There and Back Again: Elliptic Curves, Modular Forms, and L-Functions

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Abstract

L-functions form a connection between elliptic curves and modular forms. The goals of this thesis will be to discuss this connection, and to see similar connections for arithmetic functions.

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Introduction: A Road Map

In this thesis, we will be exploring what helps make an L-function an L-function. In particular, we are focusing on showing that the Dirichlet series for given arithmetic functions are in fact L-functions.

Chapter 1 is intended to introduce the reader to elliptic curves, which helped provide the initial motivation for this project. The properties of elliptic curves that we are interested in provide an alternate view on L-functions when we put them in context.

Chapter 2 introduces Dirichlet series, which are similar to L-functions. They are highly generalized, so while all L-functions can be seen as Dirichlet series, not every Dirichlet series can be viewed as an L-function. In the context of this thesis, Dirichlet series are viewed as a series rather than as a function. This chapter will also introduce the Dirichlet series that we are primarily interested in for the sake of this thesis. The theorems given in this chapter are stated in other texts, but the proofs provided are our own.

Chapter 3 is to provide the reader with a definition of an L-function that will be used for the remainder of the thesis by providing us a list of criteria to check when showing that a Dirichlet series is an L-function.

Chapter 4 provides a small introduction to modular forms. The modular forms will then be used when showing some of the analytical properties of L-functions.

Chapter 5 will state many of the results of this thesis. Theorems 5.1, 5.2, 5.3, and 5.4 are stated in other sources, but the proof given here are the writings of the author of the thesis. All other theorems in this chapter are original work, but follow similarly to the proofs given for the original theorems.

Chapter 6 is meant to give extensions and applications of this thesis, and should be considered as a guide for anyone looking to continue the work presented herein.

Chapter 1

Elliptic Curves

An *algebraic curve* is defined via a polynomial or a set of polynomials. If we were to consider the algebraic curves defined in two dimensions, we would then have curves define my linear combinations of monomials of the form $x^m y^n$, where m and n we both non-negative integers. We say the degree, d, of a polynomial is defined to be $d = \max_{x^m y^n}\{m+n\}$. We now consider this interesting theorem about polynomials and their degrees.

Theorem 1.1 (Ash and Gross (2012) Theorem 6.1)**.** *Let F be a polynomial with integer coefficients of degree d, defining a non-singular projective curve C by $F(x,y) = 0$. Then if $d = 1$ or 2, $C(\mathbb{Q})$ is either empty or infinite. If $d \geq 4$, then $C(\mathbb{Q})$ is always finite.*

This theorem says that we know what happens if $d = 1$ or 2 or if $d \geq 4$. The study of *elliptic curves* is the study of curves whose defining polynomial is of degree 3. The goal of this chapter will be to introduce elliptic curves, and present how we consider points on elliptic curves.

Definition 1.1. *An **elliptic curve**, E, is a non-empty, smooth variety $V(F)$ where* $\deg F = 3$.

Typically, an elliptic curve is given by an equation of the form

$$E: \quad y^2 = x^3 + Ax + B,$$

where A and B are constant coefficients in some field, like $\mathbb{Q}$. Even when the curve does not take this form it is usually (though not always) possible to use a change of variables to convert it to this form. The points on this curve form act on each other under an operation that will be described in the next section.

1.1 The Operation

When we discuss elliptic curves, we primarily focus on the points on a curve E over a field K, i.e. the points in the set

$$E_F(K) = \{(x, y) \in K \times K | F(x, y) = 0\}.$$

For this section, we will assume that E is non-singular over K; in other words, we assume that at least one of the derivatives of E is non-zero in K.

For an elliptic curve E, there exists a point $\mathcal{O}$ that is always in $E(K)$. $\mathcal{O}$ is defined to be the point that all vertical lines intersect, which we refer to as the point "at infinity". Since we know that our set is non-empty, we can use two (not necessarily distinct) elements in $E(K)$ to find another point in $E(K)$, using a new operation. This is done by intersecting E with a line, L, which goes through the two elements. In order to define the operation on $E(\overline{K})$ we will need the simplified version of Bézout's Theorem.

Theorem 1.2 (Bézout's Theorem)**.** *For a cubic curve E and a line L, $L \cap E$ has exactly three points counted over $\overline{K}$ with respect to multiplicity .*

Let P and Q be in $E(K)$. Then we define the operation $P + Q$ as follows:

(1) Let L be the line connecting P and Q. If $P = Q$, then L is just the line tangent to E at P.

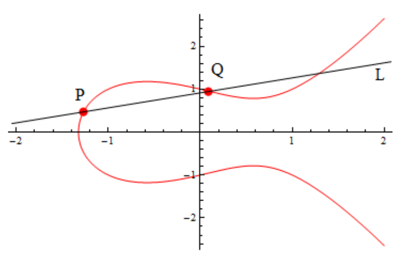

Figure 1.1 Step 1 of the Operation for elliptic curves: Picking P and Q and defining L.

(2) By Bézout's Theorem, we know that L must intersect E at a third point. Call this point $R = P \oplus Q$.

(3) We then find $P + Q$ by defining it to be $R \oplus \mathcal{O}$.

This concludes the explanation of the operation that acts on the points in $E(K)$.

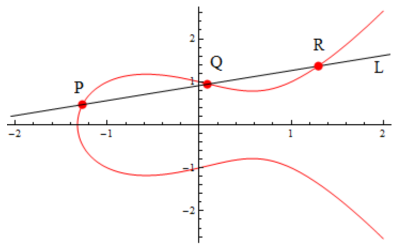

Figure 1.2 Step 2 of the Operation for elliptic curves: Identifying $R = P \oplus Q$.

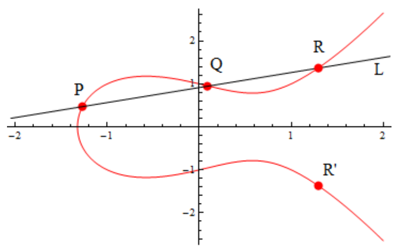

Figure 1.3 Step 3 of the Operation for elliptic curves: Identifying $R' = P + Q$.

1.2 Counting Points

This operation is a great way to find points on a given elliptic curve when we only know a one or two. In particular, we are interested in knowing the total number of points on a curve E on a given finite field.

1.2.1 Finite Fields

A *finite field* is a field which contains only a finite number of elements. An easy example of a finite field is, denoted $\mathbb{F}_p$ for a prime p. In the case where p is prime, we can say that $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$. $\mathbb{F}_q$ where $q = p^r$ for some positive integer r is also a finite field, and a demonstration of how these types of finite fields look can be found in Dummit and Foote (2004).

Counting points on a finite field, $\mathbb{F}_p$, is "easy", because there are only a finite number of points that need to be checked. Over $\mathbb{F}_{p^r}$ for $r > 1$ there are only a finite number of points, but the arithmetic is tricky. It is possible to avoid the tricky parts because the number of points in $\mathbb{F}_{p^r}$ is related to the number of points in $\mathbb{F}_p$. This relation is described in Ash and Gross (2012).

Example 1.1. *Consider the curve*

$$E : y^2 = x^3 + 11x + 17.$$

We will look for points on E in the field $\mathbb{F}_5$. We first consider the table of squares in $\mathbb{F}_5$.

y	y^2
0	0
1	1
2	4
3	4
4	1

Table 1.1 The squares in $\mathbb{F}_5$, which is also the left hand-side for E.

