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On a multivariable extension for the extended Jacobi polynomials

Abdullah Altın [a], Rabia Aktaş [b], Esra Erkuş-Duman [c,*]

[a] Eastern Mediterranean University, Faculty of Arts and Sciences, Department of Mathematics, Gazi Magusa, Mersin 10, Turkey
[b] Ankara University, Faculty of Science, Department of Mathematics, Tandoğan TR-06100, Ankara, Turkey
[c] Gazi University, Faculty of Sciences and Arts, Department of Mathematics, Teknikokullar TR-06500, Ankara, Turkey

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ABSTRACT

The main object of this paper is to construct a systematic investigation of a multivariable extension of the extended Jacobi polynomials and give some relations for these polynomials. We derive various families of multilinear and multilateral generating functions. We also obtain relations between the polynomials extended Jacobi polynomials and some other well-known polynomials. Other miscellaneous properties of these general families of multivariable polynomials are also discussed. Furthermore, some special cases of the results are presented in this study.

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1. Introduction

The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ of degree n are defined by the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n(1-x)^{-\alpha}(1+x)^{-\beta}}{2^n n!}D_x^n\left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}\quad\left(D_x:=\frac{d}{dx}\right) \tag{1.1}$$

or, equivalently, by

$$P_n^{(\alpha,\beta)}(x)=\binom{\alpha+n}{n}{}_2F_1\left(-n,\alpha+\beta+n+1;\alpha+1;\frac{1-x}{2}\right), \tag{1.2}$$

where ${}_2F_1$ denotes the familiar (Gauss) hypergeometric function which corresponds to the special case $r-1=s=1$ of the generalized hypergeometric function ${}_rF_s$ with r numerator and s denominator parameters. The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are generated by

$$\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)t^n=\frac{2^{\alpha+\beta}}{\rho}(1-t+\rho)^{-\alpha}(1+t+\rho)^{-\beta}, \tag{1.3}$$

where $\rho=(1-2tx+t^2)^{1/2}$ [2]. These polynomials are orthogonal over the interval $(-1,1)$ with respect to the weight function

* Corresponding author.
E-mail addresses: abdullah.altin@emu.edu.tr (A. Altın), raktas@science.ankara.edu.tr (R. Aktaş), eduman@gazi.edu.tr (E. Erkuş-Duman).

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$$\omega(x) = (1-x)^{\alpha}(1+x)^{\beta}.$$

In fact, we have the following relation

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\,dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}\delta_{m,n}$$
$$\big(\min\{\mathbb{R}(\alpha), \mathbb{R}(\beta)\} > -1;\ m,n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\big),$$

where $\delta_{m,n}$ denotes the Kronecker delta.

In order to give a unified presentation of the classical orthogonal polynomials (especially Jacobi, Laguerre and Hermite polynomials), Fujiwara [4] studied the polynomial $F_n^{(\alpha,\beta)}(x;a,b,c)$ so-called extended Jacobi polynomial (EJP) and defined it by the Rodrigues formula

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \frac{(-c)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta} D_x^n\big\{(x-a)^{n+\alpha}(b-x)^{n+\beta}\big\} \quad (c>0). \tag{1.4}$$

The polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ are essentially those that were considered by Szegö himself [9, p. 58], who showed (by means of a simple linear transformation) that these polynomials are just a constant multiple of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. By comparing the Rodrigues representations (1.1) and (1.4), it is not difficult to rewrite Szegö's observation [9, p. 58, Eq. (4.1.2)] in the form (cf., e.g., [8, p. 388, Problem 11], [6]):

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \big\{c(a-b)\big\}^n P_n^{(\alpha,\beta)}\left(\frac{2(x-a)}{a-b}+1\right) \tag{1.5}$$

or, equivalently,

$$P_n^{(\alpha,\beta)}(x) = \big\{c(a-b)\big\}^{-n} F_n^{(\alpha,\beta)}\left(\frac{1}{2}\big\{a+b+(a-b)x\big\}; a,b,c\right). \tag{1.6}$$

Thus, as already pointed out by Srivastava and Manocha [8], the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ may be looked upon as being equivalent to (and not as a generalization of) the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. Furthermore, by recourse to certain limiting processes, it is easily verified that the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ would give rise to the Laguerre and Hermite polynomials (and indeed also the Bessel polynomials) just as the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ do. Consequently, the main purpose of Fujiwara's investigation [4] is already served by the classical Jacobi polynomials themselves.

In terms of the hypergeometric function, we find from (1.2) and (1.5) that

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \big\{c(a-b)\big\}^n \binom{\alpha+n}{n} {}_2F_1\left(-n, \alpha+\beta+n+1; \alpha+1; \frac{x-a}{b-a}\right). \tag{1.7}$$

The EJPs $F_n^{(\alpha,\beta)}(x;a,b,c)$ are orthogonal over the interval (a,b) with respect to the weight function $\omega(x;a,b) = (x-a)^{\alpha}(b-x)^{\beta}$. In fact, we have

$$\int_a^b (x-a)^{\alpha}(b-x)^{\beta} F_n^{(\alpha,\beta)}(x;a,b,c) F_m^{(\alpha,\beta)}(x;a,b,c)\,dx$$
$$= \frac{c^{m+n}(-1)^{\alpha+\beta+1}(a-b)^{m+n+\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}\delta_{m,n}$$
$$\big(\min\{\mathbb{R}(\alpha), \mathbb{R}(\beta)\} > -1;\ m,n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\big). \tag{1.8}$$

The main object of this paper is to construct a systematic investigation of a multivariable extension of the EJPs $F_n^{(\alpha,\beta)}(x;a,b,c)$ and give some relations satisfied by these polynomials. We derive various families of multilinear and multilateral generating functions for the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$. We also obtain relations between the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ and some other known multivariable polynomials.

2. Multivariable EJPs and their properties

With the help of the products of the EJPs $F_n^{(\alpha,\beta)}(x;a,b,c)$, we define the multivariable EJPs with degree n as follows:

$$F_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_s;\beta_1,\ldots,\beta_s)}(\mathbf{x}) := F_{n_1,\ldots,n_s}^{(\alpha_1,\ldots,\alpha_s;\beta_1,\ldots,\beta_s)}(x_1,\ldots,x_s)$$
$$= F_{n_1}^{(\alpha_1,\beta_1)}(x_1;a_1,b_1,c_1)\ldots F_{n_s}^{(\alpha_s,\beta_s)}(x_s;a_s,b_s,c_s), \tag{2.1}$$

where $\mathbf{x} = (x_1,\ldots,x_s)$ and $\mathbf{n} = n_1 + \cdots + n_s$; $n_1,\ldots,n_s \in \mathbb{N}_0$.

The following results can easily be proved by using (1.6) and (1.7).

