times 1_{n \geq 2} + \frac{2n+k}{k} K_n^k. \tag{2.10}$$

Proposition 2.3. Let $f(\zeta) = \zeta(1 + b_1\zeta + b_2\zeta^2 + \dots)$. For $k \neq 0$, we have

$$\phi_k(\zeta) = \frac{\zeta f'(\zeta)}{f(\zeta)} \times h(\zeta)^k = \sum_{n \geq 0} \frac{k+n}{k} K_n^k \zeta^n. \tag{2.11}$$

If $k \geq 1$,

$$\phi_k(\zeta) = \frac{(-1)^k}{k!} \left[\frac{\partial^k}{\partial b_1^k} \left(\sum_{n \geq 0} F_{k+n} \zeta^n \right) \right] (G_1(b_1), G_2(b_1, b_2), \dots) \tag{2.12}$$

and the function $\sum_{n \geq 0} F_{k+n}(b_1, b_2, \dots, b_{k+n})z^n$, $k \geq 1$, is given by the line integral

$$\sum_{n \geq 0} F_{k+n}(b_1, b_2, \dots)z^n = -\frac{1}{2i\pi} \int \frac{h'(\zeta)}{\zeta^{k-1}(\zeta - z)h(\zeta)} d\zeta. \tag{2.13}$$

If $k \geq 1$, the function $\phi_{-k}(\zeta) = \frac{\zeta f'(\zeta)}{f(\zeta)} \times \frac{\zeta^k}{f(\zeta)^k} = \sum_{n \geq 0} \frac{k-n}{k} K_n^{-k} \zeta^n$ is given by

$$\phi_{-k}(\zeta) = \frac{(-1)^k}{k!} \left[\frac{\partial^k}{\partial b_1^k} \left(\sum_{n \geq 0} \frac{k-n}{k+n} F_{k+n} \zeta^n \right) \right] (b_1, b_2, \dots). \tag{2.14}$$

Proof. For (2.11), we use the recursion formula (2.7) or give a direct proof since $\phi_k(\zeta) = h(\zeta)^k + \frac{\zeta}{k} \frac{d}{d\zeta} h(\zeta)^k$. For (2.12), we have from (T1), for $k > 0$ and $j \geq 0$,

$$\frac{(j+k)}{k} K_j^{-k} = \frac{(-1)^k}{k!} \frac{\partial^k}{\partial b_1^k} F_{k+j}. \tag{2.15}$$

The proof of (2.13) is classical for Laurent series: let $l(\zeta) = \sum_{n \in \mathbb{Z}} \alpha_n z^n$, then

$$\sum_{n \geq k} \alpha_n z^n = \sum_{n \geq k} \frac{z^n}{2i\pi} \int \frac{l(\zeta)}{\zeta^{n+1}} d\zeta = \frac{z^k}{2i\pi} \int \frac{l(\zeta)}{\zeta^k(\zeta - z)} d\zeta.$$

We take $l(\zeta) = -\frac{\zeta h'(\zeta)}{h(\zeta)}$. For ϕ_{-k} , we proceed in the same way. \square

Remark 2.2. By composition of maps, see (1.31)–(1.32), we can define $(K_n^k)_{n \geq 0}$ for any $k \in \mathbb{R}$ with $K_0^k = 1$ and $(K_1^k, K_2^k, \dots, K_n^k, \dots) = F^{-1} \circ k \times \text{Id} \circ F$. From Proposition 2.3, we see that for fixed $n \geq 1$, we have $\lim_{k \rightarrow 0} \frac{n-k}{k} K_n^{-k} = F_n$. Since $\lim_{k \rightarrow 0} K_n^{-k} = 0$, we obtain for

$$\lim_{k \rightarrow 0} \frac{1}{k} K_n^{-k} = \frac{1}{n} \times F_n \quad \text{for } n \geq 1. \tag{2.16}$$

Proposition 2.4. Assume that f^{-1} is the inverse of f , $f(f^{-1}(z)) = z$. For $f(\zeta) = \zeta[1 + \sum_{n \geq 1} b_n \zeta^n]$ and $\phi_k(\zeta) = \frac{\zeta f'(\zeta)}{f(\zeta)} \times h(\zeta)^k$, then

$$\phi_k(f^{-1}(z)) = 1 + \sum_{n \geq 1} P_n^k(b_1, b_2, \dots, b_n) z^n \quad \text{for } k \neq 0, \tag{2.17}$$

$$P_n^k(b_1, \dots, b_n) = \sum_{0 \leq s \leq n} K_s^2(-F_1(b_1), \dots, -F_s(b_1, \dots, b_s)) \times K_{n-s}^{k-n}(b_1, \dots, b_{n-s}). \tag{2.18}$$

Proof. See [1, (A.1.2)]. In particular,

$$P_n^n(b_1, \dots, b_n) = K_n^2(-F_1(b_1), \dots, -F_n(b_1, \dots, b_n)),$$

$$P_n^n(G_1(b_1), \dots, G_n(b_1, \dots, b_n)) = K_n^2(F_1(b_1), \dots, F_n(b_1, \dots, b_n))$$

and $P_n^n(b_1, \dots, b_n) - P_n^n(G_1(b_1), \dots, G_n(b_1, \dots, b_n)) = -4F_n(b_1, b_2, \dots, b_n)$. \square

(2.19) – Expressions of (P_n^k)

If $n \neq k$,

$$\begin{aligned}
 P_n^k(b_1, \dots, b_n) &= -\frac{1}{k-n} \times \sum_{j=0}^n (k-j) F_j(b_1, \dots, b_j) K_{n-j}^{k-n}(b_1, \dots, b_{n-j}) \\
 &= \frac{k}{k-n} K_n^{k-n} - \frac{1}{k-n} \times \sum_{j=1}^n (k-j) F_j(b_1, b_2, \dots, b_j) \\
 &\quad \times K_{n-j}^{k-n}(b_1, b_2, \dots, b_{n-j}).
 \end{aligned}
 \tag{E1}$$

If $n = k$ (with $F_0 = -1$)

$$P_n^n(b_1, b_2, \dots, b_n) = \sum_{j=0}^n F_j(b_1, b_2, \dots, b_j) \times F_{n-j}(b_1, b_2, \dots, b_{n-j}).
 \tag{E2}$$

Remark 2.3. If $k = 1$, $f'(f^{-1}(z)) = 1 + \sum_{n \geq 1} P_n^1(b_1, b_2, \dots) z^n$ with

$$P_n^1 = \frac{1}{n-1} \sum_{0 \leq j \leq n} (1-j) F_j K_{n-j}^{1-n}.
 \tag{E3}$$

Proof of (E)₁ and (E)₂. From [1, (A.1.1)],

$$\frac{zf'(z)}{f(z)} [h(z)]^k \times \frac{zf'(z)}{f(z)} [h(z)]^p = 1 + \sum_{n \geq 1} P_n^{n+k+p} z^n.$$

On the other hand,

$$\frac{zf'(z)}{f(z)} [h(z)]^k = z^{1-k} f'(z) f(z)^{k-1} = z^{1-k} \frac{1}{k} \frac{d}{dz} f(z)^k = \frac{1}{k} \sum_{n \geq 0} (n+k) K_n^k z^n.$$

Multiplying $\frac{zf'(z)}{f(z)} [h(z)]^k \times \frac{zf'(z)}{f(z)} [h(z)]^p$, we obtain for any $k, p \in \mathbb{Z}, k \neq 0, p \neq 0$,

$$P_n^{n+k+p} = \sum_{0 \leq j \leq n} (j+k)(n-j+p) \times \frac{1}{pk} K_j^k K_{n-j}^p.
 \tag{E4}$$

We make $p \rightarrow 0$ as in Remark 2.2, it gives P_n^k . To obtain P_n^n , we make $k \rightarrow 0$. \square

Remark 2.4. From $\frac{(j+k)}{k} K_j^{-k} = \frac{(-1)^k}{k!} \frac{\partial^k}{\partial b_1^k} F_{k+j}$ and from (E)₄, we deduce that for $k > 0, p > 0$,

$$\begin{aligned}
 &P_n^{n+k+p}(b_1, b_2, \dots) \\
 &= \sum_{0 \leq j \leq n} \frac{(-1)^{k+p}}{k!p!} \left(\frac{\partial^k}{\partial b_1^k} F_{k+j} \right) (c_1, c_2, \dots) \times \left(\frac{\partial^p}{\partial b_1^p} F_{p+n-j} \right) (c_1, c_2, \dots).
 \end{aligned}
 \tag{E5}$$

From (E)₁, we obtain for $k > n$,

$$\begin{aligned}
 &P_n^k(b_1, b_2, \dots, b_n) \\
 &= \frac{(-1)^{k-n+1}}{(k-n)!} \sum_{j=0}^n F_j(b_1, b_2, \dots, b_j) \frac{\partial^{k-n}}{\partial b_1^{k-n}} F_{k-j}(c_1, c_2, \dots, c_{k-j})
 \end{aligned} \tag{E}_6$$

with $(c_1, c_2, \dots, c_k, \dots) = (G_1(b_1), G_2(b_1, b_2), \dots, G_k(b_1, b_2, \dots), \dots)$. When $k < n$, expressions of P_n^k in terms of partial derivatives of the Faber polynomials are more complicated.

Theorem 2.5. *For the Koebe function $f(z) = \frac{z}{(1-z)^2}$ or $f(z) = \frac{z}{(1+z)^2}$, we have $P_{n-k}^{-k} = P_{n+k}^k$ for $n \geq 1, k \geq 1$,*

$$\begin{aligned}
 &P_{n-k}^{-k}(b_1, b_2, \dots, b_{n-k}) = P_{n+k}^k(b_1, b_2, \dots, b_{n+k}) \\
 &f(b_1, b_2, \dots) = (2, 3, 4, 5, 6, \dots), \quad b_n = n + 1.
 \end{aligned} \tag{2.20}$$

Conversely, let $f(z) = z + b_1z^2 + \dots + b_nz^{n+1} + \dots$, if $P_{n-k}^{-k} = P_{n+k}^k$ for $n \geq 1, k \geq 1$, then $f(z) = \frac{z}{(1-\epsilon z)^2}$, $\epsilon = 1$ or $\epsilon = -1$.

Moreover, for the Koebe function, we have $K_{n-j}^{-n} \times 1_{n \geq j} = K_{n+j}^{-n} \times 1_{n+j \geq 0}$, $n, j \in \mathbb{Z}$. This last relation is the same as the classical $C_{n-j}^{2n} = C_{n+j}^{2n}$ on the binomial coefficients.

We verify (2.20) with $n = 3, k = 2$. With [1, (A.1.7)], we calculate $P_1^{-2} = -b_1$,

$$P_5^2 = 7b_5 - 20b_1b_4 + 30b_1b_2^2 + 35b_1^2b_3 - 50b_1^3b_2 + 14b_1^5 - 16b_2b_3.$$

When $b_n = (n + 1)$, we find $P_1^{-2} = P_5^2 = -2$. With [1, (A.1.7)],

$$P_4^2(b_1, b_2, b_3, b_4) = 6b_4 - 12b_1b_3 - 5b_2^2 + 16b_1^2b_2 - 5b_1^4$$

and $P_4^2(2, 3, 4, 5) = P_0^{-2} = 1$.

Proof of Theorem 2.5. Let $f(z) = \frac{z}{(1-z)^2}$, we have $\frac{zf'(z)}{f(z)} = \frac{1+z}{1-z}$ and

$$\left(\frac{zf'(z)}{f(z)}\right)^2 [h(z)]^k = \frac{(1+z)^2}{(1-z)^{2k+2}} = 1 + \sum_{n \geq 1} P_n^{n+k} z^n. \tag{i}$$

It gives the line integral

$$\begin{aligned}
 P_n^{n+k} &= \frac{1}{2i\pi} \int \left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 [h(\zeta)]^k \frac{d\zeta}{\zeta^{n+1}} - \sum_{\zeta \neq 0} \text{Residue} \\
 &= \frac{1}{2i\pi} \int \frac{(1+\zeta)^2}{(1-\zeta)^{2k+2}} \frac{d\zeta}{\zeta^{n+1}} - \text{Residue at } \zeta = 1.
 \end{aligned} \tag{ii}$$

With (ii), we find $P_{n-j}^{-j} = \frac{1}{2i\pi} \int \lambda(\zeta) \zeta^j \frac{d\zeta}{\zeta}$ and $P_{n+j}^j = \frac{1}{2i\pi} \int \lambda(\zeta) \zeta^{-j} \frac{d\zeta}{\zeta}$ with

$$\lambda(\zeta) = \left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 [f(\zeta)]^{-n} = \frac{(1+\zeta)^2}{(1-\zeta)^{2-2n}\zeta^n} \tag{iii}$$

since for $n \geq 1$, the function $\lambda(\zeta)$ has only a pole at $\zeta = 0$. We can calculate the two line integrals on the circle $|\zeta| = 1$. Since the function $\lambda(\zeta)$ is such that $\lambda(\frac{1}{\zeta}) = \lambda(\zeta)$, we put $\zeta = \frac{1}{\xi}$ and we see that the two line integrals P_{n-j}^{-j} and P_{n+j}^j are equal. Consider any $f(z)$ and let the function

$k(z) = f(\frac{1}{z})$, then $(\frac{zk'(z)}{k(z)})^2 = (\frac{uf'(u)}{f(u)})^2$ at $u = \frac{1}{z}$. The Koebe function satisfies $f(z) = f(\frac{1}{z})$. This proves (2.20).

Conversely, assume that $P_{n-k}^{-k} = P_{n+k}^k$ for $n \geq 1, k \geq 1$.

Taking $n = 1$, we find $P_2^1 = P_0^{-1} = 1$ and $P_{1+j}^j = 0$ for $j \geq 2$. It gives $P_n^{n-1} = 0$ for $n \geq 3$ and

$$\frac{1}{f(z)} \times \frac{z^2 f'(z)^2}{f(z)^2} = \frac{1}{z} + P_1^0 + z \tag{2.21}$$

then $f(z) = f(\frac{1}{z})$.