Next we consider the cubes and the right hand side of our defining equation for E in $\mathbb{F}_5$.

x	x^3	$x^3 + x + 2$
0	0	2
1	1	4
2	3	2
3	2	2
4	4	0

Table 1.2 The cubes in $\mathbb{F}_5$, and the right hand side for E

We note that the right-most columns in both Table 1.1 and Table 1.2 are the left- and right-hand sides of our equation for E, respectively. Therefore, we can see that there are 4 points in $E(\mathbb{F}_5)$, and they are (1,2), (1,3), (4,0), and $\mathcal{O}$.

Counting points on an elliptic curve is simple and relatively quick in a finite field compared to the rationals, and an example of code used to compute these points in $\mathbb{F}_p$ can be found in Appendix A.

1.2.2 Singular Points

There are sometimes bad points in an elliptic curve which are called singular points.

Definition 1.2. *A point P is a* ***singular point*** *on a curve C if there is no algebraically defined tangent line to C at P, in other words, all the derivatives of C at P are equal to* 0*.*

These points are handled by the operation given in Section 1.1 differently than non-singular points. The details for applying the operation to these points can be found in Ash and Gross (2012). An elliptic curve, E, is singular in a field K if E has a singular point. However, E will have at most one singular point over a given field. For example, two elliptic curves that are singular curves over $\mathbb{R}$ are shown in Figures 1.4 and 1.5. Furthermore, when we count points in $E(K)$, we do not include the singular point in our total.

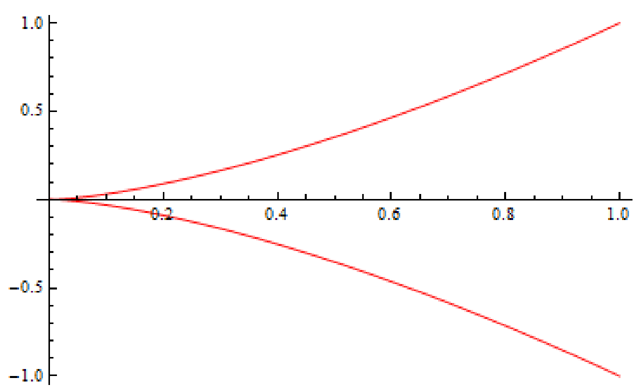

Figure 1.4 $y^2 = x^3$

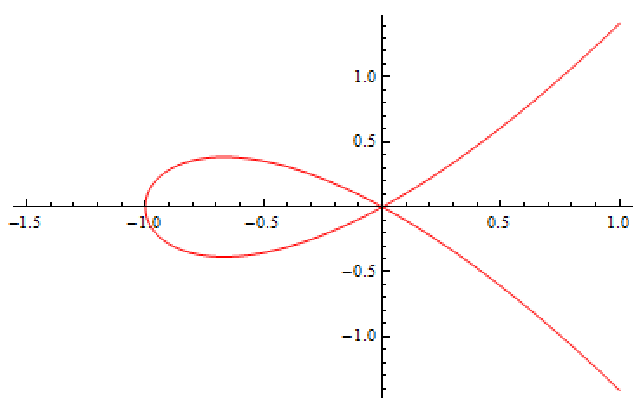

Figure 1.5 $y^2 = x^3 + x^2$

1.2.3 Discriminant

A quick way to check if an elliptic curve, E, is singular is by looking at the *discriminant*.

Definition 1.3. *For a generalized elliptic curve*

$$E : y^2 = x^3 + Ax + B,$$

*the **discriminant** is defined to be*

$$\Delta_E = -16(4A^3 + 27B^2).$$

We say E is singular in a field K if $\Delta_E = 0$ in K.

Example 1.2. *Consider the curve*

$$E : y^2 = x^3 + 11x + 17.$$

The discriminant of E is

$$\Delta_E = -16(4(11)^3 + 27(17)^2) = -210032 = -2^4(13127).$$

If we consider the fields of the form $\mathbb{F}_p$ where p is a prime, E is singular in $\mathbb{F}_2$ and $\mathbb{F}_{13127}$.

We usually refer to primes p such that a curve E is singular in $\mathbb{F}_p$ as *bad primes*.

1.3 The p-Defect

We expect that the number of points on an elliptic curve in $\mathbb{F}_p$ to be $p + 1$, because the actual number of points on $E(\mathbb{F}_p)$ is generally very close to this value, as will be shown. For non-singular elliptic curves, the p-defect is defined to be

$$a_p = p + 1 - N_p. \tag{1.1}$$

For singular elliptic curves, the p-defect is

$$a_p = p - N_p. \tag{1.2}$$

We still expect the number of points to be$p + 1$, but as we do not count the singular points, we instead expect the number of points we actually count to be p.

Example 1.3. *Returning to our curve in Example 1.1,*

$$E : y^2 = x^3 + 11x + 17,$$

we now wish to calculate some of the a_p's. For this example, we will consider $p = 5$ and $p = 2$. Let's start with $p = 5$. Since $5 \nmid \Delta_E$, we need to use (1.1). From the code in Appendix A, we get that $N_5 = 4$. Thus, we get that the 5-defect is

$$a_5 = 5 + 1 - 4 = 2.$$

If $p = 2$, then we use (1.2) to get that

$$a_2 = 2 - N_2 = 2 - 2 = 0.$$

We have now provided the method by which we calculate the p-defect. The possible values for the p-defect are bounded based upon p.

Theorem 1.3 (Hasse's Theorem)**.** *The number a_p satisfies the inequalities*

$$-2\sqrt{p} \leq a_p \leq 2\sqrt{p}.$$

A proof of Hasse's Theorem can be found in Silverman (2009). This theorem tells us that a_p is relatively small compared to p. We will eventually exploit the bound that Hasse places on the a_p's. Considering the a_p's provides us with better information about elliptic curves when we view them in other contexts, one of which we will see in the next chapter.

Chapter 2

Dirichlet Series

What are now known as Dirichlet series were originally studied by Johann Dirichlet. (See Ash and Gross (2012).) They are defined as follows:

Definition 2.1. *Let $\{a_n\}$ be a sequence of numbers. Then, a **Dirichlet series** is an expression written in terms of a complex variable s of the form*

$$D(s, \{a_n\}) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Sometimes, $\{a_n\}$ is given by a function $f(n)$. Also, When we need to refer to the real or complex components of s, we will write $s = \omega + it$. For this chapter, we view Dirichlet series as formal expressions; but later we will consider them as complex variable functions. For this reason we need to be concerned about the convergence of these series.

As usual, we say that if

$$\lim_{k\to\infty} \sum_{n=1}^{k} \frac{a_n}{n^s}$$

exists and is equal to some $J \in \mathbb{C}$, then the summation converges to J. For the Dirichlet series that we will consider, the following theorems will be useful for determining convergence.