Theorem 2.1. *The following relation between the polynomials* $F_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_s;\beta_1,\ldots,\beta_s)}(\mathbf{x})$ *and the classical Jacobi polynomials* $P_{n_i}^{(\alpha_i,\beta_i)}(x_i)$ *holds*:

$$F_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_s;\beta_1,\ldots,\beta_s)}(\mathbf{x})=\prod_{i=1}^{s}\bigl\{c_i(a_i-b_i)\bigr\}^{n_i}P_{n_i}^{(\alpha_i,\beta_i)}\left(\frac{2(x_i-a_i)}{a_i-b_i}+1\right).$$

Theorem 2.2. *By means of the hypergeometric function* ${}_2F_1$, *we have*

$$F_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_s;\beta_1,\ldots,\beta_s)}(\mathbf{x})=\prod_{i=1}^{s}\bigl\{c_i(a_i-b_i)\bigr\}^{n_i}\binom{\alpha_i+n_i}{n_i}\times{}_2F_1\left(-n_i,\alpha_i+\beta_i+n_i+1;\alpha_i+1;\frac{x_i-a_i}{b_i-a_i}\right). \tag{2.2}$$

Corollary 2.3. *We have*

$$P_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_s;\beta_1,\ldots,\beta_s)}(\mathbf{x})=P_{n_1}^{(\alpha_1,\beta_1)}(x_1)\ldots P_{n_s}^{(\alpha_s,\beta_s)}(x_s)=\prod_{i=1}^{s}\bigl\{c_i(a_i-b_i)\bigr\}^{-n_i}F_{n_i}^{(\alpha_i,\beta_i)}\left(\frac{1}{2}\bigl\{a_i+b_i+(a_i-b_i)x_i\bigr\};a_i,b_i,c_i\right),$$

where $\mathbf{x}=(x_1,\ldots,x_s)$ *and* $\mathbf{n}=n_1+\cdots+n_s$; $n_1,\ldots,n_s\in\mathbb{N}_0$.

Now we have the following

Theorem 2.4. *The multivariable EJPs* $F_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_s,\beta_1,\ldots,\beta_s)}(\mathbf{x})$ *given by* (2.2) *are orthogonal with respect to the weight function*

$$\omega(\mathbf{x})=\omega_1(x_1;a_1,b_1)\ldots\omega_s(x_s;a_s,b_s)=(x_1-a_1)^{\alpha_1}(b_1-x_1)^{\beta_1}\ldots(x_s-a_s)^{\alpha_s}(b_s-x_s)^{\beta_s}$$

over the domain

$$\Omega=\bigl\{\mathbf{x}=(x_1,\ldots,x_s):\ a_i\leqslant x_i\leqslant b_i;\ i=1,2,\ldots,s\bigr\}.$$

Proof. By (1.8) and (2.1), we have

$$\begin{aligned}&\int_{\Omega}F_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_s;\beta_1,\ldots,\beta_s)}(\mathbf{x})F_{\mathbf{m}}^{(\alpha_1,\ldots,\alpha_s;\beta_1,\ldots,\beta_s)}(\mathbf{x})\omega(\mathbf{x})\,d\mathbf{x}\\&\quad=\int_{a_1}^{b_1}F_{n_1}^{(\alpha_1,\beta_1)}(x_1;a_1,b_1,c_1)F_{m_1}^{(\alpha_1,\beta_1)}(x_1;a_1,b_1,c_1)(x_1-a_1)^{\alpha_1}(b_1-x_1)^{\beta_1}\,dx_1\\&\qquad\times\cdots\times\int_{a_s}^{b_s}F_{n_s}^{(\alpha_s,\beta_s)}(x_s;a_s,b_s,c_s)F_{m_s}^{(\alpha_s,\beta_s)}(x_s;a_s,b_s,c_s)(x_s-a_s)^{\alpha_s}(b_s-x_s)^{\beta_s}\,dx_s\\&\quad=\prod_{i=1}^{s}c_i^{m_i+n_i}(-1)^{\alpha_i+\beta_i+1}\frac{(a_i-b_i)^{\alpha_i+\beta_i+m_i+n_i+1}}{n_i!(\alpha_i+\beta_i+2n_i+1)}\frac{\Gamma(\alpha_i+n_i+1)\Gamma(\beta_i+n_i+1)}{\Gamma(\alpha_i+\beta_i+n_i+1)}\delta_{m_i,n_i},\end{aligned}$$

where $d\mathbf{x}=dx_1\ldots dx_s$ and $\min\{\mathbb{R}(\alpha_i),\mathbb{R}(\beta_i)\}>-1$; $m_i,n_i\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$; $i=1,2,\ldots,s$. The proof is completed. □

3. Relations between EJP and Chan–Chyan–Srivastava multivariable polynomials

Recently, Chan, Chyan and Srivastava [1] have introduced and investigated the multivariable extension of the classical Lagrange polynomials $g_n^{(\alpha,\beta)}(x,y)$ [2, p. 267] generated by

$$\prod_{j=1}^{r}\bigl\{(1-x_jt)^{-\alpha_j}\bigr\}=\sum_{n=0}^{\infty}g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)t^n, \tag{3.1}$$

where $|t|<\min\{|x_1|^{-1},\ldots,|x_r|^{-1}\}$.

In this section, we give various relations between Lagrange and EJP, Lagrange and multivariable EJP, Chan–Chyan–Srivastava and Jacobi polynomials. We also obtain an equality for Chan–Chyan–Srivastava polynomials with respect to hypergeometric function ${}_2F_1(x)$.

It is well known that the following equality between Jacobi and Lagrange polynomials [1]:

$$P_n^{(-\alpha-n,-\beta-n)}\left(\frac{x+y}{x-y}\right)=(y-x)^{-n}g_n^{(\alpha,\beta)}(x,y) \tag{3.2}$$

holds. We first get the following result.

Theorem 3.1. *For the EJPs $F_n^{(\alpha,\beta)}(x;a,b,c)$, we have*

$$F_n^{(-\alpha-n,-\beta-n)}\left(\frac{ax-by}{x-y};a,b,c\right)=\left\{c(a-b)\right\}^n(y-x)^{-n}g_n^{(\alpha,\beta)}(x,y).$$

Proof. In (1.5), taking $\frac{2(x-a)}{a-b}+1\rightarrow\frac{x+y}{x-y}$, $\alpha\rightarrow-\alpha-n$ and $\beta\rightarrow-\beta-n$, we obtain

$$F_n^{(-\alpha-n,-\beta-n)}\left(\frac{ax-by}{x-y};a,b,c\right)=\left\{c(a-b)\right\}^nP_n^{(-\alpha-n,-\beta-n)}\left(\frac{x+y}{x-y}\right).$$

Using (3.2) in the right-hand side of the last equality, we find the desired result. □

The next result can be obtained from Theorem 3.1, immediately.