Taking $n = 2$, we obtain $P_4^2 = 1$ and $P_{2+j}^j = 0$ for $j \geq 3$. Thus $P_n^{n-2} = 0$ for $n \geq 4$ and from (1.4),

$$\frac{1}{f(z)^2} \times \frac{z^2 f'(z)^2}{f(z)^2} = \frac{1}{z^2} + P_1^{-1} \frac{1}{z} + P_2^0 + P_3^1 z + z^2. \tag{2.22}$$

We have $P_1^{-1} = P_3^1$ ($n = 2, j = 1$). Taking the ratio of (2.20) by (2.21), we see that

$$f(z) = z \times \frac{1 + P_1^0 z + z^2}{1 + P_1^{-1} z + P_2^0 z^2 + P_1^{-1} z^3 + z^4}.$$

Let $f(z) = zh(z)$ and $l(z) = \frac{1}{h(z)} = \frac{1 + P_1^{-1} z + P_2^0 z^2 + P_1^{-1} z^3 + z^4}{1 + P_1^0 z + z^2}$. With (2.21), $l(z) = 1 + G_1 z + G_2 z^2 + \dots + G_n z^n + \dots$ must satisfy

$$l(z) \left(1 - \frac{zl'(z)}{l(z)}\right)^2 = 1 + P_1^0 z + z^2 \tag{i}$$

then

$$(l(z) - zl'(z))^2 = 1 + P_1^{-1} + P_2^0 z^2 + P_1^{-1} z^3 + z^4. \tag{ii}$$

Using the identity (2.2), we have $P_1^{-1} = 0$. With (ii), we see that $1 + P_1^0 z + z^2$ must have a double root. It implies that $(P_1^0)^2 = 4$. Identifying the coefficients in (ii) gives $P_2^0 = -2$ and $f(z) = \frac{z}{(1-\epsilon z)^2}$, $\epsilon = 1$ or $\epsilon = -1$. \square

For the Koebe function, we prove $K_{n-j}^{-n} = K_{n+j}^{-n}$ in the same way and then apply (2.2)(ix).

Remark 2.5. Eq. (2.21) has other solutions than the Koebe function, but (2.21) and $K_{1-j}^{-1} = K_{1+j}^{-1}, \forall j \geq 1$ or equivalently $K_2^{-1} = 1, K_n^{-1} = 0, \forall n \geq 3$, implies that $f(z) = \frac{z}{(1-\epsilon z)^2}$, $\epsilon = +1, -1$. According to (4.8) below, the condition $K_n^{-1} = 0$ for $n \geq 3$ means that $\frac{\partial}{\partial b_1} F_n = 0$ for $n \geq 4$.

Remark 2.6. When we write the expressions (E)₁, (E)₂, ... of (P_n^k) in the case of the Koebe function, with (2.2)(iii), (iv), (x) we obtain relations between the binomial coefficients.

3. The composition of maps

We consider the polynomials

$$\begin{aligned} (b_1, b_1, \dots, b_n) &\rightarrow F_n(b_1, b_1, \dots, b_n) \\ (b_1, b_1, \dots, b_n) &\rightarrow G_n(b_1, b_1, \dots, b_n) \end{aligned} \quad n \geq 1,$$

as functions of (b_1, b_2, \dots, b_n) and we take composition of maps. We denote $F_n(G_1, \dots, G_n)$ the composition of maps

$$(b_1, b_2, \dots, b_n) \rightarrow F_n(G_1(b_1), G_2(b_1, b_2), \dots, G_n(b_1, b_2, \dots, b_n)).$$

In the same way, $G_n(G_1, \dots, G_n)$ is the composition of maps

$$(b_1, b_2, \dots, b_n) \rightarrow G_n(G_1(b_1), G_2(b_1, b_2), \dots, G_n(b_1, b_2, \dots, b_n))$$

and $G_n(F_1, \dots, F_n)$ is the composition of maps

$$(b_1, b_2, \dots, b_n) \rightarrow G_n(F_1(b_1), F_2(b_1, b_2), \dots, F_n(b_1, b_2, \dots, b_n)).$$

For example, we have $F_1(G_1)(b_1) = F_1(-b_1) = b_1, \dots$

$$G_1(F_1)(b_1) = b_1,$$

$$G_2(F_1, F_2)(b_1, b_2) = F_1^2 - F_2 = 2b_2,$$

$$G_3(F_1, F_2, F_3)(b_1, b_2, b_3) = b_1b_2 + 3b_3,$$

$$G_4(F_1, F_2, F_3, F_4) = 2b_2^2 + 2b_1b_3 + 4b_4,$$

$$G_5(F_1, F_2, F_3, F_4, F_5) = b_1b_2^2 + 7b_3b_2 + 3b_1b_4 + 5b_5,$$

$$G_6(F_1, F_2, F_3, F_4, F_5, F_6) = 4b_1b_2b_3 + 2b_2^3 + 6b_3^2 + 10b_2b_4 + 4b_1b_5 + 6b_6,$$

$$G_7(F_1, F_2, F_3, F_4, F_5, F_6, F_7)$$

$$= 17b_3b_4 + b_1b_2^3 + 13b_2b_5 + 11b_2^2b_3 + 4b_1b_3^2 + 5b_1b_6 + 6b_1b_2b_4 + 7b_7,$$

$$G_8(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8)$$

$$= 6b_3b_1b_2^2 + 8b_5b_1b_2 + 12b_4b_1b_3 + 2b_2^4 + 12b_4^2 + 16b_4b_2^2 + 16b_6b_2$$

$$+ 22b_5b_3 + 20b_3^2b_2 + 6b_1b_7 + 8b_8.$$

Proposition 3.1. For $n \geq 1$,

$$G_n(G_1, G_2, \dots, G_n) = b_n, \tag{3.1}$$

$$F_n(b_1, b_2, \dots, b_n) + F_n(G_1, G_2, \dots, G_n) = 0. \tag{3.2}$$

Proof of (3.1) and (3.2). For (3.1), let $\tilde{h}(w) = \frac{1}{h(w)} = \sum_{n \geq 0} G_n w^n$, then $\frac{1}{\tilde{h}(w)} = h(w)$. In particular, writing $G_2(G_1, G_2) = b_2$ and $G_3(G_1, G_2, G_3) = b_3$, we obtain $G_1^2 - G_2 = b_2$ and $-G_1^3 + 2G_1G_2 - G_3 = b_3, \dots$. The second relation is an immediate consequence of (1.23) and (1.24). \square

Remark 3.1. 1. The second formula (3.2) can also be proved by recurrence: it is true for $n = 1$. We shall use the two recursion formulae (2.5) and (2.3),

$$F_n + b_1F_{n-1} + b_2F_{n-2} + \dots + b_{n-1}F_1 + nb_n = 0, \tag{3.3}$$

$$F_n + G_1F_{n-1} + G_2F_{n-2} + \dots + G_{n-1}F_1 - nG_n = 0.$$

From the first relation, we have $F_2(G_1, G_2) + G_1F_1(G_1) + 2G_2 = 0$ and from the second relation $F_2(b_1, b_2) + G_1F_1(b_1) - 2G_2 = 0$. Using that (3.2) is true for $n = 1$ and adding the two relations above, we obtain (3.2) for $n = 2$ and the formula by recurrence on n .

2. We can also find this formula with the recursion formula for the $(G_n)_{n \geq 0}$,

$$G_1 + b_1 = 0, \quad G_2 + b_1 G_1 + b_2 = 0, \quad G_3 + b_1 G_2 + b_2 G_1 + b_3 = 0 \tag{i}$$

and in general, we have $G_n + b_1 G_{n-1} + b_2 G_{n-2} + \dots + b_{n-1} G_1 + b_n = 0$ as follows. From (2.4) $F_2(b_1, b_2) = -(b_1 G_1 + 2b_2)$ and $F_2(G_1, G_2) = -(b_1 G_1 + 2G_2)$. Adding and using (i) gives the result. We proceed in the same way for F_n .

3. Another proof of (3.2) is to show that $G = F^{-1} \circ S \circ F$ as follows. From (1.23)–(1.24), we have the map $\phi : h \rightarrow u = 1 - z \frac{h'}{h}$

$$(b_1, b_2, \dots, b_n, \dots) \rightarrow \left(F_1(b_1), \frac{F_2(b_1, b_2)}{2}, \dots, \frac{F_j(b_1, b_2, \dots, b_j)}{j}, \dots \right).$$

Its inverse map gives the Schur polynomials. We calculate h from u with the relation $h(z) = \exp(\frac{1-u(z)}{z})$. The map

$$F : (b_1, b_2, \dots, b_n, \dots) \rightarrow (F_1(b_1), F_2(b_1, b_2), \dots, F_j(b_1, \dots, b_j), \dots)$$

is a bijection. The map $S : 1 - z \frac{h'}{h} \rightarrow 1 + z \frac{\tilde{h}'}{\tilde{h}} = 1 - z \frac{\tilde{h}'}{\tilde{h}}$ with $\tilde{h} = \frac{1}{h}$ is also a bijection. Then the map $\phi^{-1} \circ S \circ \phi$ is just $h \rightarrow \tilde{h} = \frac{1}{h}$. This gives (1.26). To calculate F_n^{-1} , we have to solve the system in (b_1, b_2, \dots, b_n) ,

$$F_1(b_1) = c_1, \quad F_2(b_1, b_2) = c_2, \quad F_n(b_1, b_2, \dots, b_n) = c_n, \quad \dots \tag{3.4}$$

Proof of (1.31)–(1.32). For $p \geq 1$, we consider the map $\phi_p : h \rightarrow u = 1 - pz \frac{h'}{h}$ which allows us to calculate h^p . \square

More identities similar to (3.1) and (3.2) can be found.

Theorem 3.2.

$$F_n(-F_1(b_1), -F_2(b_1, b_2), \dots, -F_k(b_1, b_2, \dots, b_k), -F_n(b_1, b_2, \dots, b_n)) = F_n(2b_1, 3b_2, 4b_3, \dots, (n+1)b_n) - F_n(b_1, b_2, b_3, \dots, b_n), \tag{3.5}$$

$$F_n(F_1, F_2, \dots, F_n) = F_n(0, -b_2, -2b_3, -3b_4, \dots, -(n-1)b_n) - F_n(b_1, b_2, \dots, b_n), \tag{3.6}$$

$$G_n(-F_1, -F_2, \dots, -F_n) = \sum_{k=0}^n b_k G_{n-k}(2b_1, 3b_2, 4b_3, \dots, (j+1)b_j, \dots), \tag{3.7}$$

$$G_n(F_1, F_2, \dots, F_n) = \sum_{k=0}^n b_k G_{n-k}(0, -b_2, -2b_3, \dots, -(j-1)b_j, \dots). \tag{3.8}$$

For $p \in \mathbb{Z}$, $p \neq 0$,

$$K_n^p(-F_1, -F_2, \dots, -F_n) = \sum_{k=0}^n K_{n-k}^{-p}(b_1, b_2, \dots, b_n) \times K_k^p(2b_1, 3b_2, \dots, (j+1)b_j, \dots), \tag{3.9}$$

$$\begin{aligned}
 K_n^p(F_1, F_2, \dots, F_n) &= \sum_{k=0}^n K_{n-k}^{-p}(b_1, b_2, \dots, b_n) \\
 &\quad \times K_k^p(0, -b_2, -2b_3, \dots, -(j-1)b_j, \dots).
 \end{aligned}
 \tag{3.10}$$

Remark 3.2. Consider the maps D^1 and D^{-1} from \mathcal{M} to \mathcal{M} ,

$$D^1 : (b_1, b_2, \dots, b_k, \dots) \rightarrow (2b_1, 3b_2, 4b_3, \dots, (n+1)b_n, \dots),$$

$$D^{-1} : (b_1, b_2, \dots, b_k, \dots) \rightarrow (0, b_2, 2b_3, \dots, (n-1)b_n, \dots),$$

then (3.5)–(3.7) can be written as $F \circ S \circ F = F \circ D^1 - F$ and $F \circ F = F \circ S \circ D^{-1}$. Remark that $K^p, F, G,$ and S are bijection while D^{-1} is not.

Proof of Theorem 3.2. To prove (3.5), we consider $f(z) = zh(z)$.

$$\frac{\left(\frac{zf'}{f}\right)'}{\frac{zf'}{f}} = - \sum_{k \geq 1} F_k(-F_1, -F_2, \dots, -F_k)z^{k-1}.
 \tag{i}$$

On the other hand

$$\begin{aligned}
 \frac{d}{dz} \log\left(\frac{zf'}{f}\right) &= \frac{1}{z} + \frac{\left(\frac{f'}{f}\right)'}{\frac{f'}{f}} = \frac{1}{z} + \left(\frac{f''}{f} - \frac{(f')^2}{f^2}\right) \times \frac{f}{f'} \\
 &= \frac{1}{z} + \frac{f''}{f'} - \frac{f'}{f} = \frac{1}{z} - \frac{f'}{f} - \sum_{k \geq 1} F_k(2b_1, 3b_2, \dots, (k+1)b_k)z^{k-1}.
 \end{aligned}$$

Using the expression of $\frac{1}{z} - \frac{f'}{f}$, we deduce

$$\frac{\left(\frac{zf'}{f}\right)'}{\frac{zf'}{f}} = \sum_{k \geq 1} F_k(b_1, b_2, \dots, b_k)z^{k-1} - \sum_{k \geq 1} F_k(2b_1, 3b_2, \dots, (k+1)b_k)z^{k-1}.
 \tag{ii}$$

The comparison of (i) and (ii) gives (3.5).

For (3.6), let $g(z) = \frac{z}{h(z)}$. Then $u(z) = \frac{zg'(z)}{g(z)} = 1 + \sum_{n \geq 1} F_n(b_1, b_2, \dots, b_n)z^n$ satisfies

$$\frac{u'(z)}{u(z)} = - \sum_{k \geq 1} F_k(F_1(b_1), F_2(b_1, b_2), \dots, F_k(b_1, b_2, \dots, b_k))z^{k-1}.
 \tag{i}$$

Thus we obtain $F_k(F_1, F_2, \dots, F_k)$ from this expansion. On the other hand, we calculate

$$\frac{u'(z)}{u(z)} = \frac{d}{dz} \log(u(z)) = \frac{1}{z} \left(- \sum_{k \geq 1} F_k z^k + z \frac{g''(z)}{g'(z)} \right).$$

Since $g'(z) = \frac{h(z) - zh'(z)}{h(z)^2}$, we obtain

$$z \frac{g''(z)}{g'(z)} = z \frac{(h - zh')'}{h - zh'} - 2z \frac{h'}{h} = z \frac{(h - zh')'}{h - zh'} + 2 \sum_{k \geq 1} F_k z^k.$$

Using $h(z) - zh'(z) = 1 + \sum_{n \geq 2} (1 - n)b_n z^n$, we deduce

$$\frac{u'(z)}{u(z)} = \frac{1}{z} \left(\sum_{k \geq 1} F_k z^k - \sum_{k \geq 1} F_k(0, -b_2, -2b_3, \dots, -(k-1)b_k) z^k \right). \tag{ii}$$

Then we compare the two identities (i) and (ii).