Theorem 2.1. *(Ash and Gross, 2012: Theorem 11.6) Suppose there exists some constant K such that $|a_n| < Kn^r$ for all n. Then the Dirichlet series $\sum a_n n^{-s}$ converges if $\omega > r + 1$.*

Example 2.1. *Now consider an even simpler example, where $a_n = 1$ for all n. Then, the Dirichlet series will be*

$$\sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We can clearly see that there exists a constant K such that $|a_n| = 1 < K$. *Therefore, we use Theorem 2.1, where* $r = 0$, *to see that this Dirichlet series converges for* $\omega > 1$.

Example 2.2. *Consider the sequence* $\{a_n\}$ *defined by* $a_n = n$. *Then, the Dirichlet series looks like*

$$\sum_{n=1}^{\infty} \frac{n}{n^s}.$$

Furthermore, we can easily see that $|a_n| = n < Kn$ *for any fixed constant* $K > 1$. *For example,* $K = 2$ *works. Then, by Theorem 2.1 we get that the Dirichlet series converges for* $\omega > 2$.

Dirichlet series are also useful, because they sometimes have a form associated to them called an Euler product.

2.1 Euler Products

In this section, we will not consider Dirichlet series associated to any sequence $\{a_n\}$, but instead we will focus on a special case.

Definition 2.2. *An **arithmetic sequence** is a sequence* $\{a_n\}$ *such that*

$$a_m a_n = a_{mn}$$

for m and n relatively prime, and $a_1 = 1$.

If we were to consider the arithmetic sequence $\{a_n\}$ to corresponding to a function $f(n)$, then f is referred to as an *arithmetic function*.

We can utilize the properties of an arithmetic sequence to write the associated Dirichlet series as an Euler product

Definition 2.3. *An **Euler product** of a Dirichlet series is given by*

$$D(s, \{a_n\}) = \prod_{p \text{ prime}} \sum_{k=0}^{\infty} \frac{a_{p^k}}{p^{ks}}.$$

We see that this definition utilizes the fact that any prime power is relatively prime to the power of another prime. To demonstrate how this works, we consider the examples of Dirichlet series that we have already seen.

Example 2.3. *Consider the Dirichlet series from Example 2.1, where $a_n = 1$ for all n. Then, we can write*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{2^{2s}} + \cdots\right)\left(\frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{3^{2s}} + \cdots\right)$$
$$\left(\frac{1}{1^s} + \frac{1}{5^s} + \frac{1}{5^{2s}} + \cdots\right)\cdots$$

because all integers $n \geq 1$ have a unique prime factorization which shows up when we multiply all the terms together. Therefore, we can say

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \sum_{k=0}^{\infty} \frac{1}{p^{ks}}.$$

Example 2.4. *We can make this a little more complicated by contemplating the Dirichlet series from Example 2.2, where $a_n = n$ for all n. This sequence utilizes the Fundamental Theorem of Arithmetic, which gives us unique factorization in $\mathbb{N}$. We can write*

$$\sum_{n=1}^{\infty} \frac{n}{n^s} = \left(\frac{1}{1^s} + \frac{2}{2^s} + \frac{2^2}{2^{2s}} + \cdots\right)\left(\frac{1}{1^s} + \frac{3}{3^s} + \frac{3^2}{3^{2s}} + \cdots\right)$$
$$\left(\frac{1}{1^s} + \frac{5}{5^s} + \frac{5^2}{5^{2s}} + \cdots\right)\cdots$$

by similar reasoning to Example 2.3. Thus, we have

$$\sum_{n=1}^{\infty} \frac{n}{n^s} = \prod_{p\ prime} \sum_{k=0}^{\infty} \frac{p^k}{p^{ks}}.$$

In general, the Euler product would look like

$$\left(\frac{a_1}{1^s} + \frac{a_2}{2^s} + \frac{a_{2^2}}{2^{2s}} + \cdots\right)\left(\frac{a_1}{1^s} + \frac{a_3}{3^s} + \frac{a_{3^2}}{3^{2s}} + \cdots\right)\left(\frac{a_1}{1^s} + \frac{a_5}{5^s} + \frac{a_{5^2}}{5^{2s}} + \cdots\right)\cdots$$

but we usually just say that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p \text{ prime}} \sum_{k=0}^{\infty} \frac{a_{p^k}}{p^{ks}}.$$

We note that by how we've defined an Euler product that any Dirichlet series based on an arithmetic sequence or arithmetic function has an Euler product expansion. We can use Euler products to provide a simpler way of viewing Dirichlet series, as we then only need to understand a_{p^k}. In particular, it can be easier to find a closed form of $\sum a_{p^k} p^{-ks}$ than to find a closed form of $\sum a_n n^{-s}$.

2.2 The Riemann Zeta-Function

We wish to now focus on a special case of a Dirichlet series, presented initially in Example 2.1, where $a_n = 1$ for all n. This special case is also referred to as the Riemann zeta-function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

As demonstrated in Example 2.3, we can write $\zeta(s)$ as an Euler product

$$\zeta(s) = \prod_{p \text{ prime}} \sum_{k=0}^{\infty} \frac{1}{p^{ks}}.$$

We can define each of the terms in the product to be

$$\zeta_p(s) = \sum_{k=0}^{\infty} \frac{1}{p^{ks}}.$$

We note that we can rewrite this using *geometric series* to say that

$$\zeta_p(s) = \frac{1}{1 - p^{-s}}.$$

Thus, the Riemann zeta-function can be written as

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}. \tag{2.1}$$

We will be returning to the Riemann zeta-function throughout the rest of this thesis. Various properties of this function will be useful as we study other related functions. For now, we will relate other Dirichlet series to this function as we will see in the next section.

2.3 Arithmetic Functions

We will expand our collection of examples by looking at Dirichlet series of arithmetic sequences which which are given by arithmetic functions. For the remainder of this section let $n = p_1^{r_1} \cdots p_k^{r_k}$ where the p_i are the unique prime factors.

2.3.1 The Möbius Function

We begin with the Möbius function, $\mu(n)$. In general, the Mobius function is defined by the formula

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^k & r_i = 1 \text{ for all } i \\ 0 & \text{otherwise} \end{cases}. \tag{2.2}$$

For example,

$$\mu(105) = \mu(3)\mu(5)\mu(7) = (-1)^3 = -1.$$

Proposition 2.1. *Suppose that $\mu(n)$ is the Möbius function defined in (2.2). Then,*

$$D(s,\mu) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

which converges when $\omega > 1$.

Proof. Initially, our Dirichlet series is

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

By changing this over to its Euler product form, we now have

$$\prod_p \sum_{k=0}^{\infty} \frac{\mu(p^k)}{p^{ks}}.$$

We will note that by the definition of $\mu(n)$, we have $\mu(p^k) = 0$ for $k > 2$. Therefore, we can say that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \sum_{k=0}^{1} \frac{\mu(p^k)}{p^{ks}}.$$

By simply doing the summation, we can then conclude that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p (1 - p^{-s}).$$

Then, by (2.1)

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

We will further note that since $|\mu(n)| \leq 1$ for all n, we can apply Theorem 2.1 to get that this series converges when $\omega > 1$. $\square$

2.3.2 Euler's Totient Function

Next, we have Euler's Totient function, which for a number n gives the number of numbers less than n that are relatively prime to n. There is a simple formula for the totient function given by

$$\phi(n) = \prod_{i=1}^{k} p_i^{r_i - 1}(p_i - 1). \tag{2.3}$$

For example,

$$\phi(315) = \phi(9)\phi(5)\phi(7) = (3(3-1))(5-1)(7-1) = 144.$$

We wish to start by relating the values of $\phi(n)$ to the Möbius function.