Theorem 3.2. *For the multivariable EJPs, we have*

$$\begin{aligned}F_{\mathbf{n}}^{(\alpha_1,\dots,\alpha_s;\,\beta_1,\dots,\beta_s)}(\mathbf{x};\mathbf{y})&=\prod_{i=1}^{s}F_{n_i}^{(-\alpha_i-n_i,-\beta_i-n_i)}\left(\frac{a_ix_i-b_iy_i}{x_i-y_i};a_i,b_i,c_i\right)\\&=\prod_{i=1}^{s}\left\{c_i(a_i-b_i)\right\}^{n_i}(y_i-x_i)^{-n_i}g_{n_i}^{(\alpha_i,\beta_i)}(x_i,y_i),\end{aligned}$$

where $\mathbf{x}=(x_1,\dots,x_s)$, $\mathbf{y}=(y_1,\dots,y_s)$, $\mathbf{n}=n_1+\cdots+n_s$; $n_1,\dots,n_s\in\mathbb{N}_0$ and $F_{\mathbf{n}}^{(\alpha_1,\dots,\alpha_s;\beta_1,\dots,\beta_s)}(\mathbf{x};\mathbf{y})$ is multivariable EJP of degree $\mathbf{n}$ with $2s$ variables.

Theorem 3.3. *The Chan–Chyan–Srivastava multivariable polynomials $g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ satisfy the following equality*

$$\begin{aligned}g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)=&\sum_{n_1+\cdots+n_{r-1}=n}\prod_{i=1}^{r-2}\binom{-\alpha_i}{n_i}(-1)^{n_i}x_i^{n_i}(x_{r-1}-x_r)^{n_{r-1}}\\&\times P_{n_{r-1}}^{(-\alpha_r-n_{r-1},-\alpha_{r-1}-n_{r-1})}\left(\frac{x_r+x_{r-1}}{x_r-x_{r-1}}\right).\end{aligned}$$

Proof. For each $j=1,\dots,r$, taking Taylor expansions of the functions $(1-x_jt)^{-\alpha_j}$ and after some simple calculations, it follows from (3.1) that the equality

$$g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)=\sum_{k_1+\cdots+k_r=n}(\alpha_1)_{k_1}\dots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\dots\frac{x_r^{k_r}}{k_r!}$$

holds (see [1]), where $(\lambda)_k:=\lambda(\lambda+1)\dots(\lambda+k-1)$ and $(\lambda)_0:=1$ denotes the Pochhammer symbol. From this expression, we may write that

$$\begin{aligned}g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)&=\sum_{k_1+\cdots+k_r=n}(\alpha_1)_{k_1}\dots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\dots\frac{x_r^{k_r}}{k_r!}\\&=\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\dots\sum_{k_{r-1}=0}^{k_{r-2}}(\alpha_1)_{n-k_1}(\alpha_2)_{k_1-k_2}\dots(\alpha_{r-1})_{k_{r-2}-k_{r-1}}(\alpha_r)_{k_{r-1}}\\&\quad\times\frac{x_1^{n-k_1}}{(n-k_1)!}\frac{x_2^{k_1-k_2}}{(k_1-k_2)!}\dots\frac{x_{r-1}^{k_{r-2}-k_{r-1}}}{(k_{r-2}-k_{r-1})!}\frac{x_r^{k_{r-1}}}{k_{r-1}!}\end{aligned}$$

$$= \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{r-1}=0}^{k_{r-2}} \binom{-\alpha_1}{n-k_1} \cdots \binom{-\alpha_{r-1}}{k_{r-2}-k_{r-1}} \binom{-\alpha_r}{k_{r-1}} (-1)^n \times x_1^{n-k_1} x_2^{k_1-k_2} \ldots x_r^{k_{r-1}}.$$

Multiplying and dividing by $(x_{r-1}-x_r)^{k_{r-1}+k_{r-2}}$ the last equality and making the necessary arrangements, we can obtain that

$$g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{r-2}=0}^{k_{r-3}} \binom{-\alpha_1}{n-k_1} \cdots \binom{-\alpha_{r-2}}{k_{r-3}-k_{r-2}} (-1)^{n-k_{r-2}} \times x_1^{n-k_1} x_2^{k_1-k_2} \ldots x_{r-2}^{k_{r-3}-k_{r-2}} (x_{r-1}-x_r)^{k_{r-2}} \times P_{k_{r-2}}^{(-\alpha_r-k_{r-2},-\alpha_{r-1}-k_{r-2})}\left(\frac{x_r+x_{r-1}}{x_r-x_{r-1}}\right)$$

$$= \sum_{k_1+\cdots+k_{r-1}=n} \prod_{i=1}^{r-2} \binom{-\alpha_i}{k_i} (-1)^{k_i} x_i^{k_i} (x_{r-1}-x_r)^{k_{r-1}} \times P_{k_{r-1}}^{(-\alpha_r-k_{r-1},-\alpha_{r-1}-k_{r-1})}\left(\frac{x_r+x_{r-1}}{x_r-x_{r-1}}\right).$$

Therefore, taking n_i instead of k_i $(i=1,\ldots,r-1)$ in the last statement, the proof is completed. □

Theorem 3.4. *The Chan–Chyan–Srivastava multivariable polynomials $g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)$ have the following relations*:

$$g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) = \sum_{n_1+\cdots+n_{r-1}=n} \prod_{i=1}^{r-2} \binom{-\alpha_i}{n_i} (-1)^{n_i} x_i^{n_i} (x_{r-1}-x_r)^{n_{r-1}} \{c(a-b)\}^{-n_{r-1}} \times F_{n_{r-1}}^{(-\alpha_r-n_{r-1},-\alpha_{r-1}-n_{r-1})}\left(\frac{ax_r-bx_{r-1}}{x_r-x_{r-1}}; a,b,c\right)$$

and

$$g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) = \sum_{n_1+\cdots+n_{r-1}=n} \prod_{i=1}^{r-2} \binom{-\alpha_i}{n_i} (-1)^{n_i} x_i^{n_i} (x_{r-1}-x_r)^{n_{r-1}} \binom{-\alpha_r}{n_{r-1}} \times {}_2F_1\left(-n_{r-1}, -\alpha_r-\alpha_{r-1}-n_{r-1}+1; -\alpha_r-n_{r-1}+1; \frac{x_{r-1}}{x_{r-1}-x_r}\right).$$

Proof. In (1.5), taking $\frac{2(x-a)}{a-b}+1 \to \frac{x_r+x_{r-1}}{x_r-x_{r-1}}$, $n \to n_{r-1}$, $\alpha \to -\alpha_r-n_{r-1}$ and $\beta \to -\alpha_{r-1}-n_{r-1}$, we obtain

$$P_{n_{r-1}}^{(-\alpha_r-n_{r-1},-\alpha_{r-1}-n_{r-1})}\left(\frac{x_r+x_{r-1}}{x_r-x_{r-1}}\right) = \{c(a-b)\}^{-n_{r-1}} \times F_{n_{r-1}}^{(-\alpha_r-n_{r-1},-\alpha_{r-1}-n_{r-1})}\left(\frac{ax_r-bx_{r-1}}{x_r-x_{r-1}}; a,b,c\right).$$

