Proceedings of Symposia in Pure Mathematics
Volume 34, 1979

CONNECTION COEFFICIENT PROBLEMS AND PARTITIONS

by

George E. Andrews[1]

ABSTRACT. We examine the connection coefficients a_{nk} in the identity $p_n(x) = \sum a_{nk}r_k(x)$ where $p_n(x)$ is an arbitrary family of polynomials and $r_k(x)$ is the k-th little q-Jacobi polynomial. From this study we obtain many of the results derived by Rogers, Bailey and Slater. We also discover "dual" identities most of which previously seemed to be unrelated either to Rogers-Ramanujan type identities or to connection coefficient problems.

1. Introduction. There are numerous problems in combinatorics concernin the determination of the connection coefficients c_{nk} between two sequences c polynomials $p_n(x)$ and $r_n(x)$, viz.

$$p_n(x) = \sum_{k=0}^{n} a_{nk}r_k(x). \tag{1.1}$$

Rota et al. [28], [29], [30], considered this problem in detail for polynomials that they call "of binomial type." Askey [9] devotes Lecture 7 to connection coefficients especially when the polynomials involved are the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. For reference we note that

$$P_n^{(\alpha,\beta)}(x) = \binom{\alpha+n}{n} {}_2F_1\left[\begin{matrix}-n, n+\alpha+\beta+1; & \frac{1-x}{2}\\ \alpha+1 & \end{matrix}\right] , \tag{1.2}$$

where the ${}_2F_1$-function is the hypergeometric function defined in general by

(1) Partially supported by National Science Foundation Grant

I wish to thank Professor R. Askey for numerous useful conversations on thi paper and especially for his comments on L.J. Rogers's solutions to connectior coefficient problems.

$$(1.3)\qquad {}_rF_s\begin{bmatrix}\alpha_1,\ldots,\alpha_r;t\\ \beta_1,\ldots,\beta_s\end{bmatrix} = 1+\sum_{n=1}^{\infty}\frac{t^n}{n!}\prod_{j=0}^{n-1}\frac{(\alpha_1+j)(\alpha_2+j)\ldots(\alpha_r+j)}{(\beta_1+j)(\beta_2+j)\ldots(\beta_s+j)}.$$

In [7] and [8], Askey and I considered (1.1) for the little q-Jacobi polynomials

$$(1.4)\qquad p_n(x;a,b|q) = {}_2\phi_1\begin{pmatrix}q^{-n},abq^{n+1};q,xq\\ aq\end{pmatrix},$$

where the ${}_2\phi_1$-function is a basic hypergeometric function defined in general

$$(1.5)\qquad {}_{s+1}\phi_s\begin{pmatrix}a_1,\ldots,a_{s+1};q,t\\ b_1,\ldots,b_s\end{pmatrix} = 1+\sum_{n=1}^{\infty}\frac{t^n}{(q)_n}\frac{(a_1)_n(a_2)_n\ldots(a_{s+1})_n}{(b_1)_n(b_2)_n\ldots(b_s)_n},$$

where

$$(1.6)\qquad (a)_n = (a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1}),$$

and

$$(1.7)\qquad (a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a)_n.$$

In particular we showed [7] that if in (1.1) $p_n(x) = p_n(x;\gamma,\delta|q)$ and $r_k(x) = p_k(x;\alpha,\beta|q)$, then

$$(1.8)\qquad a_{nk} = \frac{(-1)^k q^{k(k+1)/2}(\gamma\delta q^{n+1})_k(q^{-n})_k(\alpha q)_k}{(q)_k(\gamma q)_k(\alpha\beta q^{k+1})_k}\cdot {}_3\phi_2\begin{pmatrix}q^{-n+k},\gamma\delta q^{n+k+1},\alpha q^{k+1};q,q\\ \gamma q^{k+1},\alpha\beta q^{2k+2}\end{pmatrix}.$$

Utilizing this result, we obtained a general transformation of basic hyper- geometric series which allowed us to deduce (among other things) the Rogers Ramanujan identities:

$$1 + \sum_{n=1} \frac{q^{n^2}}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} ; \tag{1.9}$$

$$1 + \sum_{n=1} \frac{q^{n^2+n}}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} . \tag{1.10}$$

These celebrated identities have a fascinating history (see [6; Ch. 7]) con nected with Ramanujan's meteoric rise to prominence. They first arose how in a paper by L.J. Rogers [25] in 1894. Rogers had devised a technique fo establishing numerous series-product identities. Subsequently W.N. Bailey [14] and L.J. Slater [31], [32] extended Rogers's work and gave many more se product identities like (1.9) and (1.10). Recently, Askey and Ismail [11] made an extensive study of a further work of Rogers [26]. In particular, point out that the generalized Legendre polynomials incompletely treated b Feldheim [19] and Lancevickiǐ [23] were actually discovered and investigat L.J. Rogers in his Third Memoir on the Expansion of Certain Infinite Produ [26]. These polynomials, $C_n(x;\beta|q)$, are defined by

$$\sum_{n=0}^{\infty} C_n(x;\beta|q)r^n = \frac{(\beta e^{i\theta}r)_\infty(\beta e^{-i\theta}r)_\infty}{(e^{i\theta}r)_\infty(e^{-i\theta}r)_\infty} , \quad \text{where} \quad x = \cos\theta; \tag{1.11}$$

Askey and Ismail call them "continuous q-ultraspherical polynomials" for q compelling reasons, and they go on to relate these polynomials to many imp tant results in the theory of orthogonal polynomials as well as to the num q-series identities collected by Rogers [23] which have subsequently becom important in the theory of partitions [1], [3], [6]. Among the many impor results Rogers found for the $C_n(x;\gamma|q)$ was a solution to (1.1) for this of polynomials from which the Rogers-Ramanujan identities (1.9) and (1.10) follow as Askey and Ismail indicate.