Proposition 2.2. *Let $\phi(n)$ be Euler's Totient function defined by (2.3), and let $\mu(n)$ be the Möbius function. Then,*

$$\phi(n) = \sum_{d|n} d\mu\left(\frac{n}{d}\right).$$

Proof. We can better define $\phi(n)$ as

$$\phi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right) = n - \sum_{p|n}\frac{n}{p} + \sum_{\substack{p_i,p_j|n \\ p_i \neq p_j}} \frac{n}{p_i p_j} - \cdots$$

The last version is the most important. Since $\mu(m) = 0$ if m has a divisor that is a square, we can say

$$n - \sum_{p|n}\frac{n}{p} + \sum \frac{n}{p_i p_j} - \cdots = \sum_{d|n} n\frac{\mu(d)}{d}$$

by the definition of $\mu(m)$. Then, because we can switch places of the d and $\frac{n}{d}$, we get

$$n - \sum_{p|n}\frac{n}{p} + \sum \frac{n}{p_i p_j} - \cdots = \sum_{d|n} d\mu\left(\frac{n}{d}\right).$$

Therefore, we can say that

$$\phi(n) = \sum_{d|n} d\mu\left(\frac{n}{d}\right).$$

□

It should also be noted that for sums across α_n and β_n, if

$$\sum \gamma_n = \left(\sum \alpha_n\right)\left(\sum \beta_n\right),$$

then

$$\gamma_n = \sum_{d_1 d_2 = n} \alpha_{d_1} \beta_{d_2}. \tag{2.4}$$

Now we are ready to simplify the Dirichlet series associated to Euler's Totient function.

Proposition 2.3. *Let $\phi(n)$ be Euler's Totient function, then*

$$D(s,\phi) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)},$$

which converges when $\omega > 2$.

Proof. We start by looking at the right hand side of our equation, which contains Riemann ζ-function. By (2.1) and Proposition 2.1, we have

$$\frac{\zeta(s-1)}{\zeta(s)} = \left(\sum_{n\geq 1} \frac{n}{n^s}\right) \sum_{n\geq 1} \frac{\mu(n)}{n}.$$

Using Equation 2.4, we can then get that

$$\left(\sum_{n\geq 1} \frac{n}{n^s}\right) \sum_{n\geq 1} \frac{\mu(n)}{n} = \sum_{n\geq 1} \left(\frac{1}{n^s} \sum_{d|n} d\mu\left(\frac{n}{d}\right)\right).$$

Then, by Proposition 2.2, we get that

$$\sum_{n\geq 1} \left(\frac{1}{n^s} \sum_{d|n} d\mu\left(\frac{n}{d}\right)\right) = \sum_{n\geq 1} \frac{\phi(n)}{n^s}.$$

Therefore, we can conclude that

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

We can see by inspection that $|\phi(n)| < n$ for all n. Therefore, by Theorem 2.1, we have that the series converges when $\omega > 2$. $\square$

2.3.3 The Sum of Divisors Function

Now, we look at the sum of divisors function $\sigma(n)$, which can be written as

$$\sigma(n) = \sum_{d|n} d. \tag{2.5}$$

Proposition 2.4. *Let $\sigma(n)$ be the sum of divisors function defined by (2.5). Then,*

$$D(s,\sigma) = \sum_{n\geq 1} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1),$$

which converges when $\sigma > 3$.

Proof. We begin by transforming our sum into its Euler product form to get

$$\sum_{n\geq 1} \frac{\sigma(n)}{n^s} = \prod_p \left(\sum_{k\geq 1} \frac{\sigma(p^k)}{p^{ks}} \right).$$

We can further transform our sum by using the definition of $\sigma(n)$ to form

$$\prod_p \left(\sum_{k\geq 1} \frac{\sum_{n=0}^{k} p^n}{p^{ks}} \right).$$

Multiplying by $\frac{p-1}{p-1}$, we now have

$$\prod_p \left(\frac{1}{p-1} \sum_{k\geq 1} \frac{p^{k+1}-1}{p^{ks}} \right).$$

Using geometric series we can then get

$$\prod_p \left(\frac{p}{(p-1)(1-p^{1-s})} - \frac{1}{(p-1)(1-p^{-s})} \right).$$

After combining the two terms we are left with

$$\prod_p \left(\frac{1}{(1-p^{1-s})(1-p^{-s})} \right),$$

which by definition of the Riemann zeta-function is

$$\zeta(s)\zeta(s-1).$$

Thus,

$$\sum_{n\geq 1} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1).$$

While not a strict upper bound, we can observe that $|\sigma(n)| < n^2$ for all n. Therefore, by Theorem 2.1, we have that the series converges when $\omega > 3$. □

2.3.4 The Sum of k-th Powers of Divisors Function

We can extend the previous proposition to a more generalized case by looking at the sum of the k-th powers of the divisors $\sigma_k(n)$, which is given by the formula

$$\sigma_k(n) = \sum_{d|n} d^k. \tag{2.6}$$

Proposition 2.5. *Let $\sigma_k(n)$ be defined by (2.6). Then,*

$$D(s, \sigma_k) = \sum_{n\geq 1} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k),$$

which converges when $\omega > k+2$.

Proof. By the definition of σ_k, we have

$$\sum_{n\geq 1} \frac{\sigma_k(n)}{n^s} = \sum_{n\geq 1} \left(\frac{1}{n^s} \sum_{d|n} d^k \right).$$

Utilizing the idea from equation 2.4, we get

$$\left(\sum_{n\geq 1} \frac{1}{n^s} \right) \left(\sum_{n\geq 1} \frac{n^k}{n^s} \right).$$

This by definition is

$$\zeta(s)\zeta(s-k)$$

Therefore, we have that

$$\sum_{n\geq 1} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k).$$

Again, this is not a strict upper bound, but we can observe that $|\sigma_k(n)| < n^{k+1}$ for all n. Therefore by Theorem 2.1, we know that the series converges when $\omega > k+2$. □

We can note that $\sigma(n)$ is a special case of this final proposition when $k = 1$. We will return to these propositions later on when we discuss modular forms associated to L-functions.

This covers several examples of Dirichlet series that we wish to explore in this thesis. Primarily, we are interested in what other properties these series have. These properties will be explained in depth in the next chapter.

2.4 Dirichlet Series Associated to Elliptic Curves

Before we move on to the net chapter, there is one more example of Dirichlet series that we wish to consider, the Dirichlet series which comes from the p-defects of a fixed elliptic curve. Recall, from Chapter 1, that an elliptic curve is a cubic curve typically of the form

$$E: \quad y^2 = x^3 + Ax + B.$$

Also, recall the p-defect, a_p, from Section 1.3. As the values of a_p exist for only when p is a prime, we want to be able to define the remaining a_n terms when n is not prime. We can create a relationship between the values of the a_n's to the p-defects as follows:

(i) for $n = 1$ let $a_1 = 1$,

(ii) if p is prime let a_p be the p-defect,

(iii) if p is prime, $k \geq 2$, and p is a good prime let $a_{p^k} = a_{p^{k-1}} a_p - p a_{p^{k-2}}$, and

(iv) if m and n are relatively prime, then set $a_{mn} = a_m a_n$.