To prove (3.8), we write $\frac{h(w)}{h(w) - wh'(w)}$ in two different ways,

$$\begin{aligned} \frac{1}{1 - \frac{wh'(w)}{h(w)}} &= \sum_{n \geq 0} G_n(F_1(b_1), F_2(b_1, b_2), \dots, F_n(b_1, b_2, \dots, b_n)) w^n, \\ \frac{h(w)}{h(w) - wh'(w)} &= (1 + b_1 w + b_2 w^2 + \dots) \times \sum_{n \geq 0} G_n(0, -b_2, \dots, -(n-1)b_n, \dots) w^n \end{aligned}$$

since

$$\begin{aligned} h(w) - wh'(w) &= (1 + b_1 w + b_2 w^2 + \dots + b_n w^n + \dots) \\ &\quad - (b_1 w + 2b_2 w^2 + \dots + nb_n w^n + \dots) \\ &= 1 - b_2 w^2 - 2b_3 w^3 - \dots - (n-1)b_n w^n - \dots. \end{aligned}$$

To prove (3.7), we consider $\frac{h(w)}{h(w) + wh'(w)}$. To prove (3.9),

$$\left(1 + z \frac{h'(z)}{h(z)} \right)^p = \left(1 - \sum_{k \geq 1} F_k z^k \right)^p = \sum_{n \geq 0} K_n^p(-F_1, -F_2, \dots, -F_n) z^n.$$

This is also equal to $\frac{(h(z) + zh'(z))^p}{h(z)^p}$. For (3.10), we take $(1 - z \frac{h'(z)}{h(z)})^p$. \square

Remark 3.3. With (3.7)–(3.10) we define differential operators on Faber polynomials. For (3.7)–(3.8), we have (see (1.16)) $G_{n-k} = -\frac{1}{n} \frac{\partial}{\partial b_k} F_n$, thus

$$\begin{aligned} G_n(-F_1, -F_2, \dots, -F_n) \\ = -\frac{1}{n+1} \left[\sum_{k \geq 0} \frac{b_k}{k+1} \frac{\partial F_{n+1}}{\partial b_{k+1}} \right] (2b_1, 3b_2, 4b_3, \dots, (n+1)b_n). \end{aligned} \tag{3.11}$$

For (3.9), we take for example $p = 2$. See Proposition 2.4. With (1.16), we deduce that

$$K_{n-k}^{-2} = \frac{1}{n+1} \frac{\partial}{\partial b_k} \left(\frac{\partial}{\partial b_1} F_{n+1} \right) = \frac{1}{n+2} \frac{\partial}{\partial b_k} \left(\frac{\partial}{\partial b_2} F_{n+2} \right)$$

and

$$\begin{aligned} K_n^2(-F_1, -F_2, \dots, -F_n) &= \frac{1}{n+1} U_2 \left(\frac{\partial}{\partial b_1} F_{n+1} \right) + K_n^{-2}(b_1, \dots, b_n) \\ &= \frac{1}{n+2} U_2 \left(\frac{\partial}{\partial b_1} F_{n+2} \right) + \frac{1}{n+2} \frac{\partial^2 F_{n+2}}{\partial b_1^2}, \end{aligned} \tag{3.12}$$

where U_2 is the differential operator

$$\begin{aligned} U_2 &= \sum_{k \geq 1} K_k^2(2b_1, 3b_2, \dots, (k+1)b_k) \frac{\partial}{\partial b_k} \\ &= 4b_1 \frac{\partial}{\partial b_1} + (6b_2 + 4b_1^2) \frac{\partial}{\partial b_2} + (8b_3 + 12b_1 b_2) \frac{\partial}{\partial b_3} + \dots. \end{aligned} \tag{3.13}$$

We have

$$U_2 \frac{\partial}{\partial b_1} - \frac{\partial}{\partial b_1} U_2 = -4L_1 \tag{3.14}$$

where L_1 is given by (1.13). In the same way, in (3.10), we put

$$\begin{aligned} T_2 &= \sum_{k \geq 1} K_k^2(0, -b_2, -2b_3, \dots, -(k-1)b_k) \frac{\partial}{\partial b_k} \\ &= -2b_2 \frac{\partial}{\partial b_2} - 4b_3 \frac{\partial}{\partial b_3} + (b_2^2 - 6b_4) \frac{\partial}{\partial b_4} + (4b_2b_3 - 8b_5) \frac{\partial}{\partial b_5} + \dots \end{aligned} \tag{3.15}$$

Then $T_2[h(z)] = (h(z) - zh'(z))^2 - 1$. We have

$$\begin{aligned} K_n^2(F_1, F_2, \dots, F_n) &= \frac{1}{n+1} T_2 \frac{\partial}{\partial b_1} F_{n+1} + K_n^{-2}(b_1, \dots, b_n) \\ &= \frac{1}{n+1} \frac{\partial}{\partial b_1} T_2 F_{n+1} + K_n^{-2}(b_1, b_2, \dots, b_n) \\ &= \frac{1}{n+2} T_2 \frac{\partial}{\partial b_2} F_{n+2} + \frac{1}{n+2} \frac{\partial^2 F_{n+2}}{\partial b_1^2}. \end{aligned} \tag{3.16}$$

By (2.18), $K_n^2(-F_1, -F_2, \dots) - K_n^2(F_1, F_2, \dots) = -4F_n = -\frac{4}{n+1} L_1(F_{n+1})$, see (7.11). With (3.14), it gives $\frac{\partial}{\partial b_1}(U_2 - T_2)F_n = 0$, thus $(U_2 - T_2)F_n$ does not depend on b_1 for $n \geq 2$. For $p \geq 1$, $T_2 \frac{\partial}{\partial b_p} - \frac{\partial}{\partial b_p} T_2 = -2(p-1) \sum_{k \geq p} (k-p-1)b_{k-p} \frac{\partial}{\partial b_k}$

$$T_2 \frac{\partial}{\partial b_p} - \frac{\partial}{\partial b_p} T_2 = -2(p-1) \sum_{n \geq 0} (n-1)b_n \frac{\partial}{\partial b_{p+n}} = -2(p-1)(L_p + 2W_p). \tag{3.17}$$

Corollary 3.3. Let $(\mathcal{P}_k)_{k \geq 2}$ be the coefficients of the Schwarzian derivative as in (C1), and let

$$\mathcal{H} = T_2 \frac{\partial}{\partial b_2} + \frac{\partial^2}{\partial b_1^2} \tag{3.18}$$

then $\mathcal{P}_k(b_1, b_2, b_3, \dots, b_k) + (k-2)F_k(2b_1, 3b_2, \dots, (k+1)b_k)$ is equal to

$$-\frac{1}{2(k+2)} [\mathcal{H}F_{k+2}](2b_1, 3b_2, \dots, (j+1)b_j, \dots). \tag{3.19}$$

Corollary 3.4. Let $\mathcal{T} = \frac{\partial^2}{\partial b_1^2} + T_2 \frac{\partial}{\partial b_2} + (\frac{\partial}{\partial b_2} - b_1 \frac{\partial}{\partial b_3} + \frac{\partial}{\partial b_4})$. The condition (2.21) is equivalent to

$$(\mathcal{T}F_n)(G_1, G_2, \dots, G_{n-2}) = 0 \quad \forall n \geq 4. \tag{3.20}$$

Proof. The condition (2.21) is the same as

$$K_n^2(-F_1, -F_2, \dots) = b_n + b_1 b_{n-1} + b_{n-2} \quad \forall n \geq 1. \tag{3.21}$$

From (1.26), we have $K^p \circ S \circ F = K^p \circ F \circ G$. Thus

$$K_n^2(-F_1, -F_2, \dots) = \frac{1}{n+2} \left(T_2 \frac{\partial}{\partial b_2} F_{n+2} \right) \circ G + K_n^{-2} \circ G. \tag{3.22}$$

Since $G \circ G = \text{Identity}$ and $G_1(b_1) = -b_1$, we can write the right side in (3.21) as $(G_n - b_1 G_{n-1} + G_{n-2})$ at the point (G_1, G_2, \dots, G_n) . With (1.16), we have

$$G_n - b_1 G_{n-1} + G_{n-2} = -\frac{1}{n+2} \left(\frac{\partial}{\partial b_2} - b_1 \frac{\partial}{\partial b_3} + \frac{\partial}{\partial b_4} \right) F_{n+2}. \tag{3.23}$$

Then (3.22) and (3.23) imply (3.20). The condition (3.20) is always satisfied for $n = 1, 2, 3$. For $n = 4$, it gives $3b_2 - 2b_1^2 = 1$, for $n = 5$, $5b_3 - 3b_1 b_2 - b_1 = 0$. \square

4. The polynomials and their derivatives. Proof of the Main Theorem

4.1. The partial derivatives $(\frac{\partial}{\partial b_k})_{k \geq 1}$

Theorem 4.1. For $p \geq 1, n \geq 0$,

$$(n+p)G_n = -\frac{\partial F_{n+p}}{\partial b_p}. \tag{4.1}$$

In particular, $\frac{\partial F_p}{\partial b_p} = -p$, and $\frac{\partial F_n}{\partial b_k} = -nG_{n-k}$ if $k \leq n$. Let $(F_n^{-1})_{n \geq 1}$ be the inverse Faber polynomials, then

$$\frac{\partial}{\partial b_k} F_p^{-1} = -\frac{1}{k} F_{p-k}^{-1} \times 1_{k \leq p}. \tag{4.2}$$

Proof. Let $\psi(w) = w + b_1 + \frac{b_2}{w} + \frac{b_3}{w^2} + \dots + \frac{b_p}{w^{p-1}} + \dots$ and

$$\psi_p(w) = \psi(w) - \frac{t}{w^{p-1}} \quad p \geq 1.$$

We have $w^p \psi'_p(w) = w^p \psi'(w) + (p-1)t$,

$$\frac{w \psi'_p(w)}{\psi_p(w)} = 1 + \sum_{n \geq 1} F_n(b_1, \dots, b_{p-1}, b_p - t, b_{p+1}, \dots) \times \frac{1}{w^n}.$$

We differentiate this equation with respect to t and we make $t = 0$,

$$\phi(w) = \frac{d}{dt} \Big|_{t=0} \frac{w \psi'_p(w)}{\psi_p(w)} = \sum_{n \geq 1} \frac{\partial F_n}{\partial b_p} \times \frac{1}{w^n}.$$

On the other hand

$$\frac{w \psi'_p(w)}{\psi_p(w)} = \frac{w^p \psi'(w) + (p-1)t}{w^{p-1} \psi(w) - t}.$$

We calculate ϕ with this expression

$$\frac{d}{dt} \frac{w \psi'_p(w)}{\psi_p(w)} = w \left[\frac{(p-1)w^{p-2} \psi(w) + w^{p-1} \psi'(w)}{(w^{p-1} \psi(w) - t)^2} \right] = -w \frac{d}{dw} \frac{1}{(w^{p-1} \psi(w) - t)}.$$

At $t = 0$,

$$\phi(w) = -w \frac{d}{dw} \left(\frac{1}{w^{p-1} \psi(w)} \right) = -w \frac{d}{dw} \sum_{n \geq 0} G_n \times \frac{1}{w^{n+p}} = \sum_{n \geq 0} G_n (n+p) \frac{1}{w^{n+p}}.$$

Comparing the two expressions of ϕ and since F_n does not contain b_p when $n < p$, we obtain the result. To calculate the derivatives of the map F^{-1} , we take

$$h(z) = 1 + \sum_{j \geq 1} F_j^{-1}(b_1, b_2, b_3, \dots, b_j) z^j$$

since $F \circ F^{-1} = \text{Identity}$, we have $\frac{d}{dz} \log(h(z)) = -\sum_{k \geq 1} b_k z^{k-1}$. We differentiate with respect to b_k , for $k \geq 1$,

$$-z^{k-1} = \frac{\partial}{\partial b_k} \left(\frac{h'(z)}{h(z)} \right) = \frac{d}{dz} \left(\frac{\frac{\partial}{\partial b_k} h(z)}{h(z)} \right).$$

We integrate this identity with respect to z ,

$$-\frac{1}{k} z^k = \frac{\frac{\partial}{\partial b_k} h(z)}{h(z)} + C(b_1, b_2, \dots)$$

where $C(b_1, b_2, \dots)$ is constant in z . Making $z = 0$, we see that $C = 0$. thus

$$-\frac{1}{k} z^k \times h(z) = \frac{\partial}{\partial b_k} h(z) = \sum_{j \geq 1} \left(\frac{\partial}{\partial b_k} F_j^{-1} \right) z^j.$$

Since $-\frac{1}{k} z^k \times h(z) = -\frac{1}{k} z^k (1 + \sum_{j \geq 1} F_j^{-1} z^j)$, we obtain the partial derivatives of F^{-1} . \square

Corollary 4.2. For $n \geq 0, p \geq 1$,

$$\frac{\partial F_{n+p}}{\partial b_p}(G_1(b_1), G_2(b_1, b_2), \dots, G_n(b_1, b_2, \dots, b_n)) = -(n + p)b_n. \tag{4.3}$$

Proof. Since $G \circ G = \text{Identity}$, it is a consequence of (4.1). \square

Corollary 4.3.

$$\frac{\partial G_n}{\partial b_k} = -K_{n-k}^{-2} \times 1_{n \geq k}. \tag{4.4}$$

Proof. We differentiate $G = F^{-1} \circ S \circ F$.

$$\frac{\partial G_n}{\partial b_k} = \sum_{j \geq 1} \frac{\partial F_n^{-1}}{\partial b_j} (S \circ F) \times \left(-\frac{\partial F_j}{\partial b_k} \right) = \sum_{1 \leq j \leq n} \left(-\frac{1}{j} F_{n-j}^{-1} (S \circ F) \right) \times (j G_{j-k}).$$

After simplification by j , and since $F^{-1} \circ S \circ F = G$, we find

$$\frac{\partial G_n}{\partial b_k} = - \sum_{1 \leq j \leq n} G_{n-j} G_{j-k} = -K_{n-k}^{-2}. \quad \square$$

The following operators up to a minus sign, $(Z_k)_{k \geq 0}$ were introduced in [2].

