A CLASS OF GENERATING FUNCTIONS FOR A NEW GENERALIZATION OF EULERIAN POLYNOMIALS WITH THEIR INTERPOLATION FUNCTIONS

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Abstract

Motivated by a number of recent investigations, we define and investigate the various properties of a new family of the Eulerian polynomials. We derive useful results involving these Eulerian polynomials including (for example) their generating functions, new series and L-type functions.

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1. Preliminaries

The Eulerian polynomials have been studied from Euler's time to the present, which have been extensively investigated in many different contexts in the mathematics and computer science literature (see [1-21] for a systematic work).

Recently, Kim *et al* have studied on some identities of the Eulerian polynomials in connection with Genocchi and Tangent numbers using the fermionic p-adic integral on $\mathbb{Z}_p$ in [10]. Kim and Kim introduced a new definition of Eulerian polynomials and gave their symmetric relations (for details, see [11], [12]). Araci *et al* also introduced the generalizations of the Eulerian-type polynomials using the fermionic p-adic q-integral on $\mathbb{Z}_p$ and derived some new interesting identities *cf.* [1], [3], [4].

Leonard Euler gave the Eulerian polynomials in 1749 by the rule:

$$\sum_{k=0}^{\infty}(k+1)^n x^k=\frac{\mathcal{A}_n(x)}{(1-x)^{n+1}}. \tag{1.1}$$

Euler introduced the Eulerian polynomials in an attempt to evaluate the Dirichlet eta function

$$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} \tag{1.2}$$

at negative integers. It is well known in [6] that Dirichlet eta functions are closely related to Riemann zeta function as follows:

$$\zeta(s):=\begin{cases}\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1-2^{-s}}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s} & (\Re(s)>1)\\ \frac{1}{1-2^{1-s}}\eta(s) & (\Re(s)>0;s\neq 0).\end{cases} \tag{1.3}$$

Combining the Eq. (1.1) with the Eq. (1.3), it reduces to

$$\mathcal{A}_n(-1)=\left(2^{n+1}-4^{n+1}\right)\zeta(-n)=\frac{\left(4^{n+1}-2^{n+1}\right)B_{n+1}}{n+1}\ \text{(see [10])}$$

where B_n are the Bernoulli numbers defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\ \ |t|<2\pi.$$

The Eulerian polynomials $\mathcal{A}_n(x)$ are defined by means of the following exponential generating series:

$$e^{\mathcal{A}(x)t}=\sum_{n=0}^{\infty}\mathcal{A}_n(x)\frac{t^n}{n!}=\frac{1-x}{e^{t(1-x)}-x} \tag{1.4}$$

in which the usual convention about replacing $\mathcal{A}^n(x)$ by $\mathcal{A}_n(x)$. Hereby, we note that generating functions transform problems about *sequences* into problems about *polynomials*. By this way, generating functions are important to solve all sorts of counting problems.

The Eulerian polynomials can be computed by the recurrence relation:

$$(\mathcal{A}(x)+(x-1))^n-x\mathcal{A}_n(x)=\begin{cases}1-x, & \text{if } n=0\\ 0, & \text{if } n>0\end{cases} \tag{1.5}$$

where the usual convention about replacing $\mathcal{A}^n(x)$ by $\mathcal{A}_n(x)$, (for more information, see [1], [3], [4], [11], [10], [7]).

Let p be a fixed odd prime number. Throughout this paper, we always make use of the following notations: $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^*=\mathbb{N}\cup\{0\}$.

Let v_p be normalized exponential valuation of $\mathbb{C}_p$ such that

$$|p|_p=p^{-v_p(p)}=\frac{1}{p}.$$

When one talks of q-extension, q-can be regarded as an indeterminate, a complex number $q\in\mathbb{C}$, or a p-adic number $q\in\mathbb{C}_p$; it is always clear from the context. If $q\in\mathbb{C}$, then one usually assumes that $|q|<1$. If $q\in\mathbb{C}_p$, then one usually assumes that $|q-1|_p<1$, and hence $q^x=\exp(x\log q)$ for $x\in\mathbb{Z}_p$. In this work, we also use the notations:

$$[x]_q=\frac{1-q^x}{1-q}\ \ and\ \ [x]_{-q}=\frac{1-(-q)^x}{1+q},$$

(see, for details, [1], [3], [8], [9], [15]). We note that $\lim_{q\to 1}[x]_q=x$ for any x with $|x|_p\le 1$ in the present p-adic case.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For a positive integer d with $(d,p)=1$, set

$$\begin{aligned} X &= X_d=\varprojlim_{n}\mathbb{Z}/dp^n\mathbb{Z},\ X_1=\mathbb{Z}_p \\ X^* &= \bigcup_{\substack{0<a<dp\\(a,p)=1}} a+dp\mathbb{Z}_p \end{aligned}$$

and

$$a+dp^n\mathbb{Z}_p=\{x\in X\mid x\equiv a\,(\mathrm{mod}\,dp^n)\},$$

where $a\in\mathbb{Z}$ satisfies the condition $0\le a<dp^n$.