There are still, however, some aspects of the general theory of these q-series identities that have yet to fit into the general connection coefficient theorems of these q-orthogonal polynomials. In particular, this paper was motivated as an attempt to fit the general multiple series expansion of a very well-poised basic hypergeometric series (see eq. (5.1) in Section 5) into the framework of connection coefficient theory. While we have not been totally successful in fulfilling our original goal, we have managed to show that the fundamental identities of Rogers [27], Bailey [13], [14] and Slater [31], [32] are in fact direct corollaries of (1.1) when $p_n(x)$ is an arbitrary polynomial and $r_n(x)$ is $p_n(x;a,b|q)$ (see Lemma 1 of Section 2). As an immediate corollary (Lemma 3 of Section 2), we find that the special case $p_n(x) = \alpha_n x^n$ allows us to invert the so-called "Bailey transform" which in turn allows us to produce a family of identities that are dual to the numerous results given by Rogers, Bailey and Slater; this duality is explored in Sections 4 and 5. Perhaps one of the most interesting new implications of Lemma 3 is the following ${}_3F_2$ summation given in Section 4:

$$(1.12)\qquad {}_3F_2\left[\begin{matrix}-k, a+k, \frac{a}{3}; 3/4\\ \frac{a}{2}, \frac{a}{2}+\frac{1}{2}\end{matrix}\right] = \begin{cases} 0 & k \not\equiv 0 \pmod 3 \\ \dfrac{(3n)!\,(\frac{a}{3}+1)_n}{n!\,(a+1)_{3n}}, & k = 3n\,. \end{cases}$$

As we shall see the following pure connection coefficient theorem is essentially equivalent to Bailey's transform:

Theorem 1. If $p_n(x) = \sum_{k=0}^{n} b_{nk} r_k(x)$, where

$$(1.13)\qquad p_n(x) = {}_3\phi_2\left(\begin{matrix}\rho_1, \rho_2, q^{-n} & ; q, xq\\ \alpha q, \rho_1\rho_2 q^{-n-1}/\alpha\beta & \end{matrix}\right)$$

and

$$r_k(x) = p_k(x;\alpha,\beta|q), \tag{1.14}$$

then

$$b_{nk} = \frac{(\alpha\beta q^2/\rho_1)_n(\alpha\beta q^2/\rho_2)_n(\alpha\beta q^2)_{k-1}(1-\alpha\beta q^{2k+1})(\rho_1)_k(\rho_2)_k(q^{-n})_k\ \frac{\alpha\beta}{\rho}}{(\alpha\beta q^2)_n(\alpha\beta q^2/\rho_1\rho_2)_n(q)_k(\alpha\beta q^2/\rho_1)_k(\alpha\beta q^2/\rho_2)_k(\alpha\beta q^{n+2})_k} \tag{1.15}$$

Before continuing, let us exhibit the limiting case of Theorem 1 that volves the Jacobi polynomials:

$$(1.16)\qquad {}_3F_2\left[\begin{matrix}-n, r_1+\alpha+\beta+n+1, r_2; & \frac{1-x}{2}\\ \alpha+1, r_1+r_2 & \end{matrix}\right]$$

$$= \frac{[r_1]_n[\alpha+\beta+2-r_2]_n}{[\alpha+\beta+2]_n[r_1+r_2]_n}\sum_{k=0}^{n}\frac{[\alpha+\beta+1]_k(\alpha+\beta+2k+1)[r_1+\alpha+\beta+n+1]_k[r_2][-n]_k P_k^{(\alpha,\beta)}(x)}{[\alpha+1]_k(\alpha+\beta+1)[1-r_1-n]_k[\alpha+\beta+2-r_2]_k[\alpha+\beta+n+2]_k},$$

where $[A]_k = A(A+1)\dots(A+k-1)$. To obtain (1.16) from Theorem 1 we replac ρ_1, ρ_2, α and β by $q^{r_1+\alpha+\beta+n+1}, q^{r_2}, q^{\alpha}$ and q^{β} respectively; we the $q \to 1$. This result contains as special or limiting cases the connection c efficient theorems for the Jacobi polynomials when only one parameter is a to vary $(I : r_1 = \gamma-\alpha,\ r_2 = \alpha+1;\ II : r_1 = \delta-\beta,\ r_2 \to \infty)$ (see [34; Ch. 9] [9; Ch. 7] for reference to the primary literature). Askey has pointed ou me that (1.16) can also be derived by applying a β-integration to either o original connection coefficient theorems for Jacobi polynomials.

In Section 2 we shall prove the fundamental lemmas and Theorem 1. In Section 3, we consider the relationship of Theorem 1 to the work of Rogers. Bailey and Slater. In Section 4 we discuss the identities dual to those o Rogers, Bailey and Slater. In Section 5, the implications of our work for multiple series expansions of certain hypergeometric functions. We concluc with a brief look at some of the combinatorial implications of this work.

2. The Connection Coefficient Theorems. We begin with a lemma which effectively determines the connection coefficients a_{nk} in (1.1) when $p_n(x)$ is arbitrary and $r_k(x)$ is the little q-Jacobi polynomial (1.4).

Lemma 1. Let $G_k = \sum_{j=k}^{n} \frac{D_j}{(q)_{j-k}(\alpha\beta q^2)_{k+j}}$,

then

$$(2.1)\quad \sum_{j=0}^{n} \frac{D_j x^j}{(q)_j(\alpha q)_j} = \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}(\alpha\beta q^2)_{k-1}(1-\alpha\beta q^{2k+1})G_k p_k(x;\alpha,\beta|q)}{(q)_k}.$$

Proof. This result like [7; Th. 10] is easily deduced from the following two orthogonality properties of the little q-Jacobi polynomials:

$$(2.2)\quad \sum_{i=0}^{\infty} \frac{\alpha^i q^i (q^{i+1})_\infty}{(\beta q^{i+1})_\infty} p_n(q^i;\alpha,\beta|q)p_m(q^i;\alpha,\beta|q)$$

$$= \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{\alpha^n q^n (q)_\infty(\alpha\beta q^{n+1})_\infty (q)_n}{(\beta q^{n+1})_\infty(\alpha q)_\infty(\alpha q)_n(1-\alpha\beta q^{2n+1})}, & \text{if } m = n, \end{cases}$$

(equation (3.8) of [7]);

$$(2.3)\quad \sum_{i=0}^{\infty} \frac{\alpha^i q^i (q^{i+1})_\infty}{(q^{i+1})_\infty} p_n(q^i;\alpha,\beta|q)q^{im}$$

$$= \frac{(q)_\infty(\alpha\beta q^{m+n+2})_\infty(q^{-m})_n \alpha^n q^{n(m+1)}}{(\beta q^{n+1})_\infty(\alpha q^{m+1})_\infty(\alpha q)_n},$$

(corrected form of top line on page 14 of [7]).

Now the coefficient of $p_k(x;\alpha,\beta|q)$ in the expansion (2.1) of $\sum D_j x^j (q)_j^{-1}(\alpha q)_j^{-1}$ is clearly

$$(2.4)\qquad \frac{\displaystyle\sum_{i=0}^{\infty} \frac{\alpha^i q^i (q^{i+1})_\infty}{(\beta q^{i+1})_\infty} p_k(q^i;\alpha,\beta|q) \sum_{j=0}^{n} \frac{D_j q^{ij}}{(q)_j(\alpha q)_j}}{\displaystyle\sum_{i=0}^{\infty} \frac{\alpha^i q^i (q^{i+1})_\infty}{(\beta q^{i+1})_\infty} p_k(q^i;\alpha,\beta|q)^2} .$$

