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On generating functions of modified Laguerre polynomials

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Abstract

In this note, we have obtained some results on bilateral and trilateral generating functions of modified Laguerre polynomials. Furthermore, some comments are made on the results of Laguerre polynomials obtained by Das and Chatterjea [1] while deriving the bilateral and trilateral generating relations by using the operator obtained by double interpretations.

Key words: Laguerre polynomials, generating functions, special functions.

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1 Introduction

In [1], Das and Chatterjea have claimed that the operator 'A', obtained by double interpretations to both the index (n) and the parameter (α) of the Laguerre polynomial in Weisner's group-theoretic method, given by

$$A = xy^{-1}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - xy^{-1}z \tag{1}$$

such that

$$A[y^{\alpha}z^{n}L_n^{(\alpha)}(x)] = (n+1)L_{n+1}^{(\alpha-1)}(x)y^{\alpha-1}z^{n+1} \tag{2}$$

and

$$\exp(aA)f(x,y,z) = \exp(-\frac{axz}{y})f(x+axy^{-1}z, y+az, z) \tag{3}$$

in obtaining the following generating function in original:

$$(1+t)^{\alpha}\exp(-xt)L_n^{(\alpha)}(x+xt) = \sum_{m=0}^{\infty}\binom{m+n}{m}L_{n+m}^{(\alpha-m)}(x)t^{m}, \qquad |t|<1. \tag{4}$$

Finally they obtained the following theorems:

Theorem 1. *If there exists a generating function of the form*

$$F(x,t)=\sum_{n=0}^{\infty} a_n L_n^{(\alpha-n)}(x)t^n, \tag{5}$$

then

$$\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)\sigma_n(y)t^n=(1+t)^{\alpha}\exp(-xt)F\left(x+xt,\frac{yt}{1+t}\right), \tag{6}$$

where

$$\sigma_n(y)=\sum_{k=0}^{\infty} a_k\binom{n}{k}y^k. \tag{7}$$

Theorem 2. *If there exists a generating function of the form*

$$F(x,y,t)=\sum_{n=0}^{\infty} a_n L_n^{(\alpha-n)}(x)g_n(y)t^n, \tag{8}$$

where $g_n(y)$ is an arbitrary polynomial, then

$$\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)\sigma_n(y,z)t^n=(1+t)^{\alpha}\exp(-xt)F\left(x+xt,y,\frac{zt}{1+t}\right), \tag{9}$$

where

$$\sigma_n(y,z)=\sum_{k=0}^{\infty} a_k\binom{n}{k}g_k(y)z^k. \tag{10}$$

Theorem 3. *If there exists a generating function of the form*

$$G(x,t)=\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x)t^n, \tag{11}$$

then

$$\sum_{n=0}^{\infty} z^n\sum_{k=0}^{n} a_k\binom{n}{k}L_n^{(\alpha-n+k)}(x)t^n=(1+z)^{\alpha}\exp(-xz)G(x+xz,tz). \tag{12}$$

To prove Theorema 1 and 2 they used result (4) and to obtain Theorem 3 they used the operator 'A' directly.

Here, we would like to mention that the authors of [1] perhaps fails to notice the work [2]. In fact, neither the operator 'A' nor the generating function (4) is new and they are found derived in [2] and the Theorems 1—3 are the direct consequences of them.

The aim at presenting this article is to obtain some results on the bilateral and trilateral generating functions involving the said polynomial. We would also like to point it out that the operator 'A', satisfying (2) and (3) (obtained by double interpretations) with the help of which Das and Chatterjea proved (4) and the theorems 1-3, is not only cumbrous but also unnecessary in the derivation of the aforementioned result (4) and the theorems. In fact, the operator obtained by single interpretation to the index in the study of modified Laguerre polynomials, $L_n^{(\alpha-n)}(x)$ by Weisner's method is very much simple and straight forward even in deriving the nice extensions of the same.

The main results of the paper are given below:

Theorem 4. *If*

$$G(x,w)=\sum_{n=0}^{\infty}a_nL_{n+r}^{(\alpha-n)}(x)w^n, \tag{13}$$

then

$$\sum_{n=0}^{\infty}L_{n+r}^{(\alpha-n)}(x)\sigma_n(z)w^n=(1+w)^{\alpha}\exp(-xw)G\left(x(1+w),\frac{zw}{1+w}\right), \tag{14}$$

where

$$\sigma_n(z)=\sum_{k=0}^{n}a_k\binom{n+r}{k+r}z^k. \tag{15}$$

Theorem 5. *If*

$$G(x,u,w)=\sum_{n=0}^{\infty}a_nL_{n+r}^{(\alpha-n)}(x)g_n(u)w^n, \tag{16}$$

then

$$\sum_{n=0}^{\infty}L_{n+r}^{(\alpha-n)}(x)\sigma_n(u,z)w^n=(1+w)^{\alpha}\exp(-xw)G\left(x(1+w),u,\frac{zw}{1+w}\right), \tag{17}$$