The p-adic q-Haar distribution is defined by Kim in [13] and [14], as follows:

$$\mu_q(x+p^n\mathbb{Z}_p)=\frac{q^x}{[p^n]_q}.$$

Thus, for $f\in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is also defined by Kim as follows:

$$I_q(f)=\int_{\mathbb{Z}_p}f(x)\,d\mu_q(x)=\lim_{n\to\infty}\sum_{x=0}^{p^n-1}f(x)\,\mu_q(x+p^n\mathbb{Z}_p)=\lim_{n\to\infty}\frac{1}{[p^n]_q}\sum_{x=0}^{p^n-1}f(x)\,q^x. \tag{1.6}$$

The bosonic integral is considered as the bosonic limit $q\to 1$, $I_1(f)=\lim_{q\to 1}I_q(f)$. In [16], similarly, the fermionic p-adic integration on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f)=\lim_{t\to -q}I_t(f)=\int_{\mathbb{Z}_p}f(x)\,d\mu_{-q}(x). \tag{1.7}$$

From the Eq. (1.7), we have the known integral equation in [16]:

$$q^nI_{-q}(f_n)+(-1)^{n-1}I_{-q}(f)=[2]_q\sum_{l=0}^{n-1}(-1)^{n-1-l}q^lf(l), \tag{1.8}$$

where $f_n(x)$ is a translation with $f(x+n)$. It follows from the Eq. (1.8) that

If n is odd, then

$$q^nI_{-q}(f_n)+I_{-q}(f)=[2]_q\sum_{l=0}^{n-1}(-1)^lq^lf(l). \tag{1.9}$$

If n is even, then we have

$$I_{-q}(f)-q^nI_{-q}(f_n)=[2]_q\sum_{l=0}^{n-1}(-1)^lq^lf(l). \tag{1.10}$$

Substituting $n=1$ into the Eq. (1.9), then it becomes

$$qI_{-q}\left(f_{1}\right)+I_{-q}\left(f\right)=\left[2\right]_{q}f\left(0\right). \tag{1.11}$$

Replacing q by q^{-1} in the Eq. (1.11), we have

$$I_{-q^{-1}}\left(f_{1}\right)+qI_{-q^{-1}}\left(f\right)=\left[2\right]_{q}f\left(0\right). \tag{1.12}$$

In [10], Kim *et al.* is considered $f(x)=e^{-x(1+q)t}$ in the Eq. (1.12), then they gave Witt's formula of Eulerian polynomials as follows: for $n\in\mathbb{N}^{*}$,

$$I_{-q^{-1}}\left(x^{n}\right)=\frac{\left(-1\right)^{n}}{\left(1+q\right)^{n}}\mathcal{A}_{n}\left(-q\right). \tag{1.13}$$

In [11], the new generalization of the Eulerian polynomials on $\mathbb{Z}_{p}$ was introduced by D. Kim and M. S. Kim, as follows: for $w\in\mathbb{N}^{*}$

$$I_{-q^{-1}}\left(q^{(1-w)x}x^{n}\right)=\frac{\left(-1\right)^{n}}{w^{n}\left(1+q\right)^{n}}\mathcal{A}_{n}\left(-q,w\right). \tag{1.14}$$

It follows from the Eq. (1.14) that

$$\lim_{w\rightarrow1}I_{-q^{-1}}\left(q^{(1-w)x}x^{n}\right)=I_{-q^{-1}}\left(x^{n}\right)=\frac{\left(-1\right)^{n}}{\left(1+q\right)^{n}}\mathcal{A}_{n}\left(-q\right).$$

By using the fermionic p-adic invariant q-integral on $\mathbb{Z}_{p}$, we consider a new generalization of the Eulerian polynomials and give some intereting properties. Actually, we are motivated from the papers of Kim *et al* [10] and Kim *et al* [11] to write this paper.