Let us first evaluate the numerator in (2.4); it equals

$$\sum_{j=0}^{n} \frac{D_j}{(q)_j(\alpha q)_j} \sum_{i=0}^{\infty} \frac{\alpha^i q^i (q^{i+1})_\infty p_k(q^i;\alpha,\beta|q) q^{ij}}{(\beta q^{i+1})_\infty}$$

$$= (q)_\infty \sum_{j=0}^{n} \frac{D_j}{(q)_j(\alpha q)_j} \frac{(\alpha\beta q^{j+k+2})_\infty (q^{-j})_k \alpha^k q^{k(j+1)}}{(\beta q^{k+1})_\infty (\alpha q^{j+1})_\infty (\alpha q)_k}$$

(by (2.3))

$$= \frac{(q)_\infty(\alpha\beta q^2)_\infty}{(\alpha q)_\infty(\beta q^{k+1})_\infty(\alpha q)_k} \sum_{j=k}^{n} \frac{D_j(-1)^k q^{\binom{k+1}{2}} \alpha^k}{(q)_{j-k}(\alpha\beta q^2)_{j+k}}$$

$$(2.5)\qquad = \frac{(-1)^k \alpha^k q^{\binom{k+1}{2}} (q)_\infty (\alpha\beta q^2)_\infty G_k}{(\alpha q)_\infty (\beta q^{k+1})_\infty (\alpha q)_k} .$$

Hence substituting (2.5) for the numerator of (2.4) and replacing the denomin-tor by the appropriate expression from (2.2) with $m = n = k$, we see that th coefficient of $p_k(x;\alpha,\beta|q)$ in the expansion (2.1) must be

$$\frac{(-1)^k \alpha^k q^{\binom{k+1}{2}} (q)_\infty (\alpha\beta q^2)_\infty G_k (\beta q^{k+1})_\infty (\alpha q)_\infty (\alpha q)_k (1-\alpha\beta q^{2k+1})}{(\alpha q)_\infty (\beta q^{k+1})_\infty (\alpha q)_k \alpha^k q^k (q)_\infty (\alpha\beta q^{k+1})_\infty (q)_k}$$

$$= \frac{(-1)^k q^{\binom{k}{2}} (\alpha\beta q^2)_{k-1} (1-\alpha\beta q^{2k+1}) G_k}{(q)_k} ,$$

as desired. □

Lemma 2.

$$\frac{x^k}{(q)_k(\alpha q)_k} = \sum_{j=0}^{k} \frac{(-1)^j q^{\binom{j}{2}} (\alpha\beta q^2)_{j-1}(1-\alpha\beta q^{2j+1}) p_j(x;\alpha,\beta|q)}{(q)_j (q)_{k-j} (\alpha\beta q^2)_{k+j}} . \tag{2.6}$$

Proof. In Lemma 1 set $D_j = 0$ for $j < n$, $D_n = 1$. Then clearly $G_k = \frac{1}{(q)_{n-k}(\alpha\beta q^2)_{n+k}}$, and our result follows immediately. □

Lemma 3. If $B_k = \sum_{j=0}^{k} \frac{A_j}{(q)_{k-j}(\alpha\beta q^2)_{k+j}}$, then

$$A_k = (1-\alpha\beta q^{2k+1}) \sum_{j=0}^{k} \frac{(\alpha\beta q^2)_{k+j-1} B_j (-1)^{k-j} q^{\binom{k-j}{2}}}{(q)_{k-j}} ,$$

and conversely.

Proof. The converse follows immediately from our first assertion, since the vector $(A_0,\ldots,A_k)$ is being transformed into the vector $(B_0,\ldots,B_k)$ by the nonsingular linear transformation $\left(\frac{1}{(q)_{i-j}(\alpha\beta q^2)_{i+j}}\right)_{k\times k}$, a lower triangular matrix; hence the converse is merely the application of the inverse mapping which is also a nonsingular linear transformation. In fact our theorem is equivalent to the assertion that

$$\left(\frac{1}{(q)_{i-j}(\alpha\beta q^2)_{i+j}}\right)^{-1}_{k\times k} = \left(\frac{(1-\alpha\beta q^{2i+1})(\alpha\beta q^2)_{i+j-1}(-1)^{i-j} q^{\binom{i-j}{2}}}{(q)_{i-j}}\right)_{k\times k} \tag{2.7}$$

However (2.7) follows immediately from Lemma 2 since $\left(\frac{1}{(q)_{i-j}(\alpha\beta q^2)_{i+j}}\right)_{k\times k}$ carries the basis $\{A_0,\ldots,A_k\}$ of the vector space of polynomials of degree $\le k$ over the complex numbers onto the basis $\{B_0,\ldots,B_k\}$, where

$$B_j = \frac{x^j}{(q)_j(\alpha q)_j}$$

and

$$(2.8) \qquad A_j = \frac{(-1)^j q^{\binom{j}{2}} (\alpha\beta q^2)_{j-1} (1-\alpha\beta q^{2j+1}) p_j(x;\alpha,\beta|q)}{(q)_j} ;$$

the inverse of this mapping is clearly the right side of (2.7) by (1.4).

Proof of Theorem 1. We apply Lemma 1 with

$$D_j = (\rho_1)_j (\rho_2)_j (q^{-n})_j q^j (\rho_1\rho_2 q^{-n-1}/\alpha\beta)_j^{-1},$$

and our theorem follows immediately once we observe that

$$G_k = \sum_{j=k}^{n} \frac{(\rho_1)_j (\rho_2)_j (q^{-n})_j q^j}{(q)_{j-k} (\alpha\beta q^2)_{k+j} (\rho_1\rho_2 q^{-n-1}/\alpha\beta)_j}$$

$$= \frac{(\rho_1)_k (\rho_2)_k (q^{-n})_k q^k}{(\alpha\beta q^2)_{2k} (\rho_1\rho_2 q^{-n-1}/\alpha\beta)_k} \, {}_3\phi_2 \left(\begin{matrix} \rho_1 q^k, \rho_2 q^k, q^{-n+k}; q, q \\ \alpha\beta q^{2k+2}, \rho_1\rho_2 q^{-n-1+k}/\alpha\beta \end{matrix} \right)$$

$$= \frac{(\rho_1)_k (\rho_2)_k (q^{-n})_k q^k}{(\alpha\beta q^2)_{2k} (\rho_1\rho_2 q^{-n-1}/\alpha\beta)_k} \frac{(\alpha\beta q^{k+2}/\rho_1)_{n-k} (\alpha\beta q^{k+2}/ \ {}_2)_{n-k}}{(\alpha\beta q^{2k+2})_{n-k} (\alpha\beta q^2/\rho_1\rho_2)_{n-k}}$$