where

$$\sigma_n(u,z)=\sum_{k=0}^{n}a_k\binom{n+r}{k+r}g_k(u)z^k. \tag{18}$$

Theorem 6. *If*

$$G(x,w)=\sum_{n=0}^{\infty}a_nL_{n+r}^{(\alpha)}(x)w^n, \tag{19}$$

then

$$\sum_{n=0}^{\infty}z^n\sum_{k=0}^{n}a_k\binom{n+r}{k+r}L_{n+r}^{(\alpha-n+k)}(x)w^k=(1+z)^{\alpha}\exp(-xz)G(x(1+z),zw). \tag{20}$$

Theorem 7. *If*

$$G(x,u,w)=\sum_{n=0}^{\infty}a_nL_{n+r}^{(\alpha)}(x)g_n(u)w^n, \tag{21}$$

then

$$\sum_{n=0}^{\infty}\sigma_n(x,u,v)w^n=(1+w)^{\alpha}\exp(-xw)G(x(1+w),u,wv), \tag{22}$$

where

$$\sigma_n(x,u,v)=\sum_{k=0}^{n}a_k\binom{n+r}{k+r}L_{n+r}^{(\alpha-n+k)}(x)g_k(u)v^k. \tag{23}$$

In the next section we first proceed to prove the result (4) by using the operator obtained by single interpretation to the index (n) of the polynomial under consideration.

2 Derivation of the results

At first we consider the following operator R :

$$R = xy\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} - xy + \alpha y \tag{24}$$

such that

$$R\left(L_n^{(\alpha-n)}(x)y^n\right) = (n+1)L_{n+1}^{(\alpha-n-1)}(x)y^{n+1}. \tag{25}$$

We now define the corresponding extended group generated by R as follows:

$$e^{wR}f(x,y) = e^{-wxy}(1+wy)^{\alpha}f\left(x(1+wy), \frac{y}{1+cy}\right). \tag{26}$$

Now using (26), we obtain

$$e^{wR}\left(L_n^{(\alpha-n)}(x)y^n\right) = e^{-wxy}(1+wy)^{\alpha-n}L_n^{(\alpha-n)}\left(x(1+wy)\right)y^n.$$

But, using (25), we obtain

$$e^{wR}\left(L_n^{(\alpha-n)}(x)y^n\right) = \sum_{m=0}^{\infty}\frac{w^m}{m!}(n+1)_m L_{n+m}^{(\alpha-n-m)}(x)y^{n+m}. \tag{27}$$

Equating (27) and (27), we get

$$e^{-wxy}(1+wy)^{\alpha-n}L_n^{(\alpha-n)}\left(x(1+wy)\right) = \sum_{m=0}^{\infty}\binom{n+m}{m}L_{n+m}^{(\alpha-n-m)}(x)(yw)^m. \tag{28}$$

Replacing wy by t in (28), we get

$$e^{-xt}(1+t)^{\alpha-n}L_n^{(\alpha-n)}\left(x(1+t)\right) = \sum_{m=0}^{\infty}\binom{n+m}{m}L_{n+m}^{(\alpha-n-m)}(x)(t)^m \tag{29}$$

Finally if we write $\alpha = \alpha + n$ on both sides of (29), we get

$$e^{-xt}(1+t)^{\alpha}L_n^{(\alpha)}\left(x(1+t)\right) = \sum_{m=0}^{\infty}\binom{n+m}{m}L_{n+m}^{(\alpha-m)}(x)(t)^m, \tag{30}$$

which is obtained by Das and Chatterjea ((1.4) of [1]).

Now we shall proceed to prove the Theorems 4-7 of this paper.

Proof of Theorem 4: Now

$$\begin{aligned}
&\sum_{n=0}^{\infty}(wy)^n L_{n+r}^{(\alpha-n)}(x)\sigma_n(z)\\
&=\sum_{n=0}^{\infty}(wy)^n L_{n+r}^{(\alpha-n)}(x)\sum_{k=0}^{n}\binom{n+r}{k+r}a_k z^k\\
&=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(wy)^{n+k} L_{n+k+r}^{(\alpha-n-k)}(x)a_k z^k\binom{n+k+r}{k+r}\\
&=\sum_{k=0}^{\infty}a_k(wyz)^k\sum_{n=0}^{\infty}\binom{n+k+r}{k+r}L_{n+k+r}^{(\alpha-n-k)}(x)(wy)^k\\
&=\sum_{k=0}^{\infty}a_k(wyz)^k e^{-xwy}(1+wy)^{\alpha-k}L_{k+r}^{(\alpha-k)}(x^{/})\\
&=e^{-xwy}(1+wy)^{\alpha}\sum_{k=0}^{\infty}a_k L_{k+r}^{(\alpha-k)}(x^{/})\left(\frac{wyz}{1+wy}\right)^k\\
&=e^{-xwy}(1+wy)^{\alpha}G\left(x^{/},\frac{wyz}{1+wy}\right)\\
&=e^{-xwy}(1+wy)^{\alpha}G\left(x(1+wy),\frac{wyz}{1+wy}\right),
\end{aligned}$$

which, on putting y=1, reduces exactly to Theorem 4.