Corollary 4.4. With the recursion $F_{j+1} = -\sum_{0 \leq r \leq j} (r + 1)b_{r+1}G_{j-r}$, see (2.4), for $k \geq 0$, we deduce

$$Z_k = \sum_{r \geq 0} (r + 1)b_{r+1} \frac{\partial}{\partial b_{r+k+1}} \quad \text{and} \quad (j + k + 1)F_{j+1} = Z_k F_{j+k+1}.$$

Proof. From Theorem 4.1, $(j + k + 1)G_{j-r} = -\frac{\partial}{\partial b_{r+k+1}} F_{j+k+1}$. Thus, if $k \geq 0$, with the recursion formula (2.4) where we replace G_{j-r} , we find

$$(j + k + 1)F_{j+1} = \sum_{0 \leq r \leq j} (r + 1)b_{r+1} \frac{\partial}{\partial b_{r+k+1}} F_{j+k+1} = Z_k(F_{j+k+1}).$$

For $k < 0$, then $\frac{\partial}{\partial b_{r+k+1}}$ is defined only if $r + k \geq 0$, i.e. $r \geq -k$. We decompose the sum $F_{j+1} = -\sum_{0 \leq r < -k} (r + 1)b_{r+1}G_{j-r} - \sum_{-k \leq r \leq j} (r + 1)b_{r+1}G_{j-r}$. \square

4.2. Proof of the main results

Proof of the Main Theorem.

$$\frac{\partial}{\partial b_k} \left(\frac{1}{h(z)} \right) = -\frac{z^k}{h(z)^2} = -z^k - \sum_{n \geq 1} K_n^{-2} z^{n+k}.$$

On the other hand, $\frac{\partial}{\partial b_k} \left(\frac{1}{h(z)} \right) = \sum_{n \geq 1} \frac{\partial}{\partial b_k} G_n z^n$. In these two last expressions, we identify the coefficients of equal power of z . It gives (1.17). We have $(n + p)G_n = -\frac{\partial F_{n+p}}{\partial b_p}$, then differentiating this expression with respect to k , we obtain

$$(p + k + n)K_n^{-2} = \frac{\partial^2 F_{p+k+n}}{\partial b_k \partial b_p} \quad \forall p \geq 1, \forall k \geq 1.$$

We deduce higher order partial derivatives of F_j from

$$\frac{\partial K_n^{-p}}{\partial b_k} = -pK_{n-k}^{-(p+1)} \times 1_{n \geq k} \quad \text{for } n \geq 1, k \geq 1, p \neq 0, p \in \mathbb{Z}. \tag{4.5}$$

$K_0^p = 1$ for any p . The proof of (4.5) or equivalently $\frac{\partial K_n^p}{\partial b_k} = pK_{n-k}^{p-1} \times 1_{n \geq k}$ is as follows,

$$\frac{\partial}{\partial b_k} \left(\frac{1}{h(z)^p} \right) = \sum_{n \geq 1} \frac{\partial}{\partial b_k} (K_n^{-p}) z^n. \tag{i}$$

On the other hand

$$\begin{aligned} \frac{\partial}{\partial b_k} \left(\frac{1}{h(z)^p} \right) &= \frac{-p \frac{\partial}{\partial b_k} (h(z))}{h(z)^{p+1}} = \frac{-pz^k}{h(z)^{p+1}} = -pz^k \sum_{n \geq 1} K_n^{-(p+1)} z^n \\ &= -p \sum_{q \geq 1} K_q^{-(p+1)} z^{q+k}. \end{aligned} \tag{ii}$$

The identification of the coefficients of z^q in the two expressions (i) and (ii) gives (4.5). We see that one can calculate as derivatives of Faber polynomials all the $(K_n^{-p})_{n \geq 1}$ for $p \geq 1$. \square

Proof of (T2). We wish to calculate K_n^p for $p \geq 2$. Let $\tilde{h}(z) = \frac{1}{h(z)} = 1 + G_1z + G_2z^2 + \dots + G_nz^n + \dots$. Then (T2) is obtained with the identification of coefficients of equal powers of z in $\tilde{h}(z)^{-p} = 1 + \sum_{n \geq 1} K_n^{-p}(G_1, G_2, \dots, G_n)z^n = h(z)^p$ with $h(z)^p = 1 + \sum_{n \geq 1} K_n^p(b_1, b_2, \dots, b_n)z^n$. We can also give a proof with the composition of maps $K_n^p = F^{-1} \circ pI \circ F$ and

$$K_n^{-p} \circ G = F^{-1} \circ pS \circ F \circ F^{-1} \circ S \circ F = F^{-1} \circ pI \circ F = K_n^p. \quad \square$$

Corollary 4.5. All the K_n^p , $n \geq 1$, $p \in \mathbb{Z}$ can be obtained as derivatives of Faber polynomials. For $p \geq 1$,

$$\begin{aligned} & (-1)^p (p-1)! (n+k_1+k_2+\dots+k_p) K_n^{-p}(b_1, b_2, \dots, b_n) \\ &= \frac{\partial^p F_{n+k_1+\dots+k_p}}{\partial b_{k_1} \partial b_{k_2} \dots \partial b_{k_p}}(b_1, b_2, \dots, b_n, \dots, b_q, \dots). \end{aligned} \tag{4.6}$$

Let

$$\phi(b_1, b_2, \dots, b_n, \dots, b_q, \dots) = \frac{\partial^p F_{n+k_1+\dots+k_p}}{\partial b_{k_1} \partial b_{k_2} \dots \partial b_{k_p}}(b_1, b_2, \dots, b_n, \dots, b_q, \dots),$$

for $p \geq 1$, we have

$$\begin{aligned} & (-1)^p (p-1)! (n+k_1+k_2+\dots+k_p) K_n^p(b_1, b_2, \dots, b_n) \\ &= \phi(G_1(b_1, b_2, \dots), G_2(b_1, b_2, \dots), \dots, G_q(b_1, b_2, \dots), \dots). \end{aligned} \tag{4.7}$$

Corollary 4.6. For $p \geq 1$,

$$(-1)^p (p-1)! (n+p) K_n^{-p}(b_1, b_2, \dots, b_n) = \frac{\partial^p F_{n+p}}{\partial b_1^p}(b_1, b_2, \dots, b_n, \dots, b_{n+p}), \tag{4.8}$$

$$\begin{aligned} & (-1)^p (p-1)! (n+p) K_n^p(b_1, b_2, \dots, b_n) \\ &= \frac{\partial^p F_{n+p}}{\partial b_1^p}(G_1(b), G_2(b_1, b_2), \dots, G_{n+p}(b_1, b_2, \dots, b_{n+p})). \end{aligned} \tag{4.9}$$

In particular

$$K_n^{-(n+1)} = \frac{(-1)^{n+1}}{n!(2n+1)} \frac{\partial^{n+1} F_{2n+1}}{\partial b_1^{n+1}}(b_1, b_2, \dots, b_q, \dots), \tag{4.10}$$

$$K_{n+1}^n = \frac{(-1)^n}{(n-1)!(2n+1)} \left(\frac{\partial^n F_{2n+1}}{\partial b_1^n} \right) (G_1(b_1), G_2(b_1, b_2), \dots, G_q(b_1, b_2, \dots, b_q)). \tag{4.11}$$

Proof of the Main Corollary. Let $f(z) = z + b_1z^2 + b_2z^3 + \dots + b_nz^{n+1} + \dots$.

$$\begin{aligned} \frac{f''(z)}{f'(z)} &= - \sum_{k \geq 1} F_k(2b_1, 3b_2, \dots, (k+1)b_k) z^{k-1}, \\ \left(\frac{f''(z)}{f'(z)} \right)' &= - \sum_{k \geq 2} (k-1) F_k(2b_1, 3b_2, \dots, (k+1)b_k) z^{k-2}. \end{aligned} \tag{i}$$

On the other hand

$$\begin{aligned} -\frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 &= -\frac{1}{2z^2} \left(\frac{zf''(z)}{f'(z)} \right)^2 \\ &= -\frac{1}{2z^2} \left(\sum_{k \geq 1} F_k(2b_1, 3b_2, \dots, (k+1)b_k) z^k + 1 - 1 \right)^2 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{2z^2} \left(1 + \sum_{k \geq 1} F_k(2b_1, 3b_2, \dots) z^k \right)^2 + \frac{1}{z^2} \left(1 + \sum_{k \geq 1} F_k(2b_1, 3b_2, \dots) z^k \right) - \frac{1}{2z^2} \\
 &= -\frac{1}{2z^2} \sum_{k \geq 2} K_k^2(F_1(2b_1), F_2(2b_1, 3b_2), \dots) z^k + \frac{1}{z^2} \sum_{k \geq 2} F_k(2b_1, 3b_2, \dots) z^k. \tag{ii}
 \end{aligned}$$

We add the two expressions (i) and (ii) to obtain the Main Corollary. \square

Remark 4.1. Let $H_k(b_1, b_2, \dots, b_k) = F_k(2b_1, 3b_2, 4b_3, \dots, (k + 1)b_k)$. With the expressions of the $(F_n)_{n \geq 1}$ in [3], we find $H_1(b_1) = F_1(2b_1) = -2b_1$ and $H_2(b_1, b_2) = F_2(2b_1, 3b_2) = 2(2b_1^2 - 3b_2)$

$$H_3(b_1, b_2, b_3) = F_3(2b_1, 3b_2, 4b_3) = 2(-4b_1^3 + 9b_1b_2 - 6b_3),$$

$$H_4(b_1, b_2, b_3, b_4) = F_4(2b_1, 3b_2, 4b_3, 5b_4) = 2(8b_1^4 - 24b_1^2b_2 + 9b_2^2 + 16b_1b_3 - 10b_4).$$

We can calculate \mathcal{P}_k with $\mathcal{P}_k = -(k - 1)H_k - \frac{1}{2} \sum_{j=1}^{k-1} H_{k-j}H_j$ or we can use (C1).

Proof of (T3). $h(z)^p = \sum_{n \geq 0} K_n^p z^n = (1 + b_1z + b_2z^2 + \dots + b_nz^n + \dots)^p,$

$$K_n^p = \frac{1}{2i\pi} \int \frac{(1 + \phi_n(\xi) + b_{n+1}\xi^{n+1} + \dots)^p}{\xi^{n+1}} d\xi = \frac{1}{2i\pi} \int \frac{(1 + \phi_n(\xi))^p}{\xi^{n+1}} d\xi$$

and we write Newton binomial formula $(1 + \phi_n(\xi))^p = 1 + p\phi_n(\xi) + \frac{p(p-1)}{2}\phi_n(\xi)^2 + \dots$. \square

Proof of (T4). If $b_1 \neq 0$, then $\phi_n(z) = b_1z + b_2z^2 + b_3z^3 + \dots + b_nz^n = b_1z(1 + \frac{b_2}{b_1}z + \dots + \frac{b_n}{b_1}z^{n-1})$. Thus

$$(\phi_n(z))^k = b_1^k z^k \left(1 + \frac{b_2}{b_1}z + \dots + \frac{b_n}{b_1}z^{n-1} \right)^k = b_1^k z^k \sum_{j \geq 1} K_j^k \left(\frac{b_2}{b_1}, \dots, \frac{b_{j+1}}{b_1} \right) z^j.$$

The coefficient of z^n in this expression is obtained for $j + k = n$. This gives (T4). We can obtain the exact expression of K_n^p as follows,

$$K_n^p = \sum_{1 \leq k_1 \leq n} C_{k_1}^p D_n^{k_1} = \sum_{1 \leq k_1 \leq n} C_{k_1}^p b_1^{k_1} \sum_{1 \leq k_2 \leq n-k_1} C_{k_2}^{k_1} \left(\frac{b_2}{b_1} \right)^{k_2} D_{n-k_1}^{k_2} \left(\frac{b_3}{b_2}, \dots \right).$$

If $b_1 \neq 0$,

$$K_1^p = C_1^p b_1, \quad K_2^p = C_1^p b_2 + C_2^p b_1^2, \quad K_3^p = C_1^p b_3 + C_2^p b_1^2 K_1^2 \left(\frac{b_2}{b_1} \right) + C_3^p b_1^3,$$

$$K_4^p = C_1^p b_4 + C_2^p b_1^2 K_2^2 \left(\frac{b_2}{b_1}, \frac{b_3}{b_1} \right) + C_3^p b_1^3 K_1^3 \left(\frac{b_2}{b_1} \right) + C_4^p b_1^4,$$

$$K_5^p = C_1^p b_5 + C_2^p b_1^2 K_3^2 \left(\frac{b_2}{b_1}, \frac{b_3}{b_1}, \frac{b_4}{b_1} \right) + C_3^p b_1^3 K_2^3 \left(\frac{b_2}{b_1}, \frac{b_3}{b_1} \right) + C_4^p b_1^4 K_1^4 \left(\frac{b_2}{b_1} \right) + C_5^p b_1^5,$$

$$\begin{aligned}
 K_6^p &= C_1^p b_6 + C_2^p b_1^2 K_4^2 \left(\frac{b_2}{b_1}, \frac{b_3}{b_1}, \frac{b_4}{b_1}, \frac{b_5}{b_1} \right) + C_3^p b_1^3 K_3^3 \left(\frac{b_2}{b_1}, \frac{b_3}{b_1}, \frac{b_4}{b_1} \right) \\
 &\quad + C_4^p b_1^4 K_2^4 \left(\frac{b_2}{b_1}, \frac{b_3}{b_1} \right) + C_5^p b_1^5 K_1^5 \left(\frac{b_2}{b_1} \right) + C_6^p b_1^6.
 \end{aligned}$$

If $b_1 = 0$, and $b_2 \neq 0$,

$$\begin{aligned} K_2^p &= C_1^p b_2, & K_3^p &= C_1^p b_3, & K_4^p &= C_1^p b_4 + C_2^p b_2^2, \\ K_5^p &= C_1^p b_5 + C_2^p b_2 K_1^2 \left(\frac{b_3}{b_2}\right), & K_6^p &= C_1^p b_6 + C_2^p b_2^2 K_2^2 \left(\frac{b_3}{b_2}, \frac{b_4}{b_2}\right) + C_3^p b_2^3, \\ K_7^p &= C_1^p b_7 + C_2^p b_2^2 K_3^2 \left(\frac{b_3}{b_2}, \frac{b_4}{b_2}, \frac{b_5}{b_2}\right) + C_3^p b_2^3 K_1^3 \left(\frac{b_3}{b_2}\right), & \dots & \quad \square \end{aligned}$$