2. On the Dirichlet's type of Eulerian polynomials

In this part, we assume that d is an odd natural number. Then we consider the following equality by using the Eq. (1.9):

$$\int_{\mathbb{Z}_{p}}f\left(x+d\right)d\mu_{-q^{-1}}\left(x\right)+q^{d}\int_{\mathbb{Z}_{p}}f\left(x\right)d\mu_{-q^{-1}}\left(x\right)=\left[2\right]_{q}\sum_{l=0}^{d-1}\left(-1\right)^{l}q^{d-l+1}f\left(l\right). \tag{2.1}$$

Let χ be a Dirichlet character with conductor d, by $p\mid d$. Then, substituting $f(x)=\chi\left(x\right)q^{(1-w)x}e^{-x(1+q)wt}$ in the Eq. (2.1), we have

$$\int_{\mathbb{Z}_{p}}\chi\left(x+d\right)q^{(1-w)(x+d)}e^{-(x+d)(1+q)wt}d\mu_{-q^{-1}}\left(x\right)+q^{d}\int_{\mathbb{Z}_{p}}\chi\left(x\right)q^{(1-w)x}e^{-x(1+q)wt}d\mu_{-q^{-1}}\left(x\right)$$
$$=\left[2\right]_{q}\sum_{l=0}^{d-1}\left(-1\right)^{l}q^{d-l+1}\chi\left(l\right)q^{(1-w)l}e^{-l(1+q)wt}.$$

After some simplifications, we see that

$$\int_{\mathbb{Z}_{p}}\chi\left(x\right)q^{(1-w)x}e^{-x(1+q)wt}d\mu_{-q^{-1}}\left(x\right)=\left[2\right]_{q}\sum_{l=0}^{d-1}\frac{\left(-1\right)^{l}q^{d-l+1}q^{(1-w)l}\chi\left(l\right)e^{-l(1+q)wt}}{q^{(1-w)d}e^{-d(1+q)wt}+q^{d}}. \tag{2.2}$$

Let $\mathcal{F}_{q}^{w}\left(t\mid\chi\right)=\sum_{n=0}^{\infty}\mathcal{A}_{n,\chi}\left(-q,w\right)\frac{t^{n}}{n!}$. Then, we state the following definition of generating function of the Dirichlet's type of the generalized Eulerian polynomials.

Definition 1. *For $n, w \in \mathbb{N}^*$, we define*

$$\sum_{n=0}^{\infty} \mathcal{A}_{n,\chi}(-q,w)\frac{t^n}{n!} = [2]_q \sum_{l=0}^{d-1} \frac{(-1)^l q^{d-l+1} q^{(1-w)l} \chi(l) e^{-l(1+q)wt}}{q^{(1-w)d} e^{-d(1+q)wt} + q^d}. \tag{2.3}$$

From the expressions of the Eq. (2.2) and the Eq. (2.3), we give the following theorem which seems to be Witt's formula for the Dirichlet's type of the generalized Eulerian polynomials.

Theorem 1. *The following equality holds:*

$$I_{-q^{-1}}\left(\chi(x) q^{(1-w)x} x^n\right) = \frac{(-1)^n}{w^n (1+q)^n} \mathcal{A}_{n,\chi}(-q,w) \tag{2.4}$$

From the Eq. (2.3), we discover

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{A}_{n,\chi}(-q,w)\frac{t^n}{n!} &= [2]_q \sum_{l=0}^{d-1} (-1)^l q^{d-l+1} q^{(1-w)l} \chi(l) \frac{e^{-l(1+q)wt}}{q^{(1-w)d} e^{-d(1+q)wt} + q^d} \\
&= [2]_q \sum_{l=0}^{d-1} (-1)^l q^{-l+1} q^{(1-w)l} \chi(l) e^{-l(1+q)wt} \sum_{m=0}^{\infty} (-1)^m q^{-mwd} e^{-mwd(1+q)t} \\
&= q[2]_q \sum_{m=0}^{\infty} \sum_{l=0}^{d-1} (-1)^{l+md} \chi(l+md) \left(q^{-w}\right)^{l+md} e^{-(l+md)(1+q)wt} \\
&= q[2]_q \sum_{m=0}^{\infty} (-1)^m \chi(m) q^{-wm} e^{-m(1+q)wt}.
\end{aligned}$$

Thus, we obtain the following theorem.

Theorem 2. *For each $w \in \mathbb{N}^*$, we have*

$$\mathcal{F}_q^w(t \mid \chi) = \sum_{n=0}^{\infty} \mathcal{A}_{n,\chi}(-q,w)\frac{t^n}{n!} = q[2]_q \sum_{m=0}^{\infty} (-1)^m \chi(m) q^{-wm} e^{-m(1+q)wt}. \tag{2.5}$$

By applying the definition of Taylor expansion of $e^{-m(1+q)wt}$ to the Eq. (2.5), we procure the following theorem.