Cor-1: If we put r=0 in Theorem 4 then we get the following result:

If

$$G(x,w)=\sum_{n=0}^{\infty}a_n L_n^{(\alpha-n)}(x)w^n,$$

then

$$\sum_{n=0}^{\infty}L_n^{(\alpha-n)}(x)\sigma_n(z)w^n=(1+w)^{\alpha}\exp(-xw)G\left(x(1+w),\frac{zw}{1+w}\right),$$

where

$$\sigma_n(z)=\sum_{k=0}^{n}a_k\binom{n}{k}z^k.$$

which is Theorem 1 of Das and Chatterjea.

Proof of Theorem 5: Proof of Theorem 5 is exactly similar to Theorem 4 and the calculation is a routine one.

Cor-2: If we put r=0 in theorem 5 then we get

$$\sum_{n=0}^{\infty}L_n^{(\alpha-n)}(x)\sigma_n(u,z)w^n=(1+w)^{\alpha}\exp(-xw)G\left(x(1+w),u,\frac{zw}{1+w}\right),$$

where

$$\sigma_n(u,z)=\sum_{k=0}^{n}a_k\binom{n}{k}g_k(u)z^k,$$

which is Theorem 2, obtained by Das and Chatterjea.

Proof of Theorem 6:

$$\begin{aligned}
&\sum_{n=0}^{\infty}(w)^n\sigma_n(x,v)\\
&=\sum_{n=0}^{\infty}(w)^n\sum_{k=0}^{n}a_k\binom{n+r}{k+r})L_{n+r}^{(\alpha-n+k)}(x)v^k\\
&=\sum_{n=0}^{\infty}(w)^{n+k}\sum_{k=0}^{\infty}a_k\binom{n+k+r}{k+r}L_{n+k+r}^{(\alpha-n)}(x)v^k\\
&=\sum_{k=0}^{\infty}a_k(wv)^k\sum_{n=0}^{\infty}\binom{n+k+r}{k+r}L_{n+k+r}^{(\alpha-n)}(x)w^n\\
&=\sum_{k=0}^{\infty}a_k(wv)^k e^{-xw}(1+w)^{\alpha}L_{k+r}^{(\alpha)}(x^{/})\\
&=e^{-xw}(1+w)^{\alpha}\sum_{k=0}^{\infty}a_kL_{k+r}^{(\alpha)}(x^{/})(wv)^k\\
&=e^{-xw}(1+w)^{\alpha}G\Big(x(1+w),wv\Big),
\end{aligned}$$

which is Theorem 6.

Cor-3: If we put r=0 in theorem 6 then we get the following result:

If

$$G(x,w)=\sum_{n=0}^{\infty}a_nL_n^{(\alpha)}(x)w^n,$$

then

$$\sum_{n=0}^{\infty}z^n\sum_{k=0}^{n}a_k\binom{n}{k}L_n^{(\alpha-n+k)}(x)w^k=(1+z)^{\alpha}\exp(-xz)G(x(1+z),zw).$$

which is Theorem 3 of Das and Chatterjea.

Proof of Theorem 7: Proof of Theorem 7 is exactly similar to Theorem 6 and the calculation is a routine one.

Cor-4: If we put r=0 in theorem 7, then we get the following result:

If

$$G(x,u,w)=\sum_{n=0}^{\infty}a_nL_n^{(\alpha)}(x)g_n(u)w^n,$$

then

$$\sum_{n=0}^{\infty}\sigma_n(x,u,v)w^n=(1+w)^{\alpha}\exp(-xw)G\Big(x(1+w),u,wv\Big),$$

where

$$\sigma_n(x,u,v)=\sum_{k=0}^{n}a_k\binom{n}{k}L_n^{(\alpha-n+k)}(x)g_k(u)v^k,$$

which is found derived in [ ].