5. Identities related to the $(W_j)_{j \geq 1}$, the $(V_j^k)_{j \geq 1}$, $k \in \mathbb{Z}$ and the $(V_j)_{j \geq 1}$

It has been proved in [3] that

$$W_j W_q = W_q W_j \quad \text{for } j \geq 1, q \geq 1. \tag{5.1}$$

For $j \geq 1, m \geq 0$,

$$W_j F_m = m \delta_{j,m} \quad \text{and} \quad W_j G_m = G_{m-j} \tag{5.2}$$

with the convention $G_p = 0$ if $p < 0$. Moreover $K_0^p = 1$ and

$$W_j (K_n^p) = 0 \quad \text{for } n < j, \quad W_j (K_n^p) = -p K_{n-j}^p \quad \text{for } n \geq j, p \in \mathbb{Z}. \tag{5.3}$$

Proof of (5.1)–(5.3). For (5.1), we remark that $W_j W_p[h(z)] = z^{p+j} h(z)$. For the other identities, let $(b_1, \dots, b_k, \dots) \rightarrow h(z) = 1 + b_1 z + b_2 z^2 + \dots + b_k z^k + \dots$, then $\frac{\partial}{\partial b_j}[h(z)] = z^j$. It is enough to calculate $W_j[h(z)]$, use (1.1), then calculate $W_j[\frac{1}{h(z)}]$, use (1.3) by equating coefficients of similar powers of z . This is done as follows. Let

$$(b_1, b_2, \dots, b_k, \dots) \rightarrow \phi(b_1, b_2, \dots, b_k, \dots).$$

Since W_j is a differential operator, $W_j[\exp(\phi)] = \exp(\phi) \times W_j[\phi]$. With (1.1), we have $W_j[h(z)] = h(z) \times (-\sum_{k=1}^{+\infty} \frac{W_j F_k}{k} z^k)$. Comparing this last expression with $W_j[h(z)] = -z^j h(z)$, we deduce that $z^j = \sum_{k=1}^{+\infty} \frac{W_j F_k}{k} z^k$. Equating the coefficients of z^k gives $W_j(F_m)$. To obtain $W_j[G_m]$, we calculate $W_j[\frac{1}{h(z)}] = -\frac{W_j[h(z)]}{h(z)^2}$. With $W_j[h(z)] = -z^j h(z)$, it gives $W_j[\frac{1}{h(z)}] = \frac{z^j}{h(z)}$. With(1.2), it gives

$$\frac{z^j}{h(z)} = \sum_{m \geq 1} W_j G_m z^m. \tag{i}$$

In (i), we replace $\frac{1}{h(z)}$ by (1.2), thus $z^j (\sum_{m \geq 0} G_m z^m) = \sum_{m \geq 1} W_j[G_m] z^m$. In this identity, equating the coefficients of z^m gives $W_j(G_m)$. In the same way,

$$W_j[h(z)^p] = p h(z)^{p-1} W_j[h(z)] = -p z^j h(z)^p = -p z^j \left(1 + \sum_{n \geq 1} K_n^p z^n\right).$$

On the other hand, $W_j[h(z)^p] = \sum_{s \geq 1} W_j[K_s^p] z^s$. Identifying the two expressions of $W_j[h(z)^p]$, we obtain $-p z^j (1 + \sum_{n \geq 1} K_n^p z^n) = \sum_{s \geq 1} W_j[K_s^p] z^s$. Equating the coefficients of equal powers of z^j gives $W_j(K_n^p)$. \square

Theorem 5.1. *The operators $(V_j^k)_{j \geq 1}$, $k \in \mathbb{Z}$ satisfy (1.11),*

$$V_k^q V_s^p + (p + 1)V_{s+k}^{p+q} = \sum_{n \geq 0, j \geq 0} K_n^{p+1} K_j^{q+1} \frac{\partial^2}{\partial b_{n+s} \partial b_{k+j}} \tag{5.4}$$

and $V_j^k = \sum_{n \geq 0} K_n^k W_{j+n}$

$$V_j^k(F_p) = pK_{p-j}^k, \quad V_j^k(K_s^q) = -qK_{s-j}^{q+k}, \tag{5.5}$$

$$V_j^k[h(z)] = -z^j [h(z)]^{k+1}. \tag{5.6}$$

The polynomials $(P_n^k)_{n \geq 0}$, $P_0^k = 1$ (see (1.4)) satisfy

$$\begin{aligned} V_p^k(P_n^{n+j}) &= -(2k + j)P_{n-p}^{n-p+j+k} + \frac{2(k-p)(n-p+k+j)}{k+j} K_{n-p}^{k+j} \quad \text{if } k+j \neq 0, \\ V_p^{-j}(P_n^{n+j}) &= jP_{n-p}^{n-p} + 2(j+p)F_{n-p}, \\ V_k(P_n^{n+j}) &= -(2k+j)P_{n-k}^{n+j} \end{aligned} \tag{5.7}$$

and the Neretin polynomials (C1), $z^2 S(f)(z) = \sum_{k \geq 2} \mathcal{P}_k z^k$,

$$V_k(\mathcal{P}_j) = -(k^3 - k)P_{j-k}^j. \tag{5.8}$$

Proof of (5.7)–(5.8). Since $V_p[h(z)] = -z^p h(z)^{p+1}$, we deduce for $p \geq 1$,

$$V_p[f(z)] = -f(z)^{p+1}, \quad V_p\left(\frac{f'(z)}{f(z)}\right) = -f(z)^{p-1} f'(z) \tag{5.9}$$

and $V_p\left(\frac{f''(z)}{f'(z)}\right) = -p(p+1)f(z)^{p-1} f'(z)$. We obtain (5.7)–(5.8) by identification of coefficients. \square

In [3], the homogeneity operator $L_0 = b_1 \frac{\partial}{\partial b_1} + 2b_2 \frac{\partial}{\partial b_2} + \dots + kb_k \frac{\partial}{\partial b_k} + \dots$ is expressed as $L_0 = \sum_{j \geq 1} F_j W_j$.

Lemma 5.2. *We have $G_n = K_n^{-1}$, $b_n = K_n^1$,*

$$kG_k = \sum_{1 \leq j \leq k} F_j G_{k-j} \quad \text{and} \quad nK_n^p = -p \sum_{1 \leq j \leq n} F_j K_{n-j}^p. \tag{5.10}$$

Proof of (5.10). From the recurrence relation for the polynomials $(F_k)_{k \geq 0}$,

$$L_0 = \sum_{k \geq 1} kb_k \frac{\partial}{\partial b_k} = - \sum_{j \geq 1} F_j \left(\sum_{k \geq j} b_{k-j} \frac{\partial}{\partial b_k} \right) = \sum_{j \geq 1} F_j W_j.$$

K_n^p is homogeneous of degree n , thus $L_0 K_n^p = nK_n^p$. Since $L_0 K_n^p = \sum_{j \geq 1} F_j W_j K_n^p = - \sum_{1 \leq j \leq n} F_j p K_{n-j}^p$, we obtain the recursion formula for K_n^p . See (2.5), (2.7). \square

Theorem 5.3. *The $(\frac{\partial}{\partial b_j})_{j \geq 1}$ are given in terms of the $(W_j)_{j \geq 1}$ with*

$$\frac{\partial}{\partial b_j} = -W_j - \sum_{k \geq 1} G_k W_{j+k}, \tag{5.11}$$

$$X_0 = - \sum_{j \geq 1} b_j \frac{\partial}{\partial b_j} = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \dots - b_k \frac{\partial}{\partial b_k} - \dots \tag{5.12}$$

satisfies

$$X_0 = - \sum_{j \geq 1} G_j W_j. \tag{5.13}$$

In particular,

$$X_0(F_n) = -nG_n. \tag{5.14}$$

Proof. For $(\frac{\partial}{\partial b_j})_{j \geq 1}$, it is enough to verify that

$$\frac{\partial}{\partial b_j} [h(z)] = -W_j[h(z)] - \sum_{k \geq 1} G_k W_{j+k}[h(z)]. \tag{i}$$

Since $W_j[h(z)] = -z^j h(z)$, we have

$$-W_j[h(z)] - \sum_{k \geq 1} G_k W_{j+k}[h(z)] = z^j (1 + G_1 z + G_2 z^2 + \dots + G_k z^k + \dots) \times h(z).$$

Thus (i) is the same as $z^j = z^j (1 + G_1 z + G_2 z^2 + \dots + G_k z^k + \dots) \times h(z)$. It is the immediate consequence of (1.3). To prove (5.13), we see that $X_0[h(z)] = 1 - h(z)$. We write X_0 as $X_0 = \sum_{j \geq 1} H_j W_j$. Applied to $h(z)$, it gives

$$X_0[h(z)] = 1 - h(z) = - \left[\sum_{j \geq 1} H_j z^j \right] h(z).$$

Thus

$$\sum_{j \geq 1} H_j z^j = \frac{h(z) - 1}{h(z)} = 1 - \frac{1}{h(z)} = 1 - \sum_{n \geq 0} G_n z^n.$$

By identification of the coefficient of z^j , we find $H_j = -G_j$. \square

Remark 5.1. If we calculate $W_{p+k} F_{n+p}$ with (5.2), we obtain with (5.11) another proof of (1.16), $\frac{\partial}{\partial b_p} F_{n+p} = -W_p F_{n+p} - \sum_{k \geq 1} G_k W_{p+k} F_{n+p} = -(n + p)G_n$.

Theorem 5.4. For $k, p \in \mathbb{Z}, j \geq 1$,

$$V_j^k = \sum_{n \geq 1} K_{n-j}^{k-p} V_n^p \quad \text{and} \quad X_0 = \sum_{n \geq 1} [K_n^{-p} - K_n^{-1-p}] V_n^p.$$

We can also express the (V_j^k) in terms of $(V_j)_{j \geq 1}$ with the inverse function $f^{-1}(z)$.

6. The composition of differentials on coefficients

6.1. The inverse function

The inverse function is important in the study of coefficients regions, see [11, p. 104]. Also asymptotics of the derivatives of the Faber polynomials are calculated with inverse functions, see [10]. Let $f(w) = wh(w) = w + b_1w^2 + b_2w^3 + \dots + b_nw^{n+1} + \dots$.

$$w \frac{f'(w)}{f(w)} = 1 + w \frac{h'(w)}{h(w)} = 1 - \sum_{k \geq 1} F_k(b_1, b_2, \dots, b_k)w^k.$$

We denote $k(z) = f^{-1}(z)$ the inverse of f , we have $(f \circ k)(z) = z$, letting $w = k(z)$, $z \frac{k'(z)}{k(z)} = \frac{f(w)}{wf'(w)}$ and

$$\begin{aligned} z \frac{k'(z)}{k(z)} &= \frac{1}{1 - \sum_{p \geq 1} F_p(b_1, b_2, \dots, b_p)k(z)^p} \\ &= 1 + \sum_{m=1}^{+\infty} G_m(-F_1, -F_2, \dots, -F_m)k(z)^m \end{aligned}$$

we have $k(z) = f^{-1}(z) = z - b_1z^2 - (b_2 - 2b_1^2)z^3 - (b_3 + 5b_1^3 - 5b_1b_2)z^4 - (b_4 - 14b_1^4 + 21b_1^2b_2 - 6b_1b_3 - 3b_2^2)z^5 + \dots$. By a residue calculus, we know that

$$f^{-1}(z) = \frac{1}{2i\pi} \int_{|\zeta|=\rho} \frac{\zeta f'(\zeta)}{f(\zeta) - z} d\zeta = \sum_{n \geq 1} \left(\frac{1}{2i\pi} \int \frac{\zeta f'(\zeta)}{f(\zeta)} \frac{d\zeta}{f(\zeta)^n} \right) z^n. \tag{6.1}$$

Theorem 6.1. *Let $f(z) = z + b_1z^2 + \dots + b_nz^n + \dots$. The inverse function of f , $f^{-1}(f(z)) = z$ is given in terms of the derivatives of the Faber polynomials of $f(z)$ with*

$$\begin{aligned} f^{-1}(z) &= z + \sum_{n \geq 1} \frac{1}{n+1} K_n^{-(n+1)} z^{n+1} \\ &= z + \sum_{n \geq 1} \frac{(-1)^{n+1}}{(2n+1)} \times \frac{1}{(n+1)!} \left[\frac{\partial^{n+1}}{\partial b_1^{n+1}} F_{2n+1}(b_1, b_2, \dots, b_q, \dots) \right] z^{n+1}. \end{aligned} \tag{6.2}$$

Let $g(z) = zh(\frac{1}{z}) = z + b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \dots + \frac{b_{n+1}}{z^n} + \dots$, then the inverse function of g is

$$\begin{aligned} g^{-1}(z) &= z - b_1 - \sum_{n \geq 1} \frac{1}{n} K_{n+1}^n \frac{1}{z^n} \\ &= z - b_1 + \sum_{n \geq 1} \frac{(-1)^{n+1}}{(2n+1)} \\ &\quad \times \frac{1}{n!} \left[\frac{\partial^n}{\partial b_1^n} F_{2n+1} \right] (G_1(b_1), G_2(b_1, b_2), \dots, G_q(b_1, b_2, \dots), \dots) \frac{1}{z^n}. \end{aligned} \tag{6.3}$$

Proof. $\frac{zf'(z)}{f(z)} = -\sum_{k \geq 0} F_k z^k$ with $F_0 = -1$ and $\frac{z^n}{f(z)^n} = \sum_{p \geq 0} K_p^{-n} z^p$, we deduce that

$$\frac{zf'(z)}{f(z)} \frac{1}{f(z)^n} = -\sum_{p \geq 0, k \geq 0} F_k K_p^{-n} z^{p+k-n}.$$

The residue is obtained for $p + k - n = -1$ and is equal to $-\sum_{0 \leq k \leq n-1} F_k K_{n-k}^{-n}$. This is the coefficient of z^n in the expression of $f^{-1}(z)$. Thus the coefficient of z^{n+1} is

$$K_n^{-(n+1)} - \sum_{1 \leq k \leq n} F_k K_{n-k}^{-(n+1)} = K_n^{-(n+1)} - \frac{n}{n+1} K_n^{-(n+1)}$$

where we have used the recursion formula (2.7). For example the coefficients of z^4 is given when $n = 3$ by $\frac{1}{4}K_3^{-4} = 5b_1b_2 - b_3 - 5b_1^3$. The expressions of the coefficients of $f^{-1}(z)$ in terms of the (K_n^k) were found in an other way in [2, (1.2.8)–(1.2.9)]. For $g^{-1}(z)$, we use [2, (1.2.8)] and (T2). \square

Proposition 6.2. *We have*

$$h(f^{-1}(z)) = 1 + b_1z - \sum_{n \geq 2} \frac{1}{n-1} K_n^{1-n} z^n \tag{6.4}$$

with $f(z) = zh(z)$. Assume that $p \geq 2$, then

$$[h(f^{-1}(z))]^p = 1 + \sum_{1 \leq n \leq p-1} \frac{p}{p-n} K_n^{p-n} z^n - F_p z^p - \sum_{n \geq p+1} \frac{p}{n-p} K_n^{p-n} z^n. \tag{6.5}$$

Assume that $p \geq 1$, then

$$[h(f^{-1}(z))]^{-p} = 1 + \sum_{n \geq 1} \frac{p}{n+p} K_n^{-(p+n)} z^n. \tag{6.6}$$

The function $\psi(z) = h(f^{-1}(z))$ has been considered in [13]. The coefficients of $[h(f^{-1}(z))]^p$, $p \in \mathbb{Z}$ have been given in [2, (1.2.4) and (0.7)].