Using this fact and Theorem 3.3, the first equality follows. Similarly, by (1.7) and the first equality of this theorem, we obtain easily the second one. □

4. Generating functions and recurrence relations for multivariable EJPs

In [7] it was shown that the Jacobi polynomials are generated by

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n} t^n = (1-t)^{-1-\alpha-\beta}\, {}_2F_1\left(\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2}; 1+\alpha; \frac{2t(x-1)}{(1-t)^2}\right).$$

Using this formula and (1.6), it is easily seen that the EJPs $F_n^{(\alpha,\beta)}(x,a,b,c)$ hold

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n \{c(a-b)\}^{-n} F_n^{(\alpha,\beta)}(x;a,b,c)}{(1+\alpha)_n} t^n = (1-t)^{-1-\alpha-\beta}\, {}_2F_1\left(\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2}; 1+\alpha; \frac{4t(x-a)}{(a-b)(1-t)^2}\right). \tag{4.1}$$

Theorem 4.1. *For the multivariable EJPs $F_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(\mathbf{x})$, we have*

$$\sum_{n_1,\ldots,n_r=0}^{\infty}\gamma(n_1,\ldots,n_r)F_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(\mathbf{x})t_1^{n_1}\ldots t_r^{n_r}$$
$$=\prod_{i=1}^{r}(1-t_i)^{-1-\alpha_i-\beta_i}\,{}_2F_1\left(\frac{\alpha_i+\beta_i+1}{2},\frac{\alpha_i+\beta_i+2}{2};1+\alpha_i;\frac{4t_i(x_i-a_i)}{(a_i-b_i)(1-t_i)^2}\right) \tag{4.2}$$

or, equivalently

$$\sum_{n_1,\ldots,n_r=0}^{\infty}\gamma(n_1,\ldots,n_r)F_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}\big((\mathbf{b}-\mathbf{a})\mathbf{v}+\mathbf{a}\big)t_1^{n_1}\ldots t_r^{n_r}$$
$$=\prod_{i=1}^{r}(1-t_i)^{-1-\alpha_i-\beta_i}\,{}_2F_1\left(\frac{\alpha_i+\beta_i+1}{2},\frac{\alpha_i+\beta_i+2}{2};1+\alpha_i;\frac{-4t_iv_i}{(1-t_i)^2}\right), \tag{4.3}$$

where

$$\gamma(n_1,\ldots,n_r)=\prod_{i=1}^{r}\frac{(\alpha_i+\beta_i+1)_{n_i}\{c_i(a_i-b_i)\}^{-n_i}}{(1+\alpha_i)_{n_i}},$$
$$(\mathbf{b}-\mathbf{a})\mathbf{v}+\mathbf{a}=\big((b_1-a_1)v_1+a_1,\ldots,(b_r-a_r)v_r+a_r\big).$$

Proof. Using (2.1) and (4.1), we obtain the first generating relation. Changing appropriate variables, we have the second one. □

Now, we need the following lemma to obtain some recurrence relations for multivariable EJPs $F_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(\mathbf{x})$.

Lemma 4.2. *Let a generating function for $f_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$ be*

$$(1-t_1)^{-d_1}\ldots(1-t_r)^{-d_r}\Psi\left(\frac{-4x_1t_1}{(1-t_1)^2},\ldots,\frac{-4x_rt_r}{(1-t_r)^2}\right)=\sum_{n_1,\ldots,n_r=0}^{\infty}f_{n_1,\ldots,n_r}(x_1,\ldots,x_r)t_1^{n_1}\ldots t_r^{n_r}, \tag{4.4}$$

where $f_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$ is a polynomial of degree n_i with respect to x_i (of total degree $n=n_1+\cdots+n_r$), provided that

$$\Psi(u_1,\ldots,u_r)=\psi_1(u_1)\ldots\psi_r(u_r);\qquad u_i=\frac{-4x_it_i}{(1-t_i)^2},\quad i=1,\ldots,r,$$
$$\psi_i(u_i)=\sum_{n_i=0}^{\infty}\varphi_{n_i}u_i^{n_i},\quad \varphi_0\neq 0.$$

Then we have

$$x_i\frac{\partial f_{n_1,\ldots,n_r}}{\partial x_i}-n_if_{n_1,\ldots,n_r}=-(d_i+n_i-1)f_{n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_r}-x_i\frac{\partial f_{n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_r}}{\partial x_i}, \tag{4.5}$$

$$x_i\frac{\partial f_{n_1,\ldots,n_r}}{\partial x_i}-n_if_{n_1,\ldots,n_r}=-d_i\sum_{k=0}^{n_i-1}f_{n_1,\ldots,n_{i-1},k,n_{i+1},\ldots,n_r}-2x_i\sum_{k=0}^{n_i-1}\frac{\partial f_{n_1,\ldots,n_{i-1},k,n_{i+1},\ldots,n_r}}{\partial x_i}, \tag{4.6}$$

$$x_i\frac{\partial f_{n_1,\ldots,n_r}}{\partial x_i}-n_if_{n_1,\ldots,n_r}=\sum_{k=0}^{n_i-1}(-1)^{n_i-k}(d_i+2k)f_{n_1,\ldots,n_{i-1},k,n_{i+1},\ldots,n_r}, \tag{4.7}$$

for $i=1,2,\ldots,r$.

Proof. Differentiating (4.4) with respect to x_i and t_i and making necessary arrangements, we obtain the desired relations. □

As a result of Lemma 4.2, considering (4.3), we can write that

$$f_{n_1,\dots,n_r}(\mathbf{v}) = \gamma(n_1,\dots,n_r) F_{\mathbf{n}}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}\big((\mathbf{b}-\mathbf{a})\mathbf{v}+\mathbf{a}\big),$$
$$\varphi_{n_i} = \frac{(\alpha_i+\beta_i+1)_{2n_i}}{2^{2n_i} n_i!(1+\alpha_i)_{n_i}}, \qquad d_i = \alpha_i+\beta_i+1, \quad i=1,\dots,r.$$

With the help of Lemma 4.2 and also considering (4.5)–(4.7), one can easily obtain the next result.