This gives us the Dirichlet series associated to the given elliptic curve:

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Example 2.5. *Consider the curve*

$$E: \quad y^2 = x^3 + 11x + 17.$$

The discriminant of E was previously calculated in Example 1.2 and is

$$\Delta_E = -2^4 \cdot 13127.$$

Therefore, we know that 2 *and* 13127 *are both bad primes.*

In Example 1.3, we showed that

$$a_2 = 2 - 2 = 0.$$

Then based on (iv) in our rules, we then know that if m is odd, then

$$a_{2m} = a_2 a_m = (0) a_m.$$

Also recall from Example 1.3, the 5*-defect for E is* $a_5 = 6 - 4 = 2$.

The remaining calculations will not be shown here to achieve the remaining primes. Once those values are known, we can then use (iii) and (iv) to calculate the remaining values for the a_n*'s. Please refer to Table 2.1 to see the values for the first 100 values of n. Note that the even numbers are not listed because their corresponding values for* a_n *all equal* 0*, as previously explained.*

n	1	3	5	7	9	11	13	15	17	19
a_n	1	3	2	2	6	0	2	6	8	-8
n	21	23	25	27	29	31	33	35	37	39
a_n	6	8	-1	9	4	2	0	4	4	6
n	41	43	45	47	49	51	53	55	57	59
a_n	-10	4	12	2	-3	24	-6	0	-24	-4
n	61	63	65	67	69	71	73	75	77	79
a_n	-15	12	4	2	24	1	4	-3	0	4
n	81	83	85	87	89	91	93	95	97	99
a_n	9	0	16	12	3	4	6	-16	18	0

Table 2.1 Table of a_n values for E

In other words, our Dirichlet series looks like

$$D(s, E) = \frac{1}{1^s} + \frac{3}{3^s} + \frac{2}{5^s} + \frac{2}{7^s} + \frac{6}{9^s} + \frac{2}{13^s} + \frac{6}{15^s} + \cdots$$

This concludes the type of Dirichlet series that we will be focusing on. The examples presented here in Sections 2.2, 2.3, and 2.4 are all considered to be L-functions which is what we will discuss in the next chapter.

Chapter 3

L-Functions

While Dirichlet series are interesting and provide a neat way to embed sequence of numbers, we instead want to be able to focus on a specific kind of Dirichlet series, an L-function. L-functions are a large and growing field of number theory.

3.1 The Basics of L-Functions

Let $X = \{a_n\}$ be some arithmetic sequence. Then, we can define the L-function as follows:

Definition 3.1. *An* L***-function of an arithmetic sequence*** X *is a complex function of* $s \in \mathbb{C}$ *given by*

$$L(s, X) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

which satisfies the following properties:

(i) $L(s, X)$ *has meromorphic continuation to* $\mathbb{C}$*;*

(ii) $L(s, X)$ *has a functional equation, i.e.* $L(s, X) = \sigma(s, X)L(k - s, \hat{X})$ *for some function* σ*, some* $k \in \mathbb{R}$ *and some related object* $\hat{X}$*;*

(iii) $L(s, X)$ *has an Euler product expansion, i.e.* $L(s, X) = \prod_p L_p(s, X)$ *for some local factors* $L_p(s, X)$*.*

We can see $L(s, X)$ is given by a Dirichlet series. In the future, when we wish to talk about the series for a fixed s we will use "Dirichlet series," but when we wish to discuss the complex function, we will use "L-function."

The properties given in the definition are the properties that we will check for when showing that a Dirichlet series is an L-function.

3.2 An Example of L-Functions: The Riemann Zeta-Function

The Riemann zeta-function has helped motivate the study of L-functions. It will demonstrated that the Riemann zeta-function has meromorphic continuation to all of $\mathbb{C}$ and that it possesses a functional equation in Chapter 5. For now we recall example 2.3, which tells us that

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

This gives us the Euler product expansion of the Riemann zeta-function.

The Riemann Zeta-Function is just one example of an L-function. The rest of this thesis will be dedicated to showing how our Dirichlet series that embed our arithmetic functions (featured in Section 2.3) are also L-functions. The proofs for these Dirichlet series will be motivated by the proofs for the Riemann zeta-function.

Chapter 4

Modular Forms

Another important mathematical object in studying number theory is *modular forms*. Modular forms provide us with the connection that we will need to be able to be able to show that our Dirichlet series have meromorphic continuation to $\mathbb{C}$ as well as helping provide us with the proof of the functional equation. This chapter will focus on defining modular forms so that we can then use them in later chapters.

4.1 The Basics of Modular forms

We note that 2×2 matrices act on $z \in \hat{\mathcal{H}}$ (where $\mathcal{H}$ denotes the upper-half complex plane and $\hat{\mathcal{H}}$ denotes $\mathcal{H} \cup \{\infty\}$) by linear fractional transformations, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

The full modular group is $\mathrm{SL}_2(\mathbb{Z})$. This group is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The types of congruence subgroups that usually considered are:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\mathrm{N}} \right\}, \quad \text{and}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{\mathrm{N}} \right\}.$$

We use Γ to denote an arbitrary congruence subgroup of $SL_2(\mathbb{Z})$. For each one of the groups $\Gamma_0(N)$ and $\Gamma_1(N)$ there are *cusps*.

Definition 4.1. *The* ***cusps of*** Γ *are the equivalence classes of points of* $\mathbb{Q} \cup \{i\infty\}$ *under the natural extension of the action of* Γ *on* $\mathcal{H}$*. Locally all of the cusps "look like" the cusp at* ∞*. For any* Γ *there are a finite number of cusps.*

Now that we have all the basic definitions, we can define a modular form.

Definition 4.2. *A* ***holomorphic modular form of (integral) weight*** $k \geq 0$ ***for*** Γ *is a function* $f : \mathcal{H} \to \mathbb{C}$ *satisfying:*

(i) [regularity] f *is holomorphic on* $\mathcal{H}$*;*

(ii) [modularity] for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *we have the modular transformation law*

$$f(\gamma z) = (cz + d)^k f(z);$$

(iii) [growth condition] f *extends holomorphically to every cusp of* Γ*.*

The regularity property gives us that the function is analytic and therefore has no poles.

The modularity property provides us with some interesting situations when we consider $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Recall, we said that S and T generate $\mathrm{SL}_2(\mathbb{Z})$. These two matrices give us a couple of interesting facts that must be true for modular forms acting over $\mathrm{SL}_2(\mathbb{Z})$. First,

$$f(z+1) = f(Tz) = (1)^k f(z) = f(z).$$

Second,

$$f\left(\frac{-1}{z}\right) = f(Sz) = (z)^k f(z).$$

The growth condition of this definition is best explained by looking at the cusp at ∞. For f to be holomorphic at the cusp ∞, we require that the Laurent expansion in a variable q has coefficients such that $a_n = 0$ for all $n < 0$, in other words

$$f(z) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

We refer to this expansion as the *q-series* for a modular form. Furthermore, there exists a similar expansion at any cusp of Γ. If we have that $a_0 = 0$ for a given q-series, then the modular form with that q-series is called a *cusp form*.