Proof. By a residue calculus,

$$\begin{aligned} h(f^{-1}(z)) &= \frac{1}{2i\pi} \int \frac{h(\xi)f'(\xi)}{f(\xi) - z} d\xi = \sum_{n \geq 0} \left(\frac{1}{2i\pi} \int_{|\xi|=\rho} \frac{\xi f'(\xi)}{f(\xi)} \frac{1}{f(\xi)^{n-1}} \frac{d\xi}{\xi^2} \right) z^n \\ &= 1 + b_1z + \sum_{n \geq 2} \left(\frac{1}{2i\pi} \int_{|\xi|=\rho} \frac{\xi f'(\xi)}{f(\xi)} \frac{1}{f(\xi)^{n-1}} \frac{d\xi}{\xi^2} \right) z^n. \end{aligned}$$

Since with (2.7), $\sum_{1 \leq k \leq n} F_k K_{n-k}^{1-n} = \frac{n}{n-1} K_n^{1-n}$, the coefficient of z^n in the expansion of $h(f^{-1}(z))$ is given by $K_n^{1-n} - \frac{n}{n-1} K_n^{1-n} = -\frac{1}{n-1} K_n^{1-n}$. Then we finish the proof as in Theorem 6.1. In the same way, we have

$$[h(f^{-1}(z))]^p = \sum_{n \geq 0} \frac{1}{2i\pi} \left(\int \frac{\xi f'(\xi)}{f(\xi)} \frac{1}{f(\xi)^{n-p}} \frac{d\xi}{\xi^{p+1}} \right) z^n.$$

We have $\frac{\xi f'(\xi)}{f(\xi)} \frac{1}{f(\xi)^{n-p}} = -\sum_{k \geq 0, j \geq 0} F_k K_j^{p-n} \xi^{k+j+p-n}$, the coefficient of ξ^p in this expression is obtained when $k + j = n$ and is equal to

$$-\sum_{0 \leq k \leq n} F_k K_{n-k}^{p-n} = K_n^{p-n} - \sum_{1 \leq k \leq n} F_k K_{n-k}^{p-n} = -\frac{p}{n-p} K_n^{p-n} z^n$$

when $p \neq n$. If $p = n$, we take the coefficient $-F_p$ of ξ^p in $\xi \frac{f'(\xi)}{f(\xi)}$. \square

Remark 6.1. Following [13], for any $p \in \mathbb{Z}$, $p \neq 0$,

$$\frac{[h(f^{-1}(z))]^p}{f'(f^{-1}(z))} = \sum_{n \geq 0} K_n^{p-(n+1)} z^n. \tag{6.7}$$

Proof. We have

$$\begin{aligned} \frac{[h(f^{-1}(z))]^p}{f'(f^{-1}(z))} &= \frac{1}{2i\pi} \int \frac{h(\xi)^p}{f(\xi) - z} d\xi = \frac{1}{2i\pi} \int \frac{h(\xi)^{p-1}}{1 - \frac{z}{f(\xi)}} \frac{d\xi}{\xi} \\ &= \sum_{n \geq 0} \left(\frac{1}{2i\pi} \int \frac{h(\xi)^{p-1}}{\xi^n h(\xi)^n} \frac{d\xi}{\xi} \right) z^n = \sum_{n \geq 0} \frac{1}{2i\pi} \int h(\xi)^{p-n-1} \frac{d\xi}{\xi^{n+1}} z^n. \end{aligned}$$

Since $h(\xi)^{p-n-1} \frac{1}{\xi^{n+1}} = \sum_{j \geq 0} K_j^{p-n-1} \frac{\xi^j}{\xi^{n+1}}$, the residue is $K_n^{p-(n+1)}$. \square

Remark 6.2. With the inverse function, we obtain also expressions of P_n^k (see Proposition 2.4),

$$\begin{aligned} P_n^k(b_1, b_2, \dots, b_n) &= \sum_{0 \leq s \leq n} \frac{k+s}{k} K_s^k(b_1, b_2, \dots, b_s) \\ &\quad \times K_{n-s}^s \left(\frac{1}{2} K_1^{-2}(b_1), \frac{1}{3} K_2^{-3}(b_1, b_2), K_3^{-4}, \dots, \frac{1}{p+1} K_p^{-(p+1)}, \dots \right). \end{aligned} \tag{6.8}$$

Proof. From Proposition 2.3, $\phi_k(\zeta) = \sum_{n \geq 0} \frac{k+n}{k} K_n^k \zeta^n$. We put $\zeta = f^{-1}(z)$. From Theorem 6.1,

$$f^{-1}(z)^n = z^n \sum_{j \geq 0} K_j^n \left(\frac{1}{2} K_1^{-2}, \frac{1}{3} K_2^{-3}, K_3^{-4}, \dots, \frac{1}{p+1} K_p^{-(p+1)}, \dots \right) z^j. \tag{6.9}$$

This gives the first expression of P_n^k . \square

Remark 6.3.

$$(f^{-1})'(z) = 1 + \sum_{k \geq 1} \frac{(-1)^k}{k!} z^k \psi_k(z) \quad \text{with } \psi_k(z) = \sum_{p \geq 0} \frac{(2k+p)!}{(k+p)!} D_{k+p}^k z^p. \tag{6.10}$$

Proof. We use (T3).

$$\begin{aligned} \frac{d}{dz} f^{-1}(z) &= 1 + \sum_{n \geq 1} K_n^{-(n+1)} z^n = 1 + \sum_{1 \leq k \leq n, 1 \leq n} (-1)^k \frac{(n+k)!}{n!k!} D_n^k z^n \\ &= 1 + \sum_{k \geq 1} \frac{(-1)^k}{k!} z^k \left(\sum_{n \geq k} \frac{(n+k)!}{n!} D_n^k z^{n-k} \right). \quad \square \end{aligned}$$

Remark 6.4. For $s \geq 1, p \in \mathbb{Z}$, then

$$\begin{aligned} V_s^p[f^{-1}(z)] &= z^{1+s} \sum_{n \geq 0} K_n^{p-s-(n+1)} z^n = \frac{z^{1+s} [h(f^{-1}(z))]^{p-s}}{f'(f^{-1}(z))} \\ &= \left(\frac{f(\zeta)^{1+s} h(\zeta)^{p-s}}{f'(\zeta)} \right) \Big|_{\zeta=f^{-1}(z)}. \end{aligned}$$

Let $\phi_k(\zeta) = \frac{\zeta f'(\zeta)}{f(\zeta)} \times h(\zeta)^k = \sum_{n \geq 0} \frac{k+n}{k} K_n^k \zeta^n$, we have

$$V_s^p[\phi_k(\zeta)] = -\zeta^s \sum_{n \geq 0} \frac{k(k+n+s)}{n+s} K_n^{k+p} \zeta^n. \tag{6.11}$$

6.2. Composition of derivations and recurrence formulae

We know (see 1.25) that $F_n(b_1, b_2, \dots, b_n) + F_n(G_1, G_2, \dots, G_n) = 0$. In the following, we show how the differentiation of this identity yields the recursion formula (2.8) with $p = -1$ and $r = -2$, i.e. $K_n^{-1} = \sum_{0 \leq j \leq n} K_j^{-2} K_{n-j}^1$. Then we prove that it gives a partial differential equation satisfied by the $(F_n)_{n \geq 1}$.

The differentiation of (1.25) with respect to b_k gives

$$\frac{\partial}{\partial b_k} F_n(b_1, b_2, \dots, b_n) + \sum_{j=1}^n \left(\frac{\partial F_n}{\partial b_j} \right) (G_1, G_2, \dots, G_n) \times \frac{\partial G_j}{\partial b_k} (b_1, b_2, \dots) = 0. \tag{6.12}$$

We know from (1.16) that $\frac{\partial F_n}{\partial b_i} (b_1, b_2, \dots, b_n) = -n G_{n-i} (b_1, b_2, \dots, b_n)$. This expression calculated at the point $(G_1(b_1), G_2(b_1, b_2), \dots, G_n(b_1, b_2, \dots, b_n))$ gives

$$\left(\frac{\partial F_n}{\partial b_j} \right) (G_1, G_2, \dots, G_n) = -n G_{n-j} (G_1, G_2, \dots, G_n) = -n b_{n-j}. \tag{6.13}$$

We replace in (6.12), we obtain

$$\frac{\partial F_n}{\partial b_k} (b_1, b_2, \dots, b_n) - \sum_{j=1}^{n-1} n b_{n-j} \frac{\partial G_j}{\partial b_k} (b_1, b_2, \dots, b_n) - n \frac{\partial G_n}{\partial b_k} = 0, \tag{6.14}$$

or equivalently

$$\frac{\partial F_n}{\partial b_k} (b_1, b_2, \dots, b_n) - \sum_{j=1}^{n-1} n b_j \frac{\partial G_{n-j}}{\partial b_k} (b_1, b_2, \dots, b_n) - n \frac{\partial G_n}{\partial b_k} = 0.$$

On the other hand, $-n G_j = \frac{\partial F_n}{\partial b_{n-j}}$. We replace in (6.14), we obtain

$$\frac{\partial F_n}{\partial b_k} (b_1, b_2, \dots, b_n) + \sum_{j=k}^{n-1} b_{n-j} \frac{\partial^2 F_n}{\partial b_{n-j} \partial b_k} (b_1, b_2, \dots, b_n) - n \frac{\partial G_n}{\partial b_k} = 0. \tag{6.15}$$

We go back to the expressions of the partial derivatives of F_n in terms of the K_n^p to see that (6.15) is the same as $K_n^{-1} = \sum_{0 \leq j \leq n} K_j^{-2} K_{n-j}^1$.

Lemma 6.3.

$$-K_{n-k}^{-2} = \frac{\partial G_n}{\partial b_k} \quad \forall n, k, n \leq k, \quad \text{and}$$

$$\frac{\partial^2 F_n}{\partial b_r \partial b_s} = -n \frac{\partial G_n}{\partial b_{r+s}} \quad r, s \geq 1, n \geq 1.$$

Proof. From (T1). \square

Theorem 6.4. Let $X_0 = -\sum_{j \geq 1} b_j \frac{\partial}{\partial b_j} = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \dots - b_k \frac{\partial}{\partial b_k} - \dots$. Then the identity

$$K_n^{-1} = \sum_{0 \leq j \leq n} K_j^{-2} K_{n-j}^1 \tag{6.16}$$

is the same as

$$\frac{\partial^2 F_n}{\partial b_r \partial b_s} = \frac{\partial}{\partial b_{r+s}} (X_0 F_n) \quad \forall r, s \geq 1, n \geq 1. \tag{6.17}$$

Proof. From (6.15), $\frac{\partial F_n}{\partial b_k} + \sum_{j \geq 1} b_j \frac{\partial^2 F_n}{\partial b_j \partial b_k} = n \frac{\partial G_n}{\partial b_k}$. The left side of this equation is

$$\frac{\partial}{\partial b_k} \left(\sum_{j \geq 1} b_j \frac{\partial}{\partial b_j} F_n \right) = -\frac{\partial}{\partial b_k} (X_0 F_n).$$

On the other hand, $n \frac{\partial G_n}{\partial b_k}$ is given by Lemma 6.3. This proves the theorem. \square

7. First order differential operators on \mathcal{M}

We have seen that the operators $(W_j)_{j \geq 1}$, $X_0, \frac{\partial}{\partial b_j}$ allow to pass from polynomials $(F_k)_{k \geq 1}$ to polynomials $(G_m)_{m \geq 1}$. In particular, we found $(n + p)G_n = -\frac{\partial F_{n+p}}{\partial b_p}$ and $W_j G_m = G_{m-j}$. Operators (Z_k) in [2] are of this type. In [9], family of vector fields related to the Virasoro algebra have been considered. We found that the operators $(V_k)_{k \geq 1}$ transforms the Neretin polynomials \mathcal{P}_j into $-(k^3 - k)P_{j-k}^k$. In the following, we construct first order differential operators on the manifold \mathcal{M} which permit to pass from one polynomial to the other.

7.1. The operators $(X_k)_{k \in \mathbb{Z}}$

The operator $X_0 = -\sum_{j \geq 1} G_j W_j = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \dots - b_n \frac{\partial}{\partial b_n} - \dots$ has appeared to be a natural operator on \mathcal{M} . We have

$$X_0 \frac{\partial}{\partial b_p} - \frac{\partial}{\partial b_p} X_0 = b_p \frac{\partial}{\partial b_p}. \tag{7.1}$$

On the other hand, $\frac{\partial}{\partial b_k} = -\sum_{j \geq 1} G_j W_{j+k}$.