Theorem 4.3. *For the multivariable EJPs $F_{\mathbf{n}}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(\mathbf{x})$, we have the following recurrence relations, respectively*:

$$\begin{aligned}
&(x_i-a_i)\gamma(n_1,\dots,n_r)\frac{\partial}{\partial x_i} F_{n_1,\dots,n_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)} \\
&\quad + (x_i-a_i)\gamma(n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_r)\frac{\partial}{\partial x_i} F_{n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)} \\
&\quad = -(\alpha_i+\beta_i+n_i)\gamma(n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_r) F_{n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)} \\
&\quad + n_i\gamma(n_1,\dots,n_r) F_{n_1,\dots,n_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)},
\end{aligned}$$

$$\begin{aligned}
&(x_i-a_i)\gamma(n_1,\dots,n_r)\frac{\partial}{\partial x_i} F_{n_1,\dots,n_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)} - n_i\gamma(n_1,\dots,n_r) F_{n_1,\dots,n_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)} \\
&\quad = -\sum_{k=0}^{n_i-1}\gamma(n_1,\dots,n_{i-1},k,n_{i+1},\dots,n_r)\bigg\{(\alpha_i+\beta_i+1) F_{n_1,\dots,n_{i-1},k,n_{i+1},\dots,n_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)} \\
&\quad + 2(x_i-a_i)\frac{\partial}{\partial x_i} F_{n_1,\dots,n_{i-1},k,n_{i+1},\dots,n_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}\bigg\},
\end{aligned}$$

and

$$\begin{aligned}
&(x_i-a_i)\gamma(n_1,\dots,n_r)\frac{\partial}{\partial x_i} F_{n_1,\dots,n_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)} - n_i\gamma(n_1,\dots,n_r) F_{n_1,\dots,n_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)} \\
&\quad = \sum_{k=0}^{n_i-1}(-1)^{n_i-k}(\alpha_i+\beta_i+1+2k)\gamma(n_1,\dots,n_{i-1},k,n_{i+1},\dots,n_r) F_{n_1,\dots,n_{i-1},k,n_{i+1},\dots,n_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}.
\end{aligned}$$

Fujiwara [4] shows that the EJPs are generated by

$$\sum_{n=0}^{\infty} F_n^{(\alpha,\beta)}(x)t^n = \frac{2^{\alpha+\beta}}{\rho}(1+\delta t+\rho)^{-\alpha}(1-\delta t+\rho)^{-\beta}, \tag{4.8}$$

where

$$\rho = \big\{1+2tX'(x)+\delta^2t^2\big\}^{1/2}, \qquad X(x) = c(x-a)(b-x),$$
$$\delta = c(b-a).$$

As a generalization of this expression, we have the following result by using (4.8).

Theorem 4.4. *For the multivariable EJPs, we have*

$$\sum_{n=0}^{\infty} H_n^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(x_1,\dots,x_r)t^n = \prod_{i=1}^{r}\frac{2^{\alpha_i+\beta_i}}{\rho_i}(1+\delta_i t+\rho_i)^{-\alpha_i}(1-\delta_i t+\rho_i)^{-\beta_i},$$

where

$$H_n^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(x_1,\dots,x_r) = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\cdots\sum_{n_{r-1}=0}^{n-n_1-\cdots-n_{r-2}} F_{n-(n_1+\cdots+n_{r-1}),n_1,\dots,n_{r-1}}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(x_1,\dots,x_r);$$

and also

$$\rho_i = \big\{1+2tX_i'(x_i)+\delta_i^2t^2\big\}^{1/2}, \qquad X_i(x_i) = c_i(x_i-a_i)(b_i-x_i),$$
$$\delta_i = c_i(b_i-a_i).$$

5. Partial differential equation for the product of EJPs

Lee [5] obtained a partial differential equation for the product of two Jacobi polynomials. Actually, the family of polynomials having the form

$$\{\Phi_n\}\big|_{n=0}^{\infty} = \big\{P_{n-k}^{(\alpha,\beta)}(x)P_k^{(\gamma,\delta)}(y)\big\}\big|_{k=0,\,n=0}^{n,\infty}$$

holds the equation

$$\begin{aligned}
&\big(1-x^2\big)^2 u_{xxxx} - 2\big(1-x^2\big)\big(1-y^2\big)u_{xxyy} + \big(1-y^2\big)^2 u_{yyyy} \\
&\quad + 2\big[\beta-\alpha-(\alpha+\beta+4)x\big]\big(1-x^2\big)u_{xxx} - 2\big[\beta-\alpha-(\alpha+\beta+2)x\big]\big(1-y^2\big)u_{xyy} \\
&\quad - 2\big[\delta-\gamma-(\delta+\gamma+2)y\big]\big(1-x^2\big)u_{xxy} + 2\big[\delta-\gamma-(\delta+\gamma+4)y\big]\big(1-y^2\big)u_{yyy} \\
&\quad + \big[A_n\big(1-x^2\big) + a_1(x)\big(a_1(x)-2x\big)\big]u_{xx} + \big[B_n\big(1-y^2\big) + b_1(y)\big(b_1(y)-2y\big)\big]u_{yy} \\
&\quad - 2b_1(y)a_1(x)u_{xy} + C_n a_1(x)u_x + D_n b_1(y)u_y + n(n+\alpha+\beta+1) \\
&\quad \times (n+\gamma+\delta+1)(n+\alpha+\beta+\gamma+\delta+2)u = 0,
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
A_n &= 2n^2 + (\gamma+\delta+2n+1)(\alpha+\beta+\gamma+\delta+2) - 2(\alpha+\beta+3), \\
B_n &= 2n^2 + (\alpha+\beta+2n+1)(\alpha+\beta+\gamma+\delta+2) - 2(\gamma+\delta+3), \\
C_n &= 2n^2 + (\gamma+\delta+2n+1)(\alpha+\beta+\gamma+\delta+2) - (\alpha+\beta+2), \\
D_n &= 2n^2 + (\alpha+\beta+2n+1)(\alpha+\beta+\gamma+\delta+2) - (\gamma+\delta+2), \\
a_1(x) &= \beta-\alpha-(\alpha+\beta+2)x, \qquad b_1(y) = \delta-\gamma-(\delta+\gamma+2)y.
\end{aligned}$$

By means of this equation and the relation between Jacobi polynomials and EJPs given by (1.5), we give a partial differential equation satisfied by the product of two EJPs as follows.