4.1.1 Vector Spaces

The set of modular forms of a fixed weight, k, for a specific congruence subgroup, Γ, is denoted, $M_k(\Gamma)$.

Proposition 4.1. *The set $M_k(\Gamma)$ is a complex vector space.*

Proof. Let $f, g \in M_k(\Gamma)$, and let $a, b \in \mathbb{C}$. We wish to show that $af + bg \in M_k(\Gamma)$. To do this we have three conditions to check: regularity, modularity, and growth conditions.

(i) We can see that the sum of holomorphic functions are still holomorphic, and that multiplying by a constant does not create any poles. Therefore, $af + bg$ is holomorphic on $\mathcal{H}$. Thus, $af + bg$ satisfies the regularity condition.

(ii) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then,

$$\begin{aligned}(af + bg)(\gamma z) &= af(\gamma z) + bg(\gamma z)\\ &= a(cz+d)^k f(z) + b(cz+d)^k g(z)\\ &= (cz+d)^k(af(z) + bg(z))\\ &= (cz+d)^k[(af+bg)(z)].\end{aligned}$$

Therefore, we have that $af + bg$ satisfies the modularity property.

(iii) $af + bg$ satisfies the growth conditions for the same reason that $af + bg$ satisfies the regularity condition

Therefore, we have that $af + bg$ is a modular form. Furthermore, by testing the modularity condition, we also confirmed that $af + bg$ is of weight k, and works with Γ. Therefore, $af + bg \in M_k(\Gamma)$, as desired. □

It is also possible to move between spaces when we have two modular forms that are associated with the same congruence subgroup. For example, let $f \in M_k(\Gamma)$ and $g \in M_\ell(\Gamma)$. Then, we get that if $\gamma \in \Gamma$, then

$$\begin{aligned}fg(\gamma z) &= f(\gamma z)g(\gamma z)\\ &= (cz+d)^k f(z)(cz+d)^\ell g(z)\\ &= (cz+d)^{k+\ell} fg(z).\end{aligned}$$

We also know that multiplying two holomorphic functions together produces another holomorphic function. Therefore, we know that fg is also a modular form for the group Γ. However, fg does not have weight k or weight l; instead, fg has weight $k + \ell$ as previously demonstrated. Note that this explanation makes use of the fact that f and g are both utilizing the same Γ.

Now we have provided an understanding for how modular forms interact with each other.

Chapter 5

Meromorphic Continuation and Functional Equations

We know from the previous chapter that given a cusp form, f, we get a function which looks like $\sum_{n=1}^{\infty} c_n q^n$. From Shimura (1994), a cusp form can be related to an L-function analytically by an integral transform denoted $\{\mathcal{A}f\}$[1] which is defined

$$\{\mathcal{A}f\}(s) = \phi(s) = \int_0^\infty f(ix)x^{s-1}\,dx.$$

The main goal of this chapter is to utilize this relationship to show that our Dirichlet series have meromorphic continuation to all of $\mathbb{C}$ and a functional equation.

5.1 L-Functions Associated to Cusp Forms

According to Apostol (1990), Hecke associated to every cusp form

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

an L-function formed from the Fourier coefficients of the q-series. The L-function is given by

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

[1]For those familiar with the Mellin Transform, $\{\mathcal{A}f\}$ is similar to the Mellin transform but uses $f(ix)$ rather than $f(x)$. For more information, see Shimura (1994).

which would converge absolutely for $\mathrm{Re}(s) > \frac{k}{2} + 1$, where k is the weight of f.

This explanation is given more formally in the following theorem.

Theorem 5.1. *Let f be a cusp form, then by applying our transformation $\mathcal{A}$ we get*

$$\{\mathcal{A}f\}(s) = \int_0^\infty f(iy)y^{s-1}\,dy = (2\pi)^{-s}\Gamma(s)D(s,f).$$

This initial statement allows one to prove both of the remaining concepts (meromorphic continuation and a functional equation) that we need to show that our Dirichlet series are L-functions.

We will return to this theorem in section 5.3 after demonstrating most of the motivational analogous results for $\zeta(s)$. Before proving this theorem, we will demonstrate an analogous result featuring the Riemann Zeta-function.

5.2 A Specific Example: the Riemann Zeta-Function

Recall that the Riemann Zeta-function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

From Garrett (2011), we learned that in Riemann's proof to for the functional equation of $\zeta(s)$, Riemann utilizes two functions that are well studied. The first of these is the Jacobi theta function which is given by

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{-\pi i n^2 z}.$$

The second well known function is the Gamma function which is defined by

$$\Gamma(s) = \int_0^\infty e^{-y}y^{s-1}\,dy.$$

The Gamma function provides us with an interesting property that is given in the following lemma.

Lemma 5.1. *Let ℓ be some constant number. Then,*

$$\Gamma(s)\frac{1}{\ell^s} = \int_0^\infty e^{-\ell y}y^{s-1}\,dy.$$

Proof. From our defining equation for $\Gamma(s)$, if we use a change of variables and replace y with ℓy for some constant ℓ, we would get

$$\begin{aligned}\Gamma(s) &= \int_0^\infty e^{-\ell y}(\ell y)^{s-1}\,d\ell y \\ &= \int_0^\infty e^{-\ell y}\ell^s y^{s-1}\,dy \\ &= \ell^s \int_0^\infty e^{-\ell y} y^{s-1}\,dy.\end{aligned}$$

We can rewrite this to get that

$$\Gamma(s)\frac{1}{\ell^s} = \int_0^\infty e^{-\ell y} y^{s-1}\,dy.$$

□

This lemma allows us to create the direct correspondence between one of the terms that we see in a Dirichlet series and an integral. We will use this lemma to help create an integral representation of $\zeta(s)$, and later to create an integral representation of our other Dirichlet series.

Theorem 5.2. *For* $\mathrm{Re}(s) > 1$,

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \int_0^\infty \frac{\theta(iy)-1}{2} y^{(s/2)-1}\,dy.$$

The result of this theorem provides us with an integral representation of ζ with a correction term, $\pi^{-s/2}\Gamma(\frac{s}{2})$.

Proof. For ease and to provide motivation, this proof will use t in place of $s/2$ until it is necessary. We begin with the right hand side of our equation, and rewrite it so that we expand the Jacobi theta function inside of it to get

$$\int_0^\infty \frac{\theta(iy)-1}{2} y^{t-1}\,dy = \int_0^\infty \frac{1}{2}\left(-1+\sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}\right) y^{t-1}\,dy.$$

The Jacobi theta function is a summation based on n^2 rather than n. Therefore, both the positive and negative value of a natural number contribute the same amount. The only value that is not repeated twice occurs when $n = 0$. In this case we note that $e^{-\pi 0^2 y} = 1$, which cancels with the -1 in front of the sum. Therefore, we now rewrite our integral as follows

$$\int_0^\infty \frac{1}{2}\left(-1+\sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}\right) y^{t-1}\,dy = \int_0^\infty \frac{1}{2}\left(2\sum_{n\in\mathbb{N}} e^{-\pi n^2 y}\right) y^{t-1}\,dy$$

$$= \int_0^\infty \left(\sum_{n\in\mathbb{N}} e^{-\pi n^2 y} \right) y^{t-1}\, dy.$$