Definition 7.1. For $k \geq 1$, we put

$$X_k = \frac{\partial}{\partial b_k} = - \sum_{j \geq 1} G_j W_{j+k} \quad \text{and} \quad X_{-k} = - \sum_{j \geq 1} G_{j+k} W_j. \tag{7.2}$$

Since $W_j = \sum_{p \geq 0} G_p \frac{\partial}{\partial G_{j+p}}$, we deduce

$$X_{-k} = - \sum_{j \geq 1, p \geq 0} G_{j+k} G_p \frac{\partial}{\partial G_{j+p}} = - \left[\sum_{0 \leq i \leq r-1} G_{r-i+k} G_i \right] \frac{\partial}{\partial G_r}. \tag{7.3}$$

Proposition 7.2. For $n, k \geq 1$, we have $X_0(F_n) = -nG_n$ and

$$X_k(F_n) = -nG_{n-k} \times 1_{k \leq n} \quad \text{and} \quad X_{-k}(F_n) = -nG_{n+k}. \tag{7.4}$$

Proof. From $W_j F_p = p \delta_{j,p}$. \square

Remark 7.1. In terms of the coordinates $(b_k)_{k \geq 1}$,

$$X_{-1} = -(b_2 - b_1^2) \frac{\partial}{\partial b_1} - (b_3 - b_1 b_2) \frac{\partial}{\partial b_2} - \dots - (b_{n+1} - b_1 b_n) \frac{\partial}{\partial b_n} - \dots,$$

$$X_{-2} = -(b_1^3 - 2b_1 b_2 + b_3) \frac{\partial}{\partial b_1} - \dots - (b_1^2 b_n - b_2 b_n - b_1 b_{n+1} + b_{n+2}) \frac{\partial}{\partial b_n} - \dots$$

In terms of the coordinates $(G_k)_{k \geq 1}$,

$$X_0 = -G_1 \frac{\partial}{\partial G_1} - (G_2 + G_1^2) \frac{\partial}{\partial G_2} + (G_n - K_n^2(G_1, G_2, \dots)) \frac{\partial}{\partial G_n} + \dots,$$

$$X_{-1} = -G_2 \frac{\partial}{\partial G_1} - (G_2 G_1 + G_3) \frac{\partial}{\partial G_2} - (G_2^2 + G_3 G_1 + G_4) \frac{\partial}{\partial G_3}$$

$$- (G_5 + 2G_2 G_3 + G_1 G_4) \frac{\partial}{\partial G_4} - (G_6 + 2G_4 G_2 + G_1 G_5 + G_3^2) \frac{\partial}{\partial G_5} - \dots$$

$$= \sum_{n \geq 1} (G_{n+1} + G_n G_1 - K_{n+1}^2(G_1, G_2, \dots, G_n, G_{n+1})) \frac{\partial}{\partial G_n}.$$

For $k \geq 2$, $X_{-k} = \sum_{n \geq 1} H_n \frac{\partial}{\partial G_n}$ with

$$H_n = G_{n+k} + G_{n+k-1} G_1 + G_{n+k-2} G_2 + \dots + G_n G_k - K_{n+k}^2(G_1, G_2, \dots, G_{n+k}).$$

From our main theorem, we see that the coefficient H_n is a sum of partial derivatives of Faber polynomials.

Lemma 7.3. The condition $X_{-k}(F_n) = -nG_{n+k}$ for $n \geq 1$ and $k \geq 0$ determines the operators X_{-k} in a unique way. Consider differential operators (\tilde{X}_{-k}) , $k \geq 0$, of the form

$$\tilde{X}_{-k} = B_1^k \frac{\partial}{\partial b_1} + B_2^k \frac{\partial}{\partial b_2} + \dots + B_n^k \frac{\partial}{\partial b_n} + \dots \quad \text{for } k \geq 0,$$

where the B_n^k are homogeneous polynomials in the variables $(b_1, b_2, \dots, b_n, \dots)$ of degree $n+k$ and such that $\tilde{X}_{-k}[F_n] = -nG_{n+k}$ for $n \geq 1$, $k \geq 0$, then $\tilde{X}_{-k} = X_{-k}$.

Moreover, $X_0[h(z)] = -h(z) + 1$, $X_{-1}[h(z)] = -\frac{h(z)}{z} + \frac{1}{z} + b_1h(z)$,

$$X_{-2}[h(z)] = -\frac{h(z)}{z^2} + \frac{1}{z^2} - \frac{G_1h(z)}{z} - G_2h(z),$$

$$X_{-3}[h(z)] = -\frac{h(z)}{z^3} + \frac{1}{z^3} - \frac{G_1h(z)}{z^2} - \frac{G_2h(z)}{z} - G_3h(z),$$

...

$$X_{-j}[h(z)] = \frac{1}{z^j} - \sum_{0 \leq k \leq j} \frac{G_k}{z^{j-k}} \times h(z).$$

Proof. Let $h(z) = 1 + b_1z + b_2z^2 + \dots + b_nz^n + \dots$. For X_0 , the condition $X_0[F_n] = -nG_n$ for $n \geq 1$ implies that

$$X_0 \frac{h'}{h} = - \sum_{k \geq 1} X_0(F_k)z^{k-1} = \sum_{k \geq 1} kG_kz^{k-1} = \frac{d}{dz} \left(\frac{1}{h(z)} \right).$$

Exchanging the order of derivation X_0 and $\frac{d}{dz}$, we have $\frac{d}{dz} \frac{X_0(h)}{h} = \frac{d}{dz} \left(\frac{1}{h} \right)$. Integrating with respect to z gives $\frac{X_0(h)}{h} = \frac{1}{h} + \alpha$ where α is a constant. If we take $\alpha = -1$, then $X_0(h) = 1 - h$. To express X_0 in the (b_k) coordinates, we have

$$X_0[h(z)] = -b_1z - b_2z^2 - \dots - b_nz^n - \dots = -b_1 \frac{\partial}{\partial b_1} h(z) - b_2 \frac{\partial}{\partial b_2} h(z) - \dots$$

To get X_0 in terms of the $(G_k)_{k \geq 1}$ coordinates, we consider $\tilde{h}(z) = \frac{1}{h(z)}$. We have

$$X_0[\tilde{h}(z)] = X_0 \left[\frac{1}{h(z)} \right] = \tilde{h}(z) - \tilde{h}(z)^2 = \sum_{n \geq 1} [G_n - K_n^2(G_1, \dots, G_k, \dots)] z^n.$$

Since $z^n = \frac{\partial}{\partial G_n} [\tilde{h}(z)]$, we obtain the result. For X_{-1} , the method is the same. From $X_{-1}(F_n) = -nG_{n+1}$ for $n \geq 1$, we deduce

$$\begin{aligned} X_{-1} \frac{h'}{h} &= - \sum_{k \geq 1} X_{-1}(F_k)z^{k-1} = \sum_{k \geq 1} kG_{k+1}z^{k-1} \\ &= \frac{1}{z} \sum_{k \geq 1} kG_kz^{k-1} - \frac{1}{z^2} \sum_{k \geq 1} G_kz^k = \frac{1}{z} \frac{d}{dz} \left(\frac{1}{h(z)} \right) - \frac{1}{z^2} \left(\frac{1}{h(z)} \right) + \frac{1}{z^2}. \end{aligned}$$

Exchanging the order of derivation X_{-1} and $\frac{d}{dz}$, we have

$$\frac{d}{dz} \frac{X_{-1}(h)}{h} = \frac{d}{dz} \left(\frac{1}{zh(z)} - \frac{1}{z} \right).$$

Integrating with respect to z gives $\frac{X_{-1}(h)}{h} = \frac{1}{zh(z)} - \frac{1}{z} + \text{constant}$. Taking the constant equal to b_1 gives X_{-1} . In the same way, $X_{-j}F_n = -nG_{n+j}$, for $n \geq 1$ implies that

$$\begin{aligned} X_{-j} \left(\frac{h'}{h} \right) &= - \sum_{k \geq 1} W_{-j} F_k z^{k-1} = \sum_{k \geq 1} kG_{k+j}z^{k-1} \\ &= \frac{1}{z^j} \sum_{k \geq 1} (k+j)G_{k+j}z^{k+j-1} - \frac{j}{z^{j+1}} \sum_{k \geq 1} G_{k+j}z^{k+j} \end{aligned}$$

$$= \frac{d}{dz} \left(\frac{1}{z^j h(z)} - \frac{1}{z^j} \sum_{0 \leq k \leq j} G_k z^k \right). \quad \square$$

7.2. The operators $(M_k)_{k \in \mathbb{Z}}$

We have $L_0 = \sum_{k \geq 1} k b_k \frac{\partial}{\partial b_k} = \sum_{j \geq 1} F_j W_j$.

Definition 7.2. For $k \geq 1$, let

$$M_k = \sum_{j \geq 1} F_j W_{j+k} \quad \text{and} \quad M_{-k} = \sum_{j \geq 1} F_{j+k} W_j. \tag{7.5}$$

Proposition 7.4. For $k \geq 1$, $M_{-k} F_n = n F_{n+k}$, $M_k(F_n) = n F_{n-k} \times 1_{n \geq k}$. Moreover $M_k M_p - M_p M_k = (k - p) M_{k+p}$ for $p, k \in \mathbb{Z}$.

Proof. From $W_j(F_n) = n \delta_{n,j}$. For the last identity, we verify that $M_k M_p F_n - M_p M_k F_n = (k - p) M_{k+p} F_n$. \square

Lemma 7.5. In terms of the coordinates $(b_k)_{k \geq 1}$, for $k \geq 1$,

$$M_k = \sum_{j \geq 1} F_j W_{j+k} = b_1 \frac{\partial}{\partial b_{k+1}} + 2b_2 \frac{\partial}{\partial b_{k+2}} + \dots + p b_p \frac{\partial}{\partial b_{k+p}} + \dots. \tag{7.6}$$

In particular if $h(z) = 1 + b_1 z + b_2 z^2 + \dots + b_p z^p + \dots$, we have

$$M_k[h(z)] = z^{k+1} h'(z) \quad \text{for } k \geq 1. \tag{7.7}$$

Proof. We verify the identity on $h(z)$. From $W_j[h(z)] = -z^j h(z)$. Thus

$$M_k[h(z)] = -z^k \left(\sum_{j \geq 1} F_j z^j \right) \times h(z) = z^{k+1} h'(z).$$

Since $(b_1 \frac{\partial}{\partial b_{k+1}} + 2b_2 \frac{\partial}{\partial b_{k+2}} + \dots + p b_p \frac{\partial}{\partial b_{k+p}} + \dots) h(z) = z^{k+1} h'(z)$, we obtain (7.6). For $k \geq 0$, the operators $M_{-k} = \sum_{j \geq 1} F_{j+k} W_j$ are given by $M_0 = L_0$,

$$\begin{aligned} M_{-1} &= (2b_2 - b_1^2) \frac{\partial}{\partial b_1} + (3b_3 - b_1 b_2) \frac{\partial}{\partial b_2} + \dots + ((n+1)b_{n+1} - b_1 b_n) \frac{\partial}{\partial b_n} + \dots, \\ M_{-2} &= \sum_{j \geq 1} [(j+2)b_{j+2} - b_{j+1} b_1 + b_j (b_1^2 - 2b_2)] \frac{\partial}{\partial b_j}, \\ &\dots \\ M_{-k} &= \sum_{j \geq 1} (b_j F_k + b_{j+1} F_{k-1} + \dots + b_{j+k-1} F_1 + (j+k)b_{j+k}) \frac{\partial}{\partial b_j}. \quad \square \end{aligned}$$

Remark 7.2. On \mathcal{M} , define the differential operators

$$L_k = M_k - W_k \quad \text{for } k \geq 1. \tag{7.8}$$

With the convention $F_0 = -1$,

$$L_k = \sum_{j \geq 0} F_j W_{j+k} = F_0 W_k + F_1 W_{k+1} + F_2 W_{k+2} + \dots \tag{7.9}$$

then $L_k = M_k - W_k, k \geq 1$, is the Kirillov operator

$$L_k = \frac{\partial}{\partial b_k} + \sum_{n \geq 1} (n+1)b_n \frac{\partial}{\partial b_{n+k}}. \tag{7.10}$$

For $f(z) = zh(z) = z + b_1 z^2 + b_2 z^3 + \dots + b_p z^{p+1} + \dots$, we have $L_k[f(z)] = z^{1+k} f'(z)$ and

$$L_k(F_n) = nF_{n-k} \times 1_{n \geq k}. \tag{7.11}$$

7.3. The operators $(V_j)_{j \geq 1}$ and $(V_j^k)_{j \geq 1}$

We do not stay anymore in the class of polynomials $(F_n), (G_n)$. For $j \geq 1$ and $k \in \mathbb{Z}$, see (1.9)–(1.10),

$$V_j = - \sum_{n \geq 0} K_n^{j+1} \frac{\partial}{\partial b_{n+j}} \quad \text{and} \quad V_j^k = - \sum_{n \geq 0} K_n^{k+1} \frac{\partial}{\partial b_{n+j}}. \tag{7.12}$$

The polynomials $(P_n^k)_{n \geq 0}$, see Proposition 2.4 and [1, (A.1.2)], are given by

$$\frac{zf'(z)}{f(z)} h(z)^k = 1 + \sum_{n \geq 1} P_n^k f(z)^n \tag{7.13}$$

where $f(z) = zh(z)$. We have the recursion formulas, for $q \in \mathbb{Z}$,

$$(n+1)b_n = \sum_{0 \leq j \leq n} P_j^q K_{n-j}^{j+1-q}, \quad \frac{n+1}{k+q} K_n^{k+q} = \sum_{j=0}^n P_j^q K_{n-j}^{j+k}, \tag{7.14}$$

$$-F_n = \sum_{0 \leq j \leq n} P_j^q K_{n-j}^{j-q}, \tag{7.15}$$

With (7.14), we replace $(n+1)b_n$ in (7.10). It gives for $L_k, k \geq 1$ (with $b_0 = 1$),

$$L_k = \sum_{0 \leq n} \sum_{0 \leq j \leq n} P_j^q K_{n-j}^{j+1-q} \frac{\partial}{\partial b_{n+k}} = \sum_{0 \leq j} P_j^q \left[\sum_{n \geq j} K_{n-j}^{j+1-q} \frac{\partial}{\partial b_{n+k}} \right] = - \sum_{j \geq 0} P_j^q V_{j+k}^{j-q}.$$