Theorem 5.1. *The family of polynomials given by*

$$\{\Phi_n\}\big|_{n=0}^{\infty} = \big\{F_{n-k}^{(\alpha,\beta)}(x;a_1,b_1,c_1)F_k^{(\gamma,\delta)}(y;a_2,b_2,c_2)\big\}\big|_{k=0,\,n=0}^{n,\infty}$$

holds

$$\begin{aligned}
&(x-a_1)^2(x-b_1)^2 u_{xxxx} + (y-a_2)^2(y-b_2)^2 u_{yyyy} \\
&\quad - 2(x-a_1)(x-b_1)(y-a_2)(y-b_2)u_{xxyy} \\
&\quad - \big[(\beta-\alpha)(a_1-b_1) - (\alpha+\beta+4)(2x-a_1-b_1)\big](x-a_1)(x-b_1)u_{xxx} \\
&\quad + \big[(\beta-\alpha)(a_1-b_1) - (\alpha+\beta+2)(2x-a_1-b_1)\big](y-a_2)(y-b_2)u_{xyy} \\
&\quad + \big[(\delta-\gamma)(a_2-b_2) - (\delta+\gamma+2)(2y-a_2-b_2)\big](x-a_1)(x-b_1)u_{xxy} \\
&\quad - \big[(\delta-\gamma)(a_2-b_2) - (\delta+\gamma+4)(2y-a_2-b_2)\big](y-a_2)(y-b_2)u_{yyy} \\
&\quad + \left\{-A_n(x-a_1)(x-b_1) + \frac{a(x)}{4}\big(a(x) - 2(2x-a_1-b_1)\big)\right\}u_{xx} \\
&\quad + \left\{-B_n(y-a_2)(y-b_2) + \frac{b(y)}{4}\big(b(y) - 2(2y-a_2-b_2)\big)\right\}u_{yy} \\
&\quad - \frac{1}{2}a(x)b(y)u_{xy} + \frac{1}{2}C_n a(x)u_x + \frac{1}{2}D_n b(y)u_y + n(n+\alpha+\beta+1) \\
&\quad \times (n+\gamma+\delta+1)(n+\alpha+\beta+\gamma+\delta+2)u = 0,
\end{aligned}$$

where A_n, B_n, C_n and D_n are the same as above; *and also*

$$\begin{aligned}
a(x) &= (\beta-\alpha)(a_1-b_1) - (\alpha+\beta+2)(2x-a_1-b_1), \\
b(y) &= (\delta-\gamma)(a_2-b_2) - (\delta+\gamma+2)(2y-a_2-b_2).
\end{aligned}$$

Proof. In (5.1), taking

$$x \to \frac{2(x-a_1)}{a_1-b_1}+1, \qquad y \to \frac{2(y-a_2)}{a_2-b_2}+1$$

and using 1.5, we easily obtain the desired equation for the product of two EJPs. □

6. Integral representations for the multivariable EJPs

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ have the following integral representation given by Feldheim [3]:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{\Gamma(\alpha+\beta+n+1)} \int_0^\infty t^{\alpha+\beta+n} e^{-t} L_n^{(\alpha)}\left(\frac{1}{2}(1-x)t\right) dt$$
$$\big(\alpha+\beta > -1,\ \forall n \in \{0,1,\dots\}\big), \tag{6.1}$$

where $L_n^{(\alpha)}$ is the Laguerre polynomial of degree n. By (6.1) and (1.5), it is easily seen that the EJPs have the following integral representation

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \frac{\{c(a-b)\}^n}{\Gamma(\alpha+\beta+n+1)} \int_0^\infty t^{\alpha+\beta+n} e^{-t} L_n^{(\alpha)}\left(\frac{x-a}{b-a}t\right) dt.$$

As a result of this formula, we obtain the next result.

Theorem 6.1. *For the multivariable EJPs $F_{\mathbf{n}}^{(\alpha_1,\dots,\alpha_s;\beta_1,\dots,\beta_s)}(\mathbf{x})$, we have the following integral representation*

$$F_{\mathbf{n}}^{(\alpha_1,\dots,\alpha_s;\beta_1,\dots,\beta_s)}(\mathbf{x})$$
$$= \int_0^\infty \int_0^\infty \dots \int_0^\infty \left\{ \prod_{i=1}^s \frac{\{c_i(a_i-b_i)\}^{n_i}}{\Gamma(\alpha_i+\beta_i+n_i+1)} t_i^{\alpha_i+\beta_i+n_i} e^{-t_i} L_{n_i}^{(\alpha_i)}\left(\frac{x_i-a_i}{b_i-a_i}t_i\right) \right\} dt$$
$$\big(\alpha_i+\beta_i > -1,\ i=1,\dots,s,\ \forall n_i \in \{0,1,\dots\}\big),$$

where $dt = dt_1 \dots dt_s$.

Theorem 6.2. *For the multivariable EJPs $F_{\mathbf{n}}^{(\alpha_1,\dots,\alpha_s;\beta_1,\dots,\beta_s)}(\mathbf{z})$, we have another integral representation formula*

$$F_{\mathbf{n}}^{(\alpha_1,\dots,\alpha_s;\beta_1,\dots,\beta_s)}(z_1,\dots,z_s)$$
$$= F_{n_1}^{(\alpha_1,\beta_1)}(z_1;a_1,b_1,c_1) \dots F_{n_s}^{(\alpha_s,\beta_s)}(z_s;a_s,b_s,c_s)$$
$$= \int_0^1 \int_0^1 \dots \int_0^1 \prod_{j=1}^s A(s) t_j^{\alpha_j+\beta_j+n_j} (1-t_j)^{-\beta_j-n_j-1} (a_j-b_j+z_jt_j-a_jt_j)^{n_j}\, dt$$
$$\left(\operatorname{Re}(\alpha_j+1) > \operatorname{Re}(\alpha_j+\beta_j+n_j+1) > 0,\ \left|\frac{z_j-a_j}{a_j-b_j}\right| < 1;\ j=1,\dots,s\right),$$

where

$$dt = dt_1 \dots dt_s, \qquad A(s) = \prod_{j=1}^s \frac{c_j^{n_j}\Gamma(\alpha_j+n_j+1)}{\Gamma(n_j+1)\Gamma(\alpha_j+\beta_j+n_j+1)\Gamma(-\beta_j-n_j)}.$$

Proof. We know that the integral representation for the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ (see [9]) is

$$P_n^{(\alpha,\beta)}(z) = \int_0^1 t^{\alpha+\beta+n}(1-t)^{-\beta-n-1}(2-t+tz)^n\, dt \frac{\Gamma(\alpha+n+1)}{2^n\Gamma(n+1)\Gamma(\alpha+\beta+n+1)\Gamma(-\beta-n)},$$
$$\operatorname{Re}(\alpha+1) > \operatorname{Re}(\alpha+\beta+n+1) > 0, \qquad \left|\frac{z-1}{2}\right| < 1.$$

Using this equality in (1.5) and (2.1), we obtain the desired result. □

7. Multilinear and multilateral generating functions

In this section, we derive several families of multilinear and multilateral generating functions for the multivariable EJPs given by (2.2).

We begin by stating the following theorem.