(by the Pfaff-Saalschutz summation [12; p. 68, eq.(1)])

$$= \frac{(\alpha\beta q^2/\rho_1)_n (\alpha\beta q^2/\rho_2)_n}{(\alpha\beta q^2)_n (\alpha\beta q^2/\rho_1\rho_2)_{n-k}} \frac{(\rho_1)_k \qquad (\rho_2)_k \qquad (q^{-n})_k \qquad q^k}{(\alpha\beta q^2/\rho_1)_k (\alpha\beta q^2/\rho_2)_k (\alpha\beta q^{n+2})_k (\rho_1\rho_2 q^{-n-1}/\alpha\beta}$$

$$(2.9) \qquad = \frac{(\alpha\beta q^2/\rho_1)_n (\alpha\beta q^2/\rho_2)_n (\rho_1)_k (\rho_2)_k (q^{-n})_k (-1)^k q^{kn+2k-\binom{k}{2}} (\alpha\beta/\rho_1\rho_2)^k}{(\alpha\beta q^2)_n (\alpha\beta q^2/\rho_1\rho_2)_n (\alpha\beta q^2/\rho_1)_k (\alpha\beta q^2/\rho_2)_k (\alpha\beta q^{n+2})_k} .$$

Substituting (2.9) into (2.1) we obtain Theorem 1. □

Askey has pointed out to me that the main results in this section could also be derived from [7; Th. 10] in that Lemma 2 is a limiting case of this result and Theorem 1 can be obtained from a q-beta integration of another special case of the same result.

3. The Identities of Rogers, Bailey and Slater. In extending the work of Rogers [25], [27], Bailey [13], [14] and Slater [31], [32] show that the following result is a fundamental identity special cases of which Rogers used to prove the vast majority of his series-product identities:

Theorem 2. (Bailey's Transform [33; p.58], [14; p.1]). Subject to suitable convergence conditions

$$\sum_{n=0}^{\infty}(Y)_n(Z)_n\left(\frac{X}{YZ}\right)^n B_n \tag{3.1}$$

$$= \frac{(X/Y)_\infty (X/Z)_\infty}{(X)_\infty (X/YZ)_\infty}\sum_{n=0}^{\infty}\frac{(Y)_n (Z)(X/YZ)^n A_n}{(X/Y)_n (X/Z)_n},$$

where

$$B_n = \sum_{r=0}^{n}\frac{A_r}{(X)_{n+r}(q)_{n-r}}. \tag{3.2}$$

Derivation of Theorem 2 from Theorem 1.

In Theorem 1 set $\rho_1 = Y$, $\rho_2 = Z$, $\alpha\beta q^2 = X$ and replace $x^j(q)_j^{-1}(\alpha q)_j^{-1}$ by B_j (a legitimate substitution since Theorem 1 is merely a polynomial identity in x). Hence

$$\sum_{j=0}^{n}\frac{(Y)_j(Z)_j(q^{-n})_j q^j B_j}{\left(\frac{YZq^{-n+1}}{X}\right)_j} \tag{3.3}$$

$$= \frac{(X/Y)_n (X/Z)_n}{(X)_n (X/YZ)_n} \sum_{k=0}^{n} \frac{(Y)_k (Z)_k (q^{-n})_k (\frac{Xq^n}{YZ})^k (-1)^k q^{-\binom{k}{2}}}{(X/Y)_k (X/Z)_k (Xq^n)_k}$$

$$\cdot \frac{(1-Xq^{2k-1})(X)_{k-1}(-1)^k q^{\binom{k}{2}}}{(q)_k} \sum_{j=0}^{k} (q^{-k})_j (Xq^{k-1})_j B_j q^j .$$

Now applying Lemma 3 to (3.2), we see that

$$(3.4) \qquad A_k = (1-Xq^{2k-1}) \sum_{j=0}^{k} \frac{(X)_{k+j-1} B_j (-1)^{k-j} q^{\binom{k-j}{2}}}{(q)_{k-j}}$$

$$= \frac{(1-Xq^{2k-1})(X)_{k-1}(-1)^k q^{\binom{k}{2}}}{(q)_k} \sum_{j=0}^{k} (q^{-k})_j (Xq^{k-1})_j B_j q^j .$$

Hence replacing the expression on the right side of (3.4) by A_k in (3.3), find that

$$(3.5) \qquad \sum_{j=0}^{n} \frac{(Y)_j (Z)_j (q^{-n})_j q^j B_j}{(\frac{YZq^{-n+1}}{X})_j}$$

$$= \frac{(X/Y)_n (X/Z)_n}{(X)_n (X/YZ)_n} \sum_{k=0}^{n} \frac{(Y)_k (Z)_k (q^{-n})_k (\frac{Xq^n}{YZ})^k (-1)^k q^{-\binom{k}{2}} A_k}{(X/Y)_k (X/Z)_k (Xq^n)_k} .$$

Theorem 2 follows directly from (3.5) if we let $n \to \infty$. □

Let us now observe that if we let $n \to \infty$ in (3.3) we obtain

$$(3.6) \qquad \sum_{j=0}^{\infty} (Y)_j (Z)_j (\frac{X}{YZ})^j B_j$$

$$= \frac{(X/Y)_\infty (X/Z)_\infty}{(X)_\infty (X/YZ)_\infty} \sum_{k=0}^{\infty} \frac{(Y)_k (Z)_k (X)_{k-1} (1-Xq^{2k-1})(-1)^k q^{\binom{k}{2}}}{(X/Y)_k (X/Z)_k (q)_k}$$

$$\times (\frac{X}{YZ})^k \sum_{j=0}^{k} (q^{-k})_j (Xq^{k-1})_j B_j q^j .$$

For the remainder of this section I hope to show that (3.3) and (3.6) can be used to simplify the derivations of some of the identities found by Rogers, Bailey and Slater.

For example, set

$$B_j = \frac{(X/CD)_j}{(q)_j (X/C)_j (X/D)_j}, \tag{3.7}$$

then

$$\sum_{j=0}^{k} (q^{-k})_j (Xq^{k-1})_j B_j q^j \tag{3.8}$$

$$= {}_3\phi_2\left(\begin{matrix} q^{-k}, Xq^{k-1}, X/CD; q, q \\ X/C, X/D \end{matrix}\right)$$

$$= \frac{(D)_k (C^{-1}q^{-k+1})_k}{(X/C)_k (X^{-1}Dq^{-k+1})_k} \quad \text{(by the Pfaff-Saalschutz summation [12; p.68, eq.(1)])}$$

$$= \frac{(D)_k (C)_k (\frac{X}{CD})^k}{(X/C)_k (X/D)_k}.$$

Applying (3.7) and (3.8) to (3.3) we obtain Watson's q-analog of Whipple's theorem [33; p.100, eq.(3.4.1.5)]:

$$ {}_4\phi_3\left(\begin{matrix} q^{-n}, Y, Z, X/CD; q, q \\ X/C, X/D, \frac{YZq^{-n+1}}{X} \end{matrix}\right) \tag{3.9}$$

$$= \frac{(X/Y)_n (X/Z)_n}{(X)_n (X/YZ)_n}\, {}_8\phi_7\left(\begin{matrix} Xq^{-1}, q^{1/2}X^{1/2}, -q^{1/2}X^{1/2}, Y, Z, C, D, q^{-n}; q, (\frac{X^2q^n}{YZCD})^k \\ q^{-1/2}X^{1/2}, -q^{-1/2}, X^{1/2}, X/Y, X/Z, X/C, X/D, Xq^n \end{matrix}\right)$$

Notice that the derivation of (3.9) from (3.3) requires only a single application of the Pfaff-Saalschutz summation while previously either a double application was necessary [7] or one had to rely on the more general Jackson-Dougal summation [13].