For $q = -k$ and $k \geq 1$, we obtain

$$L_k = - \sum_{j \geq 0} P_j^{-k} V_{j+k} = - \sum_{j \geq 1} P_{j-k}^{-k} \times 1_{j \geq k} V_j. \tag{7.16}$$

Definition 7.3. For any $k \in \mathbb{Z}$, with the convention $P_n^k = 0$ if $n < 0$, we put

$$L_k = - \sum_{j \geq 1} P_{j-k}^{-k} V_j. \tag{7.17}$$

7.4. The Kirillov operators $(L_{-p})_{p \geq 1}$

It has been proved in [1, (A.4.5)] that the vector fields $(L_{-p})_{p \geq 0}$ obtained by Kirillov in [7] are such that for $f(z) = zh(z)$, it holds

$$L_{-p}[f(z)] = \sum_{j \geq 0} P_{1+j+p}^p f(z)^{j+2}. \tag{7.18}$$

Proposition 7.6. Let L_{-p} , $p \geq 1$ the operator defined by (7.18), then L_{-p} is given by (7.17), we have

$$L_{-p} = - \sum_{j \geq 1} P_{j+p}^p V_j. \tag{7.19}$$

Proof. From (5.9), $V_j[f(z)] = -f(z)^{j+1}$. \square

Remark 7.3. We have

$$L_{-p} = \sum_{r \geq 1} A_r^p \frac{\partial}{\partial b_r} \quad \text{with } A_r^p = \sum_{1 \leq j \leq r} P_{j+p}^p K_{r-j}^{j+1}, \tag{7.20}$$

$$L_{-p} = - \sum_{r \geq 1} B_r^p W_r \quad \text{with } B_r^p = \sum_{1 \leq j \leq r} P_{j+p}^p K_{r-j}^j. \tag{7.21}$$

Proof. From (7.18), $L_{-p} = \sum_{j \geq 0, n \geq 0} P_{1+j+p}^p K_n^{j+2} \frac{\partial}{\partial b_{n+j+1}}$. Thus

$$L_{-p} = \sum_{r \geq 0} A_{r+1}^p \frac{\partial}{\partial b_{r+1}} \quad \text{with } A_{r+1}^p = \sum_{0 \leq j \leq r} P_{1+j+p}^p K_{r-j}^{j+2}.$$

This proves (7.20). We obtain (7.21) with (5.11),

$$\begin{aligned} L_{-p} &= - \sum_{j \geq 0, n \geq 0, k \geq 0} P_{1+j+p}^p K_n^{j+2} G_k W_{n+j+k+1} \\ &= - \sum_{j \geq 0, s \geq 0} P_{1+j+p}^p K_s^{j+1} W_{j+s+1}. \quad \square \end{aligned}$$

Remark 7.4. We have $L_{-k} = M_{-k} - Y_{-k}$ with

$$Y_{-k} = - \sum_{r \geq 1} J_r^k W_r \quad \text{and} \quad J_r^k = \sum_{s=0}^k P_s^k K_{r+k-s}^{s-k}. \tag{7.22}$$

In particular $L_{-1} = M_{-1} - X_{-1}$.

Proof. From (7.15), $M_{-k} = \sum_{r \geq 1} F_{j+k} W_j = - \sum_{r \geq 1} [\sum_{0 \leq s \leq j+k} P_s^q K_{k+j-s}^{s-q}] W_j$

$$\sum_{0 \leq s \leq j+k} P_s^q K_{k+j-s}^{s-q} = \sum_{0 \leq s \leq k} P_s^q K_{k+j-s}^{s-q} + \sum_{1 \leq s \leq j} P_{k+s}^q K_{j-s}^{k+s-q}.$$

With $k = q$, the second sum is J_j^k as in (7.21). The first sum gives Y_{-k} . \square

Remark 7.5. With (1.11) and (5.7), we find for any $p \in \mathbb{Z}$, $j \geq 1$,

$$L_{-p}V_j - V_jL_{-p} = \sum_{1 \leq s \leq j} (V_j(P_{s+p}^p))V_s. \tag{7.23}$$

8. Second order differential operators

Let $\Delta_0 = \sum_{p \geq 1, q \geq 1} F_{p+q}(W_{p+q} + W_pW_q)$.

Proposition 8.1. Let $L_0 = \sum_{j \geq 1} F_jW_j$ be the homogeneity operator, then

$$\Delta_0F_n = n(n - 1)F_n \quad \text{and} \quad (\Delta_0 + L_0)F_n = n^2F_n.$$

Proof. Because of (1.6), $W_pW_qF_n = 0$. On the other hand

$$\sum_{p \geq 1, q \geq 1} F_{p+q}W_{p+q}F_n = n \times \left(\sum_{p \geq 1, q \geq 1} \delta_{p+q, n} \right) F_n.$$

Then we remark that $(\sum_{p \geq 1, q \geq 1} \delta_{p+q, n}) = n - 1$. \square

Definition 8.1. We consider $\mathcal{Q}_j = \sum_{p \geq 1, q \geq 1, p+q=j} W_pW_q$ for $j \geq 2$.

Because of (1.8), Δ_0 and \mathcal{Q}_j are second order differential operators on the manifold \mathcal{M} . With the expression (1.8) of $W_{p+q} + W_pW_q$, we have

$$\frac{\partial}{\partial b_j}(W_pW_q + W_{p+q}) - (W_pW_q + W_{p+q})\frac{\partial}{\partial b_j} = W_p\frac{\partial}{\partial b_{q+j}} + W_q\frac{\partial}{\partial b_{j+p}}.$$

Since W_p and W_q commute, we have $\mathcal{Q}_2 = W_1^2 = K_2^2(W_1, 0)$, $\mathcal{Q}_3 = 2W_1W_2 = K_3^2(W_1, W_2, 0)$, $\mathcal{Q}_4 = 2W_1W_3 + W_2^2 = K_4^2(W_1, W_2, W_3, 0), \dots, \mathcal{Q}_n = K_n^2(W_1, W_2, \dots, W_{n-1}, 0)$ and $\mathcal{Q}_jW_p = W_p\mathcal{Q}_j$ for $j \geq 2$ and $p \geq 1$.

Since $W_{j-k}W_kG_n = W_{j-p}W_pG_n$, for $k \leq j$, $p \leq j$, we have $\mathcal{Q}_2G_j = G_{j-2}$, $\mathcal{Q}_3G_j = 2G_{j-3}, \dots, \mathcal{Q}_nG_j = (n - 1)G_{j-n}$.

The operator Δ_0 decomposes into $\Delta_0 = \Delta_1 + \Delta_2$ with

$$\Delta_1 = \sum_{j \geq 2} F_j\mathcal{Q}_j,$$

$$\Delta_2 = \sum_{j \geq 2} \left(\sum_{p \geq 1, q \geq 1, p+q=j} 1 \right) F_jW_j = \sum_{j \geq 2} (j - 1)F_jW_j.$$

Since $W_jF_n = n\delta_{j,n}$, we have $\mathcal{Q}_jF_n = 0$, $j \geq 2$ and since $\mathcal{Q}_j(G_n) = (j - 1)G_{n-j}$, we find

$$\Delta_1F_n = 0,$$

$$\Delta_2F_n = n(n - 1)F_n,$$

and

$$\Delta_1G_n = \Delta_2G_n = \sum_{j \geq 2} (j - 1)F_jG_{n-j}.$$

We deduce that

$$\begin{aligned} \Delta_1 Q_j &= Q_j \Delta_1, & W_j \Delta_1 - \Delta_1 W_j &= j Q_j, \\ \Delta_2 Q_j &= Q_j \Delta_2, & W_j \Delta_2 - \Delta_2 W_j &= j(j-1)W_j, \end{aligned}$$

and $\Delta_2 \Delta_1 - \Delta_1 \Delta_2 = \sum_{k \geq 2} k(k-1)F_k Q_k$.

Lemma 8.2. *Let $X_0 = -\sum_{j \geq 1} G_j W_j$ and $L_0 = \sum_{j \geq 1} F_j W_j$, then*

$$X_0 G_n = G_n - K_n^2(G_1, G_2, \dots, G_n), \quad X_0 F_n = -n G_n, \tag{8.1}$$

$$X_0 L_0 = L_0 X_0, \tag{8.2}$$

$$L_0 \Delta_2 = \Delta_2 L_0 \quad \text{and} \quad L_0 \Delta_1 - \Delta_1 L_0 = \sum_{k \geq 2} k F_k Q_k. \tag{8.3}$$

Proof. (8.1) results from the expression of X_0 in the $(G_n)_{n \geq 1}$ coordinates. Because of (1.6), we have $L_0 X_0 = \sum_{q \geq 1} [\sum_{j \geq 1} F_j G_{q-j}] W_q + \sum_{j \geq 1, q \geq 1} F_j G_q W_j W_q$ and

$$X_0 L_0 = \sum_{q \geq 1} q G_q W_q + \sum_{j \geq 1, q \geq 1} F_j G_q W_j W_q.$$

Since $\sum_{j \geq 1} F_j G_{q-j} = q G_q$, see (2.5), it proves that $X_0 L_0 = L_0 X_0$. The identities (8.3) are consequence of (1.6). \square

9. The conformal map from the exterior of the unit disk onto the exterior of $[-2, +2]$

Let $\psi(w) = w + \frac{1}{w}$ be the conformal map from the exterior of the unit disk onto the exterior of $[-2, 2]$. The Faber polynomials $F_n(z)$ of $[-2, 2]$ are given by

$$\frac{w^2 - 1}{w^2 - wz + 1} = \sum_{n=0}^{\infty} F_n(z) w^{-n}.$$

They satisfy the differential equation

$$(z^2 - 4)F_n''(z) + zF_n'(z) = n^2 F_n(z). \tag{9.1}$$

In the following, we consider Faber polynomials $F_n(b_1, b_2, 0, 0, \dots, 0)$. All the b_j are zero except b_1 and b_2 . We have $F_1(b_1) = -b_1$, $F_2(b_1, b_2) = b_1^2 - 2b_2$, $F_3(b_1, b_2, 0) = -b_1^3 + 3b_1 b_2$, $F_4(b_1, b_2, 0, 0) = b_1^4 - 4b_1^2 b_2 + 2b_2^2, \dots$

Theorem 9.1. *Faber polynomials associated to $\psi(w) = w + b_1 + \frac{b_2}{w}$ verify*

$$((z - b_1)^2 - 4b_2)F_n''(z) + (z - b_1)F_n'(z) = n^2 F_n(z). \tag{9.2}$$

In particular, if $b_1 = 0$ and $b_2 = 1$, we obtain (9.1).

To prove the theorem, we need the following lemmas.

Lemma 9.2. *Let $L = \frac{\partial^2}{\partial^2 b_1} + \frac{\partial}{\partial b_2} + \sum_{k \geq 1} b_k \frac{\partial^2}{\partial b_2 \partial b_k}$, then $LF_n = 0$.*

Proof. From (6.15). \square

Lemma 9.3. Consider $\Delta_0 = \sum_{p \geq 1, q \geq 1} F_{p+q} (W_{p+q} + W_p W_q)$ and let $\phi(b_1, b_2)$ be a function defined on \mathcal{M} , which depends only of b_1, b_2 , then

$$\Delta_0 \phi = \left[(b_1^2 - 2b_2) \frac{\partial^2}{\partial b_1^2} + 2b_1 b_2 \frac{\partial^2}{\partial b_1 \partial b_2} + 2b_2^2 \frac{\partial^2}{\partial b_2^2} \right] \phi.$$

Proof. Let ϕ depend only on the variables b_1 and b_2 . If $p > 2$ or $q > 2$, we have $[W_{p+q} + W_p W_q] \phi = 0$. If $p = 2, q = 2$, then $(W_4 + W_2^2) \phi = W_2^2 \phi = \frac{\partial^2}{\partial b_2^2} \phi$. If $p = 2, q = 1$ or $p = 1, q = 2$, $(W_3 + W_2 W_1) \phi = \frac{\partial}{\partial b_2} (\frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2}) \phi$. If $p = 1, q = 1$, then $(W_2 + W_1^2) \phi = [-\frac{\partial}{\partial b_2} + (\frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2})^2] \phi$.

We calculate $\Delta_0 \phi = [F_2(W_2 + W_1^2) + 2F_3(W_3 + W_2 W_1) + F_4(W_4 + W_2^2)] \phi$. This gives

$$\Delta_0 \phi = \left[F_2 \left(-\frac{\partial}{\partial b_2} + \left(\frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2} \right)^2 \right) + 2F_3 \left(\frac{\partial}{\partial b_2} \left(\frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2} \right) + F_4 \frac{\partial^2}{\partial b_2^2} \right) \right] \phi$$

or equivalently

$$\Delta_0 \phi = \left[F_2 \left(\frac{\partial^2}{\partial b_1^2} + 2b_1 \frac{\partial^2}{\partial b_1 \partial b_2} + b_1^2 \frac{\partial^2}{\partial b_2^2} \right) + 2F_3 \left(\frac{\partial^2}{\partial b_1 \partial b_2} + b_1 \frac{\partial^2}{\partial b_2^2} \right) + F_4 \frac{\partial^2}{\partial b_2^2} \right] \phi.$$

Replacing F_2, F_3, F_4 , we find Lemma 9.3. \square

Proof of the theorem. From Lemma 9.2, we know that

$$\left(2b_1 b_2 \frac{\partial^2}{\partial b_1 \partial b_2} + 2b_2^2 \frac{\partial^2}{\partial b_2^2} \right) F_n = \left(-2b_2 \frac{\partial}{\partial b_2} - 2b_2 \frac{\partial^2}{\partial b_2^2} \right) F_n.$$

We replace the right hand side in the expression of Δ_0 and we find

$$\left[(b_1^2 - 4b_2) \frac{\partial^2}{\partial b_1^2} - 2b_2 \frac{\partial}{\partial b_2} \right] F_n = n(n - 1) F_n. \tag{9.3}$$

Since F_n is homogeneous, $b_1 \frac{\partial}{\partial b_1} F_n + 2b_2 \frac{\partial}{\partial b_2} F_n = n F_n$. Replacing $-2b_2 \frac{\partial}{\partial b_2} F_n$ in (9.2), we find $(b_1^2 - 4b_2) \frac{\partial^2 F_n}{\partial b_1^2} + b_1 \frac{\partial F_n}{\partial b_1} = n^2 F_n$. \square

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