Since the integral of the sum is equal to the sum of integrals, we get that

$$\int_0^\infty \left(\sum_{n\in\mathbb{N}} e^{-\pi n^2 y} \right) y^{t-1}\, dy = \sum_{n\in\mathbb{N}} \int_0^\infty e^{-\pi n^2 y} y^{t-1}\, dy.$$

In Lemma 5.1, we learned a property of $\Gamma(s)$ which we will now exploit by noting that $\ell = \pi n^2$ and rewriting our expression as

$$\sum_{n\in\mathbb{N}} \int_0^\infty e^{-\pi n^2 y} y^{t-1}\, dy = \sum_{n\in\mathbb{N}} \Gamma(t)\pi^{-t}\frac{1}{n^{2t}} = \Gamma(t)\pi^{-t}\sum_{n=1}^{\infty}\frac{1}{n^{2t}}.$$

Using a substitution of variables by replacing t with $s/2$, we can then write that

$$\Gamma(\frac{s}{2})\pi^{-s/2}\sum_{n=1}^{\infty}\frac{1}{n^s} = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s),$$

which is the left hand side of our equation. Therefore, we reach our desired result

$$\int_0^\infty \frac{\theta(iy)-1}{2} y^{(s/2)-1}\, dy = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s).$$

□

5.2.1 Meromorphic Continuation

One can now use Theorem 5.2 to show that $\zeta(s)$ has meromorphic continuation to all of $\mathbb{C}$ and that $\zeta(s)$ has a functional equation. In this section we will show that ζ has meromorphic continuation to all of $\mathbb{C}$.

Theorem 5.3. *The Riemann zeta-function, ζ is a meromorphic function with a simple pole at 1.*

Proof. Recall the result from Proposition 5.2,

$$\int_0^\infty \frac{\theta(iy)-1}{2} y^{(s/2)-1}\, dy = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s).$$

We can then rewrite our equation to recall that

$$\int_0^\infty \frac{\theta(iy)-1}{2} y^{(s/2)-1}\, dy = \int_0^\infty \sum_{n=1}^{\infty} e^{-\pi n^2 y} y^{(s/2)-1}\, dy.$$

We then see that the series given on the right hand side decays rapidly as $y \to \infty$.

We can split the integral such that we then have

$$\int_0^1 \frac{\theta(iy)-1}{2} y^{(s/2)-1}\,dy + \int_1^\infty \sum_{n=1}^\infty e^{-\pi n^2 y} y^{(s/2)-1}\,dy.$$

The second of these two integrals clearly converges for all $s \in \mathbb{C}$, and is therefore an entire function, which we will henceforth refer to as $F(s)$.

For the first integral, we wish to find a way to make it more accessible. To do so, we utilize the Jacobi identity

$$\theta(iy) = y^{-1/2}\theta(-1/iy).$$

First, we will rewrite our integral so that it goes from 1 to ∞

$$\int_0^1 \frac{\theta(iy)-1}{2} y^{(s/2)-1}\,dy = \int_1^\infty \frac{\theta(-1/iy)-1}{2} y^{-(s/2)-1}\,dy$$

since we can relate $x \in [1,\infty)$ to an element in $(0,1]$ through $x \mapsto \frac{1}{x}$. We now use the Jacobi identity to get

$$\int_1^\infty \frac{y^{1/2}\theta(iy)-1}{2} y^{-(s/2)-1}\,dy.$$

To simplify things, we add in $y^{1/2}/2 - y^{1/2}/2$, to give us

$$\int_1^\infty \left(y^{1/2}\frac{\theta(iy)-1}{2} + \frac{y^{1/2}}{2} - \frac{1}{2} \right) y^{-(s/2)-1}\,dy.$$

We can separate the integral into three since the integral of the sum is the sum of the integrals,

$$\int_1^\infty \left(y^{(-(1+s)/2)} \frac{\theta(iy)-1}{2} \right) dy + \int_1^\infty \frac{y^{-(1+s)/2}}{2}\,dy - \int_1^\infty \frac{y^{-1-(s/2)}}{2}\,dy.$$

Computing the last two integrals we get

$$\int_1^\infty \left(y^{(3-s/2)} \frac{\theta(iy)-1}{2} \right) dy + \frac{1}{s-1} - \frac{1}{s}$$

The integral in this sum clearly converges and produces an entire function for similar reasons to $\int_1^\infty \sum_{n=1}^\infty e^{-\pi n^2 y} y^{(s/2)-1}\,dy$. We will denote this entire function as $G(s)$.

Thus we have that

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = F(s) + G(s) + \frac{1}{s-1} - \frac{1}{s}.$$

The right hand side of this equation is a meromorphic function with a simple pole at 1 and 0. Since $\pi^{-s/2}\Gamma(s/2)$ has no zeros, $\frac{\pi^{s/2}}{\Gamma(s/2)}$ is an entire function. Thus, we get that $\zeta(s)$ extends to a meromorphic function in the complex plane. Moreover, $\Gamma(s/2)$ has simple poles at $s = 0, -2, -4,...$ which means that $\frac{1}{\Gamma(s/2)}$ has simple zeros at those same values of s. Therefore, $\zeta(s)$ has a removable singularity at $s = 0$. In conclusion, we now have that ζ is a meromorphic continuation with a simple pole at 1. □

5.2.2 The Functional Equation

There is now only one thing left to show in order to conclude that the Riemann zeta-function is an L-function; that ζ has a functional equation.

The proof of the functional equation is highly technical. A form of this proof can be found in Milicic (2011).

Theorem 5.4. *For any $s \in \mathbb{C}$, $\zeta(s)$ is related to $\zeta(1-s)$ via the following equation:*

$$\zeta(s) = \pi^{s-1}2^s \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

This theorem now provides us with a way to calculate $\zeta(s)$ for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s) < 0$. Figure 5.1 provides a visual for where $\zeta(s)$ is properly defined. The non-shaded area in the figure is referred to as the critical strip.

We have now provided all the evidence needed to show that the Riemann zeta-function is an L-function. We will further use Theorem 5.4 in Section 5.5 to provide ourselves with a functional equation for our other Dirichlet series from Section 2.3.

5.3 The Generalized Integral Representation

We now wish to generalize the concepts presented in the previous section. To begin, we return to theorem 5.1 and prove the integral representation of $D(s, f)$. As in Section 5.2, this theorem will subsequently be used to show the meromorphic continuation to all of $\mathbb{C}$.

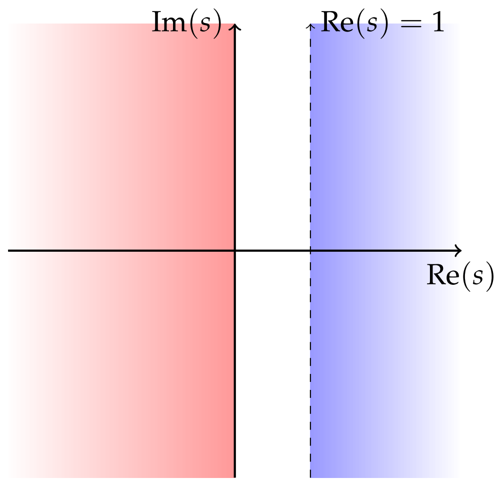

Figure 5.1 The domain of $\zeta(s)$.