4. The Dual Identities. Here we turn to an exploration of the implications of Lemma 3 whose original purpose was merely to allow us to recover Bailey's transform (Theorem 2). In the final paper by Bailey [14] on extend Rogers' work, Bailey conveniently lists for us five pairs (i)-(v) of A_n ar B_n which satisfy the initial identity of Lemma 3:

$$B_k = \sum_{j=0}^{k} \frac{A_j}{(q)_{k-j}(aq)_{k+j}}, \tag{4.1}$$

where we have replaced $\alpha\beta q$ by a. Below we list Bailey's five pairs toget with the identity implied by substituting the pair into the second formula c Lemma 3.

(i) $$A_n = \frac{(-1)^n(1-aq^{2n})(a)_n a^n q^{1/2n(3n-1)}}{(1-a)(q)_n}, \qquad B_n = \frac{1}{(q)_n}.$$

$$a^k q^{k^2} = \sum_{j=0}^{k} \frac{(q^{-k})_j (aq^k)_j q^j}{(q)_j}. \tag{4.2}$$

Identity (4.2) is merely a special case of the q-analog of the Chu-Vandermor summation [33; p.97, (3.3.].7)].

Changing a into a^2 and q into q^2 in (4.1), we next have

(ii) $$A_n = \frac{(1-aq^{2n})(a)_n(b)_n a^n q^{n^2}}{(1-a)(q)_n(aq/b)_n b^n},$$

$$B_n = \frac{(-aq^{n+1}/b)_n}{(q^2;q^2)_n(-aq)_{2n}(aq/b)_n}.$$

$$(4.3)\qquad \frac{(-1)^k (b)_k a^k q^k b^{-k} (-q)_k}{(aq/b)_k (1+aq^{2k}) (-aq)_{k-1}}$$

$$= {}_4\phi_3 \left(\begin{matrix} a^2q^{2k}, -aq/b, -aq^2/b, q^{-2k}; q^2, q^2 \\ -aq, -aq^2, a^2q^2/b^2 \end{matrix} \right).$$

This last identity I belive to be new; if we replace a by $-q^A$ and b by q^{A-B} and then let $q \to 1$, we find

$$(4.4)\qquad \frac{[A-B]_k}{(A+2k)[B+1]_{k-1}} = {}_4F_3 \left[\begin{matrix} \frac{B+1}{2}, \frac{B}{2}+1, A+k, -k; 1 \\ \frac{A+1}{2}, \frac{A}{2}+1, B+1 \end{matrix} \right],$$

a result apparently originally found by Cvetkovic and Simić [18] (see also Carlitz [17] and Gasper [20]). An interesting study of this sum was given also by Askey and Ismail [10].

Changing a into a^3 and q into q^3 in (4.1), we next have

$$(\text{iii})\qquad A_n = \frac{(-1)^n (1-aq^{2n}) (a)_n a^n q^{n(3n-1)/2}}{(1-a)(a)_n},$$

$$B_n = \frac{(aq)_{3n}}{(q^3;q^3)_n (a^3q^3;q^3)_{2n}}.$$

$$(4.5)\qquad \frac{(1-aq^{2k})(aq)_{k-1} a^k q^k (q^3;q^3)_k}{(a^3q^3;q^3)_{k-1} (q)_k (1-q^3q^{6k})}$$

$$= \sum_{j=0}^{k} \frac{(aq)_{3j} (a^3q^{3k};q^3)_j (q^{-3k};q^3)_j q^{3j}}{(q^3;q^3)_j (a^3q^3;q^3)_{2j}}$$

$$= {}_5\phi_4 \left(\begin{matrix} aq, aq^2, aq^3, a^3q^{3k}, q^{-3k}; q^3, q^3 \\ a^{3/2}q^{3/2}, -a^{3/2}q^{3/2}, a^{3/2}q^3, -a^{3/2}q^3 \end{matrix} \right)$$

again a result that appears to be new.

Next we have Bailey's fourth pair:

(iv) $$A_{2n+1} = 0,\ A_{2n} = \frac{(a;q^2)_n(f;q^2)_n(1-aq^{4n})a^n q^{2n^2}}{(q^2;q^2)_n(aq^2/f;q^2)_n(1-a)f^n},$$

$$B_n = \frac{(aq/f;q^2)_n}{(q)_n(aq;q^2)_n(aq/f)_n}.$$

(4.6) $$\sum_{j=0}^{k} \frac{(aq^k)_j(q^{-k})_j(aq/f;q^2)_j q^j}{(q)_j(aq;q^2)_j(aq/f)_j}$$

$$= {}_4\phi_3\left(\begin{matrix} aq^k, q^{-k}, (aq/f)^{1/2}, -(aq/f)^{1/2}; q, q \\ (aq)^{1/2}, -(aq)^{1/2}, aq/f \end{matrix}\right)$$

$$= \begin{cases} 0 & k \text{ odd} \\ \dfrac{(f;q)_n a^n q^n (q;q^2)_n}{(aq^2/f;q^2)_n f^n (aq;q^2)_n} & k = 2n \end{cases}.$$

This result has been given previously [5]; it is the q-analog of the termin ting form of Watson's ${}_3F_2$ summation [33; p.245, eq.(III.23)].

We now examine Bailey's fifth and final pair:

(v) $$A_{3n\pm1} = 0,\ A_{3n} = \frac{(-1)^n(aq^3;q^3)_n(1-aq^{6n})a^n q^{3n(3n-1)/2}}{(q^3;q^3)_n},$$

$$B_n = \frac{(aq^3;q^3)_{n-1}}{(q)_n(aq)_{2n-1}}.$$

$$(4.7)\qquad \sum_{j=0}^{k} \frac{(q^{-k})_j (aq^k)_j (a;q^3)_j q^j}{(q)_j (a)_{2j}}$$

$$= {}_5\phi_4 \left(\begin{matrix} q^{-k}, aq^k, a^{1/3}, e^{\frac{2\pi i}{3}} a^{1/3}, e^{\frac{4\pi i}{3}} a^{1/3}; q, q \\ a^{1/2}, -a^{1/2}, q^{1/2}a^{1/2}, -q^{1/2}a^{1/2} \end{matrix} \right)$$

$$= \begin{cases} 0 & k = 3n\pm 1 \\ \dfrac{(q;q^3)_n (q^2;q^3)_n a^n}{(aq;q^3)_n (aq^2;q^3)_n} & k = 3n, \end{cases}$$

and if we replace a by q^a and let $q \to 1$ we obtain

$$(4.8)\qquad {}_3F_2 \begin{bmatrix} -k, a+k, a/3; 3/4 \\ a/2, a/2+1/2 \end{bmatrix} = \begin{cases} 0 & k \not\equiv 0 \pmod 3 \\ \dfrac{(3n)!\,[a/3+1]_n}{n!\,[a+1]_{3n}} & k = 3n. \end{cases}$$

Apparently neither (4.7) nor (4.8) has appeared before although (4.8) resembl recent summations discovered by W. Gosper [22].