Theorem 7.1. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,\dots,y_s)$ of s complex variables $y_1,\dots,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\nu}(y_1,\dots,y_s;z):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,\dots,y_s)z^k \quad (a_k\neq 0,\ \mu,\nu\in\mathbb{C}) \tag{7.1}$$

and

$$\Theta_{n,p,\mu,\nu}(x_1,\dots,x_r;y_1,\dots,y_s;\zeta):=\sum_{k=0}^{[n/p]}a_kH_{n-pk}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(x_1,\dots,x_r)\Omega_{\mu+\nu k}(y_1,\dots,y_s)\zeta^k,\quad n,p\in\mathbb{N}. \tag{7.2}$$

Then we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\Theta_{n,p,\mu,\nu}\left(x_1,\dots,x_r;y_1,\dots,y_s;\frac{\eta}{t^p}\right)t^n\\&\quad=\prod_{i=1}^{r}\frac{2^{\alpha_i+\beta_i}}{\rho_i}(1+\delta_it+\rho_i)^{-\alpha_i}(1-\delta_it+\rho_i)^{-\beta_i}\Lambda_{\mu,\nu}(y_1,\dots,y_s;\eta)\end{aligned} \tag{7.3}$$

provided that each member of (7.3) *exists, where*

$$\rho_i=\left\{1+2tX_i'(x_i)+\delta_i^2t^2\right\}^{1/2},\qquad X_i(x_i)=c_i(x_i-a_i)(b_i-x_i),$$
$$\delta_i=c_i(b_i-a_i).$$

Proof. For convenience, let S denote the first member of the assertion (7.3) of Theorem 7.1. Then, upon substituting for the polynomials

$$\Theta_{n,p,\mu,\nu}\left(x_1,\dots,x_r;y_1,\dots,y_s;\frac{\eta}{t^p}\right)$$

from the definition (7.2) into the left-hand side of (7.3), we obtain

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_kH_{n-pk}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(x_1,\dots,x_r)\Omega_{\mu+\nu k}(y_1,\dots,y_s)\eta^kt^{n-pk}. \tag{7.4}$$

Upon inverting the order of summation in (7.4), if we replace n by $n+pk$, we can write

$$\begin{aligned}S&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_kH_{n}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(x_1,\dots,x_r)\Omega_{\mu+\nu k}(y_1,\dots,y_s)\eta^kt^{n}\\&=\sum_{n=0}^{\infty}H_{n}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(x_1,\dots,x_r)t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,\dots,y_s)\eta^k\\&=\prod_{i=1}^{r}\frac{2^{\alpha_i+\beta_i}}{\rho_i}(1+\delta_it+\rho_i)^{-\alpha_i}(1-\delta_it+\rho_i)^{-\beta_i}\Lambda_{\mu,\nu}(y_1,\dots,y_s;\eta),\end{aligned}$$

which completes the proof of Theorem 7.1. □

Theorem 7.2. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,\dots,y_s)$ of s complex variables $y_1,\dots,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\nu}(y_1,\dots,y_s;z):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,\dots,y_s)z^k \quad (a_k\neq 0,\ \mu,\nu\in\mathbb{C}). \tag{7.5}$$

Then we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2,\ldots,n_r=0}^{\infty}\sum_{k=0}^{[n_1/p]} a_k\gamma(n_1-pk,n_2,\ldots,n_r)$$
$$\times F_{n_1-pk,n_2,\ldots,n_r}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(x_1,\ldots,x_r)t_2^{n_2}\ldots t_r^{n_r}\Omega_{\mu+\nu k}(y_1,\ldots,y_s)\eta^k t_1^{n_1-pk}$$
$$=\Lambda_{\mu,\nu}(y_1,\ldots,y_s;\eta)\prod_{i=1}^{r}(1-t_i)^{-1-\alpha_i-\beta_i}$$
$$\times {}_2F_1\left(\frac{\alpha_i+\beta_i+1}{2},\frac{\alpha_i+\beta_i+2}{2};1+\alpha_i;\frac{4t_i(x_i-a_i)}{(a_i-b_i)(1-t_i)^2}\right) \tag{7.6}$$

provided that each member of (7.6) *exists, where*

$$\gamma(n_1,\ldots,n_r)=\prod_{i=1}^{r}\frac{(\alpha_i+\beta_i+1)_{n_i}\{c_i(a_i-b_i)\}^{-n_i}}{(1+\alpha_i)_{n_i}}.$$

Proof. For convenience, let S denote the first member of the assertion (7.6) of Theorem 7.2. Straightforward calculations give

$$S=\sum_{n_1,\ldots,n_r=0}^{\infty}\sum_{k=0}^{\infty}a_k\gamma(n_1,\ldots,n_r)F_{n_1,\ldots,n_r}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(x_1,\ldots,x_r)t_1^{n_1}\ldots t_r^{n_r}\Omega_{\mu+\nu k}(y_1,\ldots,y_s)\eta^k$$
$$=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,\ldots,y_s)\eta^k\sum_{n_1,\ldots,n_r=0}^{\infty}\gamma(n_1,\ldots,n_r)F_{n_1,\ldots,n_r}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(x_1,\ldots,x_r)t_1^{n_1}\ldots t_r^{n_r}.$$

If we use (4.2), then the proof of Theorem 7.2 is completed. □

8. Further consequences

By expressing the multivariable function

$$\Omega_{\mu+\nu k}(y_1,\ldots,y_s)\quad(k\in\mathbb{N}_0,\ s\in\mathbb{N})$$

in terms of simpler function of one and more variables, we can give further applications of Theorem 7.1 as well as Theorem 7.2. For example, if we set

$$s=r\quad\text{and}\quad\Omega_{\mu+\nu k}(y_1,\ldots,y_r)=g_{\mu+\nu k}^{(\gamma_1,\ldots,\gamma_r)}(y_1,\ldots,y_r)$$

in Theorem 7.1, where the Chan–Chyan–Srivastava multivariable polynomials

$$g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)$$

is generated by (3.1), then we obtain the following result which provides a class of bilateral generating functions for the Chan–Chyan–Srivastava multivariable polynomials and the multivariable EJPs defined by (2.2).

Corollary 8.1. *If* $\Lambda_{\mu,\nu}(y_1,\ldots,y_r;z):=\sum_{k=0}^{\infty}a_k g_{\mu+\nu k}^{(\gamma_1,\ldots,\gamma_r)}(y_1,\ldots,y_r)z^k$ *where* $a_k\neq 0$, $\nu,\mu\in\mathbb{N}_0$; *and*

$$\Theta_{n,p,\mu,\nu}(x_1,\ldots,x_r;y_1,\ldots,y_r;\zeta):=\sum_{k=0}^{[n/p]}a_k H_{n-pk}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(x_1,\ldots,x_r)g_{\mu+\nu k}^{(\gamma_1,\ldots,\gamma_r)}(y_1,\ldots,y_r)\zeta^k,$$

where $n,p\in\mathbb{N}$ *and*

$$H_n^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(x_1,\ldots,x_r)=\sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\cdots\sum_{n_{r-1}=0}^{n-n_1-\cdots-n_{r-2}}F_{n-(n_1+\cdots+n_{r-1}),n_1,\ldots,n_{r-1}}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(x_1,\ldots,x_r),$$