3 Applications

3.1 Application of Theorem 6

A.1: As an application of Theorem 6, we consider the following generating relation [3]:

$$\sum_{n=0}^{\infty}\binom{n+m}{n}L_{m+n}^{(\alpha)}(x)t^n=(1-t)^{-\alpha-m-1}\exp\Big(-\frac{xt}{1-t}\Big)L_m^{(\alpha)}\Big(\frac{x}{1-t}\Big) \tag{31}$$

If in our theorem, we take

$$a_n=\binom{n+m}{n},$$

then

$$G(x,t)=(1-t)^{-\alpha-m-1}\exp\Big(-\frac{xt}{1-t}\Big)L_m^{(\alpha)}\Big(\frac{x}{1-t}\Big) \tag{32}$$

Therefore by the application of our Theorem 6 we get the following generalization of the result (31):

$$\begin{aligned}&\sum_{n=0}^{\infty}z^n\sum_{k=0}^{n}\binom{m+k}{k}\binom{n+r}{k+r}L_{n+r}^{(\alpha-n+k)}(x)w^k\\&=(1+z)^{\alpha}(1-zw)^{-\alpha-m-1}\exp\Big(\frac{-xz(1+w)}{1-zw}\Big)L_m^{(\alpha)}\Big(\frac{x(1+z)}{1-zw}\Big)\end{aligned} \tag{33}$$

A.2: We now consider the following generating relation [4]:

$$\begin{aligned}&\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{(\lambda)_n}{(\alpha+m+1)_n}L_{m+n}^{(\alpha)}(x)t^n\\&=\binom{\alpha+m}{m}(1-t)^{-\lambda}\phi_2\Big[-m,\lambda;\alpha+1;x,\frac{xt}{t-1}\Big].\end{aligned} \tag{34}$$

Now, if in our theorem, we take

$$a_n=\binom{m+n}{n}\frac{(\lambda)_n}{(\alpha+m+1)_n},$$

then

$$G(x,t)=\binom{\alpha+m}{m}(1-t)^{-\lambda}\phi_2\Big[-m,\lambda;\alpha+1;x,\frac{xt}{t-1}\Big]$$

Therefore, by the application of our Theorem 6, we get the following generalization of (34)

$$\begin{aligned}&\sum_{n=0}^{\infty}z^n\sum_{k=0}^{n}\binom{m+k}{k}\frac{(\lambda)_k}{(\alpha+m+1)_k}\binom{n+r}{k+r}L_{n+r}^{(\alpha-n+k)}(x)w^k\\&=(1+z)^{\alpha}\exp(-xz)\binom{\alpha+m}{m}(1-zw)^{-\lambda}\phi_2\Big[-m,\lambda;\alpha+1;x(1+z),\frac{x(1+z)zw}{zw-1}\Big].\end{aligned}$$

3.2 Application of Theorem 7

As an application of our Theorem 7, we consider the following generating relation [4]:

$$\sum_{n=0}^{\infty} \frac{(r+n)!}{(\beta+1)_n} L_{n+r}^{(\alpha)}(x) L_n^{(\beta)}(u) w^n$$
$$= (\alpha+1)_r e^x (1-w)^{-\alpha-r-1} \psi_2 \left[\alpha+r+1; \beta+1, \alpha+1; \frac{uw}{w-1}, \frac{x}{w-1}\right] \tag{35}$$

where $\mid w \mid < 1$.

If in our Theorem 7, we take

$$a_n = \frac{(r+n)!}{(\beta+1)_n}, \qquad g_n(u) = L_n^{(\beta)}(u),$$

then

$$G(y,x,t) = (\alpha+1)_r e^x (1-w)^{-\alpha-r-1} \psi_2 \left[\alpha+r+1; \beta+1, \alpha+1; \frac{uw}{w-1}, \frac{x}{w-1}\right].$$

Therefore, by the application our Theorem 7, we get the following generalization of (35)

$$\sum_{n=0}^{\infty} \sigma_n(x,u,w) v^n$$
$$= (\alpha+1)_r (1+w)^{\alpha} e^x (1-wv)^{-\alpha-r-1} \psi_2 \left[\alpha+r+1; \beta+1, \alpha+1; \frac{uvw}{wv-1}, \frac{x(1+w)}{wv-1}\right]$$

where

$$\sigma_n(x,u,w) = \sum_{k=0}^{n} \frac{(r+k)!}{(\beta+1)_k} \binom{n+r}{k+r} L_{n+r}^{(\alpha-n+k)}(x) L_k^{(\beta)}(u) v^k.$$

References

[1] Sarama Das and Chatterjea S.K., On a partial differential operator for Laguerre polynomials, Pure Math. Manuscript, 4(1985), 187-193.

[2] Chongdar A. K., Some generating functions involving Laguerre polynomials, Bull. Cal. Math. Soc., 76(1984), 262-269.

[3] Rainville E.D., Special Functions, Macmillan (1960), New York.

[4] Srivastava H.M. and Manocha H.L., A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited) Chichester(1984), P-132.