There are of course numerous other applications of Lemma 3. In [32], Slater collects approximately 130 identities, each produced from an (A_n, B_n)-pair. Upon examination we see that many of these pairs are subsumed by speci cases of those given above; however, it is clear that those identities arisin from Slater's (and Rogers's) "Group A" are not. Since these pairs include no variables other than q, they appear to have less interest than the previous examples. Hence we only consider the pair A(2) [25; p.463]. In Lemma 3 set $\alpha = \beta = 1$, then A(2) gives the pair

$$A_k = \begin{cases} q^{6n^2-n} & k = 3n-1 \\ q^{6n^2+n} & k = 3n \\ -q^{6n^2+5n+1} - q^{6n^2+7n+2} & k = 3n+1, \end{cases}$$

$$B_k = \frac{(1-q)}{(q)_{2k+1}} .$$

Hence

$$(4.9) \qquad (1-q^{2k+1}) \sum_{j=0}^{k} \frac{(q)_{k+j}(-1)^{k-j}q^{\binom{k-j}{2}}}{(q)_{k-j}(q)_{2j+1}} = \begin{cases} q^{6n^2-n} & k \\ q^{6n^2+n} & k \\ -q^{6n^2+5n+1}-q^{6n^2+7n+1} & k \end{cases}$$

If we let $q \to 1$ we obtain the identity:

$$(4.10) \qquad (2k+1) \sum_{j=0}^{k} \binom{k+j+1}{2j+1} \frac{(-1)^{k-j}}{k+j+1} = \begin{cases} 1 & k \not\equiv 1 \pmod 3 \\ -2 & k \equiv 1 \pmod 3 \end{cases} .$$

Askey has pointed out to me that (4.10) also follows from straightforw properties of $P_k^{(1/2,-1/2)}(1/2)$; however, he notes that (4.9) does not seem follow from the corresponding q-analogues.

5. Multiple Series Identities. As mentioned in the introduction, the original intent of this paper was to fit the following generalization of (3 [4; p.199] into general results on connection coefficient problems.

For $k \geqq 1$, N a nonnegative integer,

$$(5.1) \qquad {}_{2k+4}\phi_{2k+3}\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, b_2, c_2, \ldots, b_k, c_k, q^{-N}; q, \dfrac{a^k q^{k+N}}{b_1 \ldots b_k c_1 \ldots c_k} \\ \sqrt{a}, -\sqrt{a}, aq/b_1, aq/c_1, aq/b_2, aq/c_2, \ldots, aq/b_k, aq/c_k, aq^{N+1} \end{matrix}\right)$$

$$= \frac{(aq)_N (aq/b_k c_k)_N}{(aq/b_k)_N (aq/c_k)_N} \sum_{m_1, \ldots, m_{k-1} \geqq 0} \frac{(aq/b_1c_1)_{m_1} (aq/b_2c_2)_{m_2} \cdots (aq/b_{k-1}c_{k-1})_{m_{k-1}}}{(q)_{m_1} (q)_{m_2} \cdots (q)_{m_{k-1}}}$$

$$\cdot \frac{(b_2)_{m_1} (c_2)_{m_1} (b_3)_{m_1+m_2} (c_3)_{m_1+m_2} \cdots (b_k)_{m_1+\ldots+m_{k-1}}}{(aq/b_1)_{m_1} (aq/c_1)_{m_1} (aq/b_2)_{m_1+m_2} (aq/c_2)_{m_1+m_2} \cdots (aq/b_{k-1})_{m_1+\ldots m_{k-1}}} \times$$

$$\frac{(c_k)_{m_1+\cdots+m_{k-1}}}{(aq/c_{k-1})_{m_1+\cdots+m_{k-1}}} \cdot \frac{(q^{-N})_{m_1+m_2+\cdots+m_{k-1}}}{(b_kc_kq^{-N}/a)_{m_1+m_2+\cdots+m_{k-1}}}$$

$$\frac{(aq)^{m_{k-2}+2m_{k-3}+\cdots+(k-2)m_1} q^{m_1+m_2+\cdots+m_{k-1}}}{(b_2c_2)^{m_1}(b_2c_3)^{m_1+m_2}\cdots(b_{k-1}c_{k-1})^{m_1+m_2+\cdots+m_{k-2}}} .$$

As was made clear in [4], this identity has a number of elegant special cases with applications in the theory of partitions. Recently Bressoud [16], [17] has found extensions and applications of (5.1) which further emphasize the impact of such multiple series identities in the theory of partitions.

Since (3.9) drops out of (3.6) merely by the invocation of the q-analog of the Pfaff-Saalschutz summation, it becomes clear that to obtain (5.1) from (3.6) all we need is to apply the multiple series generalization of the q-analog of the Pfaff-Saalschutz summation. Unfortunately no such generalization appears in the literature. Our duality theory allows us to produce the appropriate generalization ((5.2) below). Without an independent proof of (5.2) it would be circular reasoning to deduce (5.1) from (5.2). Hence instead of a new proof of (5.1) we wind up with the multiple series generalization of the q-analog of the Pfaff-Saalschutz summation.

To allow application of Lemma 3, we multiply (5.1) by $(q)_N^{-n}(aq)_N^{-1}$. As a result we find that the first identity in Lemma 3 may be identified with (5.1) provided $\alpha\beta q = a$:

$$A_r = \frac{(-1)^r(a)_r(1-aq^{2r})(b_1)_r(c_1)_r\cdots(b_k)_r(c_k)_r a^{kr}q^{kr+\binom{r}{2}}}{(q)_r(1-a)(aq/b_1)_r(aq/c_1)_r\cdots(aq/b_k)_r(aq/c_k)_r(b_1c_1\cdots b_kc_k)^r} ,$$

while B_N is the right hand side of (5.1) multiplied by $(aq/b_kc_k)_N(q)_N^{-1}(aq/b_k)_N^{-1}(aq/c_k)_N^{-1}$. We now fill this pair into the second identity of Lemma 3.