then we have

$$\sum_{n=0}^{\infty}\Theta_{n,p,\mu,\nu}\left(x_1,\ldots,x_r;y_1,\ldots,y_r;\frac{\eta}{t^p}\right)t^n=\prod_{i=1}^{r}\frac{2^{\alpha_i+\beta_i}}{\rho_i}(1+\delta_i t+\rho_i)^{-\alpha_i}(1-\delta_i t+\rho_i)^{-\beta_i}$$
$$\times\Lambda_{\mu,\nu}(y_1,\ldots,y_r;\eta) \tag{8.1}$$

provided that each member of (8.1) *exists.*

Remark 8.1. Using the generating relation (3.1) for the Chan–Chyan–Srivastava multivariable polynomials and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} H_{n-pk}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(x_1,\ldots,x_r) g_k^{(\gamma_1,\ldots,\gamma_r)}(y_1,\ldots,y_r)\eta^k t^{n-pk}$$
$$= \prod_{i=1}^{r} \frac{2^{\alpha_i+\beta_i}}{\rho_i}(1+\delta_i t+\rho_i)^{-\alpha_i}(1-\delta_i t+\rho_i)^{-\beta_i}(1-y_i\eta)^{-\gamma_i},$$

where

$$\left(|\eta| < \min\left\{|y_1|^{-1},\ldots,|y_r|^{-1}\right\}\right).$$

Choosing $s = r$, and $\Omega_{\mu+\nu k}(y_1,\ldots,y_r) = H_{\mu+\nu k}^{(\gamma_1,\ldots,\gamma_r;\sigma_1,\ldots,\sigma_r)}(y_1,\ldots,y_r)$ $(\mu,\nu \in \mathbb{N}_0)$, in Theorem 7.1 we obtain the following class of bilinear generating function for the multivariable EJPs.

Corollary 8.2. *If*

$$\Lambda_{\mu,\nu}(y_1,\ldots,y_r;z) := \sum_{k=0}^{\infty} a_k H_{\mu+\nu k}^{(\gamma_1,\ldots,\gamma_r;\sigma_1,\ldots,\sigma_r)}(y_1,\ldots,y_r)z^k,$$

where $a_k \neq 0$, $\mu,\nu \in \mathbb{N}_0$ *and*

$$\Theta_{n,p,\mu,\nu}(x_1,\ldots,x_r;y_1,\ldots,y_r;\zeta) := \sum_{k=0}^{[n/p]} a_k H_{n-pk}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(x_1,\ldots,x_r) H_{\mu+\nu k}^{(\gamma_1,\ldots,\gamma_r;\sigma_1,\ldots,\sigma_r)}(y_1,\ldots,y_r)\zeta^k,$$

where $n, p \in \mathbb{N}$, *then we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p,\mu,\nu}\left(x_1,\ldots,x_r;y_1,\ldots,y_r;\frac{\eta}{t^p}\right)t^n = \prod_{i=1}^{r}\frac{2^{\alpha_i+\beta_i}}{\rho_i}(1+\delta_i t+\rho_i)^{-\alpha_i}(1-\delta_i t+\rho_i)^{-\beta_i} \times \Lambda_{\mu,\nu}(y_1,\ldots,y_r;\eta) \tag{8.2}$$

provided that each member of (8.2) *exists.*

If we choose $s = 1$, $\Omega_{\mu+\nu k}(y) = P_{\mu+\nu k}^{(\gamma,\sigma)}(y)$ $(a_k \neq 0,\ \mu,\nu \in \mathbb{N}_0)$ in Theorem 7.2, we obtain the following result which provides a class of bilateral generating functions for the classical Jacobi polynomials and the multivariable EJPs.

Corollary 8.3. *If* $\Lambda_{\mu,\nu}(y;z) := \sum_{k=0}^{\infty} a_k P_{\mu+\nu k}^{(\gamma,\sigma)}(y)z^k$ $(a_k \neq 0,\ \mu,\nu \in \mathbb{N}_0)$, *then we have*

$$\sum_{n_1=0}^{\infty}\sum_{n_2,\ldots,n_r=0}^{\infty}\sum_{k=0}^{[n_1/p]} a_k\gamma(n_1-pk,n_2,\ldots,n_r)$$
$$\times F_{n_1-pk,n_2,\ldots,n_r}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(x_1,\ldots,x_r)t_2^{n_2}\ldots t_r^{n_r} P_{\mu+\nu k}^{(\gamma,\sigma)}(y)\eta^k t_1^{n_1-pk}$$
$$= \Lambda_{\mu,\nu}(y;\eta)\prod_{i=1}^{r}(1-t_i)^{-1-\alpha_i-\beta_i}$$
$$\times {}_2F_1\left(\frac{\alpha_i+\beta_i+1}{2},\frac{\alpha_i+\beta_i+2}{2};1+\alpha_i;\frac{4t_i(x_i-a_i)}{(a_i-b_i)(1-t_i)^2}\right) \tag{8.3}$$

provided that each member of (8.3) *exists, where*

$$\gamma(n_1,\ldots,n_r) = \prod_{i=1}^{r}\frac{(\alpha_i+\beta_i+1)_{n_i}\{c_i(a_i-b_i)\}^{-n_i}}{(1+\alpha_i)_{n_i}}.$$

Remark 8.2. Using (1.3) and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2,\ldots,n_r=0}^{\infty}\sum_{k=0}^{[n_1/p]}\gamma(n_1-pk,n_2,\ldots,n_r)$$
$$\times F_{n_1-pk,n_2,\ldots,n_r}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(x_1,\ldots,x_r)t_2^{n_2}\ldots t_r^{n_r}P_k^{(\gamma,\sigma)}(y)\eta^k t_1^{n_1-pk}$$
$$=\frac{2^{\gamma+\sigma}}{\rho}(1-\eta+\rho)^{-\gamma}(1+\eta+\rho)^{-\sigma}\prod_{i=1}^{r}(1-t_i)^{-1-\alpha_i-\beta_i}$$
$$\times {}_2F_1\left(\frac{\alpha_i+\beta_i+1}{2},\frac{\alpha_i+\beta_i+2}{2};1+\alpha_i;\frac{4t_i(x_i-a_i)}{(a_i-b_i)(1-t_i)^2}\right),$$

where $\rho=(1-2\eta y+\eta^2)^{1/2}$.

Furthermore, for every suitable choice of the coefficients a_k ($k\in\mathbb{N}_0$), if the multivariable function $\Omega_{\mu+\nu k}(y_1,\ldots,y_s)$ ($s\in\mathbb{N}$), is expressed as an appropriate product of several simpler functions, the assertions of Theorems 7.1 and 7.2 can be applied in order to derive various families of multilinear and multilateral generating functions for the multivariable EJPs.

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