Theorem 3. *For $n, w \in \mathbb{N}^*$, we have*

$$\frac{(-1)^n}{w^n (1+q)^{n+1}} \mathcal{A}_{n,\chi}(-q,w) = \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m) m^n}{q^{wm-1}}. \tag{2.6}$$

Combining the Eq. (2.4) with the Eq. (2.6), we arrive at the following corollary:

Corollary 1. *For $n, w \in \mathbb{N}^*$, then we get*

$$\lim_{n\to\infty} \sum_{m=1}^{p^n-1} \frac{(-1)^m \chi(m) m^n}{q^{wm}} = 2\sum_{m=1}^{\infty} \frac{(-1)^m \chi(m) m^n}{q^{wm-2}}.$$

We now derive a distribution formula for the Dirichlet's type of the generalized Eulerian polynomials using the fermionic p-adic q-integral on $\mathbb{Z}_p$, as follows:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \chi(x)\, q^{(1-w)x} x^n d\mu_{-q^{-1}}(x) \\
&= \lim_{m\to\infty} \frac{1}{[dp^m]_{-q^{-1}}} \sum_{x=0}^{dp^m-1} (-1)^x q^{(1-w)x} \chi(x)\, x^n q^{-x} \\
&= \frac{d^n}{[d]_{-q^{-1}}} \sum_{a=0}^{d-1} (-1)^a \chi(a)\, q^{-wa} \left(\lim_{m\to\infty} \frac{1}{[p^m]_{-q^{-d}}} \sum_{x=0}^{p^m-1} (-1)^x \left(\frac{a}{d}+x\right)^n q^{-dwx} \right) \\
&= \frac{d^n}{[d]_{-q^{-1}}} \sum_{a=0}^{d-1} (-1)^a \chi(a)\, q^{-wa} \int_{\mathbb{Z}_p} \left(\frac{a}{d}+x\right)^n q^{-dwx} d\mu_{-q^{-d}}(x) \\
&= \frac{d^n}{[d]_{-q^{-1}}} \sum_{a=0}^{d-1} \sum_{j=0}^{n} \binom{n}{j} (-1)^a \chi(a)\, q^{-wa} \left(\frac{a}{d}\right)^{n-j} \int_{\mathbb{Z}_p} q^{-dwx} x^j d\mu_{-q^{-d}}(x).
\end{aligned}$$

Thus, we state the following theorem.

Theorem 4. *The following identity holds true:*

$$\begin{aligned}
&\frac{(-1)^n}{w^n (1+q)^n} \mathcal{A}_{n,\chi}(-q,w) \\
&= \frac{d^n}{[d]_{-q^{-1}}} \sum_{a=0}^{d-1} \sum_{j=0}^{n} \frac{\binom{n}{j} (-1)^{a+j} \chi(a)\, q^{-wa} \left(\frac{a}{d}\right)^{n-j}}{(1+q+dw+qdw)^j} \mathcal{A}_j(-q, dw+1).
\end{aligned} \tag{2.7}$$

3. On the L-type functions

The classical Bernoulli numbers are interpolated by the Riemann zeta functions, which have profound effect on Analytic numbers theory, complex analysis and other related topics. The values of the negative integer points, also found by Euler, are rational numbers and play a vital and important role in the theory of modular forms. Many generalization of the Riemann zeta function, such as Dirichlet series, Dirichlet L-functions and L-functions, are worked in [1], [15], [17], [18], [19], [20], [21].

In this final part, our objective is to introduce a new generalization of the Eulerian-L function applying Mellin transformation to the generating function of the Eulerian polynomials. From the Eq. (2.5), for $s \in \mathbb{C}$, we consider

$$L_E^w(s \mid \chi) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathcal{F}_q^w(t \mid \chi)\, dt$$

($\Gamma(s)$ is known as Gamma function) and compute as follows:

$$\begin{aligned} L_E^w(s \mid \chi) &= q[2]_q \sum_{m=0}^{\infty} (-1)^m \chi(m) q^{-wm} \left\{ \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-m(1+q)wt} dt \right\} \\ &= \frac{q}{(1+q)^{s-1}} \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m) q^{-wm}}{(wm)^s}. \end{aligned}$$

As a result of the above applications, we give definition of the generalized Eulerian L-function as follows:

Definition 2. *For $s \in \mathbb{C}$, we have*

$$L_E^w(s \mid \chi) = \frac{q}{(1+q)^{s-1}} \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m)}{q^{wm}(wm)^s}. \tag{3.1}$$

After substituting $s = -n$ into (3.1), then the relation between the generalized Eulerian L-function and Dirichlet's type of the generalized Eulerian polynomials are given the following.

Theorem 5. *The following equality holds true:*

$$L_E^w(-n \mid \chi) = (-1)^n \mathcal{A}_{n,\chi}(-q, w) = \begin{cases} -\mathcal{A}_{n,\chi}(-q, w) & \text{if } n \text{ odd}, \\ \mathcal{A}_{n,\chi}(-q, w) & \text{if } n \text{ even}. \end{cases}$$

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