(5.2)
$$\frac{(b_1)_r(c_1)_r\cdots(b_k)_r(c_k)_r a^{kr}q^{kr}}{(aq/b_1)_r(aq/c_1)_r\cdots(aq/b_k)_r(aq/c_k)_r(b_1\cdots b_k c_1\cdots c_k)^r}$$

$$= \sum_{j=0}^{r} \frac{(q^{-r})_j(aq^r)_j(aq/b_kc_k)_j q^j}{(q)_j(aq/b_k)_j(aq/c_k)_j}$$

$$\sum_{m_1,\ldots,m_{k-1}\geq 0} \frac{(aq/b_1c_1)_{m_1}(aq/b_2c_2)_{m_2}\cdots(aq/b_{k-1}c_{k-1})_{m_{k-1}}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{k-1}}}$$

$$\frac{(b_2)_{m_1}(c_2)_{m_1}(b_3)_{m_1+m_2}(c_3)_{m_1+m_2}\cdots(b_k)_{m_1+\cdots+m_{k-1}}}{(aq/b_1)_{m_1}(aq/c_1)_{m_1}(aq/b_2)_{m_1+m_2}(aq/c_2)_{m_1+m_2}\cdots(aq/b_{k-1})_{m_1+\cdots+m_{k-1}}}$$

$$\frac{(c_k)_{m_1+\cdots+m_{k-1}}}{(aq/c_{k-1})_{m_1+\cdots+m_{k-1}}} \frac{(q^{-j})_{m_1+\cdots+m_{k-1}}}{(b_kc_kq^{-j}/a)_{m_1+\cdots+m_{k-1}}}$$

$$\frac{(aq)^{m_{k-2}+2m_{k-3}+\cdots+(k-2)m_1} q^{m_1+\cdots+m_{k-1}}}{(b_2c_2)^{m_1}(b_3c_3)^{m_1+m_2}\cdots(b_{k-1}c_{k-1})^{m_1+\cdots+m_{k-2}}}$$

In Section 5 we shall discuss a few of the implications of this formula. To conclude this section, we turn to an (A_n,B_n)-pair obtained for the study of Rogers-Ramanujan type identities related to the modulus 11 [3]. In Lemma 3 (with $a = \alpha\beta q$) we replace a by a^4 and q by q^4; then Theorem 5.1 of [3] is merely the first identity of Lemma 3 with

(5.3)
$$A_n = \frac{(-1)^n(1-aq^{2n})(a)_n a^n q^{1/2n(3n-1)}}{(1-a)(q)_n} \quad ,$$

(5.4)
$$B_n = \frac{1}{(a^4q^4;q^4)_{2n}} \sum_{j=0}^{n} \frac{(-1)^j a^{2j} q^{j(4n+1)}(aq)_{4n-2j}}{(q^2;q^2)_j(q^4;q^4)_{n-j}}$$

$$(5.5)\qquad \frac{(1-aq^{2n})(aq)_{n-1}q^{-\frac{n^2}{2}+\frac{3n}{2}}}{(1-a^4q^{8n})(q)_n}$$

$$= \sum_{j=0}^{n}\sum_{h=0}^{n} \frac{(a^4q^{8j+8h+4};q^4)_{n-j-h-1}(aq)_{4j+2h}(-1)^j a^{2h} q^{4\binom{j+h+1}{2}+(h-n)(j+h)+}}{(q^4;q^4)_{n-j-h}(q^2;q^2)_h(q^4;q^4)_j}$$

6. Applications. The types of identities treated here have numerous implications for the theory of partitions. These have been explored at length in [1], [2], [4], [6]. We describe a few here that have not been touched upon in previous expositions.

We begin by considering the cumbersome (5.2). Replace b_i by aq/b_i, c_i by aq/c_i; then replace c_i by c_iq^{-r} and a by aq^{-r}. If we now let the $c_i \to \infty$ and $r \to \infty$ we find that

$$(6.1)\qquad \frac{1}{(b_1)_\infty(b_2)_\infty\cdots(b_k)_\infty}$$

$$= \sum_{m_1,\ldots,m_k \geqq 0} \frac{q^{\sigma_1(m_1,\ldots,m_k)^2-\sigma_2(m_1,\ldots,m_k)-\sigma_1(m_1,\ldots,m_k)} b_1^{m_1}\cdots b_k^{m_k}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_k}(b_1)_{m_1}(b_2)_{m_1+m_2}\cdots(b_k)_{m_1+\cdots+m_2}}$$

where $\sigma_i(m_1,\ldots,m_k)$ is the ith elementary symmetric function of the $m_1,\ldots,m$

In (6.1) let us replace each b_i by q^i, then recalling MacMahon's formula for $\pi_k(m)$ the number of plane partitions of m with k rows [24;p.243 we see that

$$(6.2)\qquad \sum_{m=0}^{\infty}\pi_k(m)q^m = \sum_{m_1,\ldots,m_k \geqq 0} \frac{q^{\sigma_1(m_1,\ldots,m_k)^2-\sigma_2(m_1,\ldots,m_k)+m_2+2m_3+\cdots+(k-1)}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_k}(q)_{m_1}(q^2)_{m_1+m_2}\cdots(q^k)_{m_1+\cdots+}}$$

We should point out that (6.1) is relatively easy to derive from a multiple application of Cauchy's generalization of Euler's theorem (i.e. equation (5.1) with $k = 1$). However since Gordon [21] was successful in treating certain two-rowed plane partition problems by studying the case $k = 1$ of (6.1), it not unreasonable to suggest that multiple series identities such as this may prove useful in other plane partition problems especially in view of (6.2).

We close with an addition to the literature of Rogers-Ramanujan type identities. Since the full q-analog of Watson's theorem arose as (4.6), it i natural to ask what can be said if the q-analogue of Whipple's theorem [5] is employed. If we set $\alpha = \beta = 1$ in Lemma 3, then the second identity given there is just the q-analog of Whipple's theorem for

$$A_k = \begin{cases} \dfrac{q^{2n^2-n}c^n(q^2/e;q^2)_n(eq/c;q^2)_n(1-q^{4n+1})}{(eq;q^2)_n(cq^2/e;q^2)_n(1-q)}, & k = 2n \\[2ex] \dfrac{-q^{(n-1)(2n-1)}c^n(q/e;q^2)_n(e/c;q^2)_n(1-q^{4n-1})}{(e;q^2)_n(cq/e;q^2)_n(1-q)}, & k = 2n-1, \end{cases} \tag{6.3}$$

$$B_k = \frac{(c;q^2)_k}{(e)_k(cq/e)_k(q^2;q^2)_k}. \tag{6.4}$$

The first identity in Lemma 3 with this pair asserts

$$\frac{(q^2)_k(q)_k(c/q^2)_k}{(e)_k(cq/e)_k(q^2;q^2)_k} \tag{6.5}$$

$$= \sum_{j\geq 0} \frac{(q^{-k})_{2j}(q^2/e;q^2)_j(eq/c;q^2)_j(1-q^{4j+1})q^{2jk}c^j}{(q^{k+2})_{2j}(eq/q^2)_j(cq^2/e;q^2)_j(1-q)}$$

$$+ \sum_{j\geq 1} \frac{(q^{-k})_{2j-1}(q/e;q^2)_j(e/c;q^2)_j(1-q^{4j-1})q^{(2j-1)k}c^j}{(q^{k+2})_{2j-1}(e;q^2)_j(cq/e;q^2)_j(1-q)}.$$

If we let Y, Z and $n \to \infty$ in (3.5) with this (A_n, B_n) pair we obtain

(6.6)
$$\sum_{j=0}^{\infty} \frac{q^{j^2+1}(c;q^2)_j}{(e)_j(cq/e)_j(q^2;q^2)_j}$$

$$= \frac{1}{(q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^{6k^2+k}c^k(q^2/e;q^2)_k(eq/e;q^2)_k(1-q^{4k+1})}{(eq;q^2)_k(cq^2/e;q^2)_k(1-q)}$$

$$- \frac{1}{(q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^{6k^2+3k+3}c^k(q/e;q^2)_k(e/c;q^2)_k(1-q^{4k-1})}{(e;q^2)_k(cq/e;q^2)_k(1-q)} .$$

This identity can be used to derive a theta function identitys with modulus 7 (let $c \to 0$, $e = -q^{1/2}$) [1; p.443] as well as one of the mock theta function identities ($c = q^2$, $e = -q^{3/2}$) [28]. However (6.6) does not fit in with the pattern of identities given by Bailey in [14] where each result involves a single very well-poised basic hypergeometric series.

References

1. G.E. Andrews, On q-difference equations for certain well-poised basic hypergeometric series, Quart. J. Math. (Oxford Second Series), 19(1968), 433-447.

2. G.E. Andrews, A new property of partitions with applications to the Rogers-Ramanujan identities, J. Combinatorial Th., 10(1971), 266-270.

3. G.E. Andrews, On Rogers-Ramanujan type identities related to the modulus 11, Proc. London Math. Soc. (3rd. Ser.), 30(1975), 300-346.

4. G.E. Andrews, Problems and prospects for basic hypergeometric functions, from Theory and Application of Special Functions, R. Askey ed., Academic Press, New York, 1975.

5. G.E. Andrews, On q-analogs of the Watson and Whipple summations, S.I.A.M. J. Math. Anal. 7(1976), 332-336.

6. G.E. Andrews, The Theory of Partitions, Encylcopedia of Mathematics and Its Applications, Vol. 2, Addison-Wesley, Reading, 1976.

7. G.E. Andrews and R. Askey, Enumeration of partitions: the role of Eulerian series and q-orthogonal polynomials, from Higher Combinatorics, M. Aigner ed., Reidel, Dordrecht, 1977.

8. G.E. Andrews and R. Askey, The Classical Orthogonal Polynomials and Their Discrete and q-Analogs, (to appear).

9. R. Askey, Orthogonal Polynomials and Special Functions, Regional Cc Series in Appl. Math., Vol. 21, S.I.A.M., Philadelphia, 1975.

10. R. Askey and M.E.-H. Ismail, A combinatorial sum, M.R.C. Tech. Rprt #1557, July 1975.

11. R. Askey and M. Ismail, A generalization of ultraspherical polynomi (to appear).

12. W.N. Bailey, Generalized Hypergeometric Series, Cambridge Math. Tr No. 32, Cambridge University Press, London, 1935 (Reprinted: Hafner, New York 1964).

13. W.N. Bailey, Some identities in combinatory analysis, Proc. London Math. Soc. (Ser. 2), 49(1947), 421-435.

14. W.N. Bailey, Identities of the Rogers-Ramanujan type, Proc. London Math. Soc. (Ser. 2), 50(1949), 1-10.

15. D. Bressoud, A functional generalization of the Rogers-Ramanujan identities with interpretation, (to appear).

16. D. Bressoud, A new proof of Andrews's generalization of the Rogers-Ramanujan-Schur identities, (to appear).

17. L. Carlitz, Some combinatorial identities, Publ. Elektrotehn. Fak., Univ. Beograd, Ser. Math. Fiz. No. 461-497(1974), 87-90.

18. D.M. Cvetković and S.M. Simić, On enumeration of certain types of sequences, Publ. Elektrotehn. Fak., Univ. Beograd, Ser. Mat. Fiz. No. 412-46((1973), 159-164.

19. E. Feldheim, Sur les polynomes généralisés de Legendre, Izv. Akad. Nauk SSSR Ser. Mat. 5(1941), 241-248, Russian transl. ibid., 248-254, MR 3, 112.

20. G. Gasper, Positive integrals of Bessel functions, S.I.A.M. J. Math Anal., 6(1975), 868-881.

21. B. Gordon, Two new representations of the partition function, Proc. A.M.S., 13(1962), 869-873.

22. W. Gosper, Private communication.

23. I.L. Lancevickiĭ, Über die Orthogonalität der Féjèr-Szegöschen Polynome, C.R. (Dokl.) Acad. Sci. U.R.S.S., 31(1941), 199-200; MR 3, 112.

24. P.A. MacMahon, Combinatory Analysis, Vol. 2, Cambridge University Press, London, 1915 (Reprinted: Chelsea, New York, 1964).

25. L.J. Rogers, Second memoir on the expansion of certain infinite prc ducts, Proc. London Math. Soc. 25(1894), 318-343.

26. L.J. Rogers, Third memoir on the expansion of certain infinite products, Proc. London Math. Soc., 26(1894), 15-32.

27. L.J. Rogers, On two theorems of combinatory analysis and some allie identities, Proc. London Math. Soc. (Ser. 2), 16(1917), 315-336.

28. G.-C. Rota and R. Mullin, On the foundations of combinatorial theory: III. Theory of binomial enumeration, from Graph Theory and Its Applications, B. Harris ed., Academic Press, New York, 1970.

29. G.-C. Rota, D. Kahaner and A. Odlyzko, On the foundations of combinatorial theory, VIII. Finite operator calculus, J. Math. Anal. and Appl., 42 (1973), 684-760.

30. G.-C. Rota, S.M. Roman, The umbral calculus, Adv. in Math., 27(1978), 95-188.

31. L.J. Slater, A new proof of Rogers's transformations of infinite series, Proc. London Math. Soc. (Ser. 2), 53(1951), 460-475.

32. L.J. Slater, Further identities of the Rogers-Ramanujan type, Proc. London Math. Soc. (Ser. 2), 54(1962), 147-167.

33. L.J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, London, 1966.

34. G. Szego, Orthogonal Polynomials, Colloquium Publications, vol. 23, 3rd ed., A.M.S., Providence 1967.

35. G.N. Watson, The final problem: an account of the mock-theta function J. London Math. Soc., 11(1936), 55-80.

THE PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PENNSYLVANIA 16802