Proof of Theorem 5.1. Consider a modular form given by

$$f(y) = \sum_{n=1}^{\infty} c_n e^{-2\pi i n y}.$$

Then, by applying our integral transform $\mathcal{A}$ to $f(y)$, we get the left hand side of our desired equation. We then have

$$\int_0^\infty f(iy) y^{s-1}\, dy = \int_0^\infty \sum_{n=1}^{\infty} c_n e^{-2\pi n y} y^{s-1}\, dy.$$

Since the integral of the sum is equal to the sum of then integrals, we get that

$$\int_0^\infty \sum_{n=1}^{\infty} c_n e^{-2\pi n y} y^{s-1}\, dy = \sum_{n=1}^{\infty} \int_0^\infty c_n e^{-2\pi n y} y^{s-1}\, dy.$$

Since all the c_n's are constant we can pull them out of the integral, giving us

$$\sum_{n=1}^{\infty} c_n \int_0^\infty e^{-2\pi n y} y^{s-1}\, dy.$$

Then, we know from Lemma 5.1 that

$$\sum_{n=1}^{\infty} c_n \int_0^\infty e^{-2\pi n y} y^{s-1}\, dy = \sum_{n=1}^{\infty} c_n (2\pi)^{-s} \frac{1}{n^s} \Gamma(s) = (2\pi)^{-s} \Gamma(s) \sum_{n=1}^{\infty} \frac{c_n}{n^s}.$$

Therefore, we get our desired result

$$(2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{c_n}{n^s} = \int_0^{\infty} f(iy)y^{s-1}\,dy.$$

□

It should be noted, that for a given L-function there might not be a modular form that is easily accessible. This is demonstrated with the Riemann Zeta-function where instead of using $\zeta(s)$, we needed to use $\zeta(2s)$. For more notes on this, please see Section 6.1.

5.4 Meromorphic Continuation and Arithmetic Functions

We now wish to use the result for Theorem 5.3 to show whether or not our Dirichlet series from Section 2.3 have meromorphic continuation to all of $\mathbb{C}$.

5.4.1 The Möbius Function

Theorem 5.5. *Let μ be the Möbius function. Then, $D(s,\mu)$ is not a meromorphic function, as it possesses an infinite number of poles.*

Proof. Recall Proposition 2.1, which tells us that

$$D(s,\mu) = \frac{1}{\zeta(s)}.$$

The trivial zeros of the Riemann zeta function occur when s is a negative even integer, and the non-trivial zeros occur in the critical strip, where $0 < \mathrm{Re}(s) < 1$. While the non-trivial zeros are a part of the Riemann Hypothesis, the number of trivial zeros is infinite since there is a bijection between $\mathbb{N}$ and the negative even integers. We know that $D(s,\mu)$ must have a pole everywhere that $\zeta(s)$ has a zero. Therefore, $D(s,\mu)$ has an infinite number of poles, and thus is not a meromorphic function. □

5.4.2 Euler's Totient Function

Theorem 5.6. *Let ϕ be Euler's Totient function. Then, $D(s,\phi)$ is not a meromorphic function, as it possesses an infinite number of poles.*

Proof. Recall Proposition 2.3, which gives us that

$$D(s, \phi) = \frac{\zeta(s-1)}{\zeta(s)}.$$

Recall that the trivial zeros of the Riemann zeta function occur when s is a negative even integer, and the non-trivial zeros occur in the critical strip, where $0 < \mathrm{Re}(s) < 1$. Therefore, $D(s, \phi)$ has a pole everywhere $\zeta(s)$ is zero, unless $\zeta(s-1)$ is zero for the same value of s. This is never the case since $\zeta(s-1)$ is $\zeta(s)$ shifted to the right by 1, and we need a shift of an even number as even numbers are distanced from each other by an even number. Therefore, $D(s, \phi)$ has an infinite number of poles. In conclusion, we get that $D(s, \phi)$ is not a meromorphic function. □

5.4.3 The Sum of k-th Powers of Divisors

Since the sum of divisors function is a special case of the sum of k-th powers of divisors, we have combined them to one section.

Theorem 5.7. *σ_k be sum of k-th powers of divisors function. Then, if k is even or $k = 1$, $D(s, \sigma_k)$ is a meromorphic function with simple poles at 1 and $k+1$. If $k > 1$ is odd, then $D(s, \sigma_k)$ is a meromorphic function with a simple pole at $k+1$.*

Proof. Recall Proposition 2.5, which gives us that

$$D(s, \sigma_k) = \zeta(s)\zeta(s-k).$$

We know from Theorem 5.3, that $\zeta(s)$ is a meromorphic function with a simple pole at 1. Therefore, $\zeta(s-k)$ has a simple pole at $k+1$. The trivial zeros of the Riemann zeta function occur when $s = -2, -4, -6, \ldots$, and the non-trivial zeros occur in the critical strip, where $0 < \mathrm{Re}(s) < 1$. Therefore, if $k > 1$ is odd, we can get that $1 - k$ is a negative even integer. Therefore, when this is the case, $D(s, \sigma_k)$ has a removable singularity at $s = 1$. Otherwise, $s = 1$ is still a pole. Therefore, if k is even or $k = 1$, $D(s, \sigma_k)$ is a meromorphic function with simple poles at 1 and $k+1$. If $k > 1$ is odd, then $D(s, \sigma_k)$ is a meromorphic function with a simple pole at $k+1$. □

This now concludes our demonstration about whether or not our Dirichlet series from Section 2.3 have meromorphic continuation.

5.5 Functional Equations Related to Arithmetic Functions

In this section, we will show the functional equations for our Dirichlet series from Section 2.3. We will make use of the relationship between our Dirichlet series and the Riemann zeta-function, and exploit Theorem 5.4.

5.5.1 The Möbius Function

While $D(s,\mu)$ is not meromorphic, it does possess a functional equation.

Theorem 5.8. *Let μ denote the Möbius function. Then,*

$$D(s,\mu) = \pi^{1-s}2^{-s}\csc\left(\frac{\pi s}{2}\right)\frac{D(1-s,\mu)}{\Gamma(1-s)}.$$

Proof. Recall Proposition 2.1. We then see that

$$D(s,\mu) = \frac{1}{\zeta(s)}$$

can be rewritten according to Theorem 5.4 to give us

$$\frac{1}{\pi^{s-1}2^{s}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)} = \pi^{1-s}2^{-s}\csc\left(\frac{\pi s}{2}\right)\frac{1}{\Gamma(1-s)\zeta(1-s)}.$$

We can rewrite this once more using the first proposition to get

$$\pi^{1-s}2^{-s}\csc\left(\frac{\pi s}{2}\right)\frac{D(1-s,\mu)}{\Gamma(1-s)}.$$

Therefore, we can say that

$$D(s,\mu) = \pi^{1-s}2^{-s}\csc\left(\frac{\pi s}{2}\right)\frac{D(1-s,\mu)}{\Gamma(1-s)}.$$

□

As $D(s,\mu)$ converges for $\mathrm{Re}(s) > 1$, we then have that $D(s,\mu)$ is defined for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s) < 0$. Figure 5.2 provides a visual for this domain.

Recall in Section 2.1, we can stated that any Dirichlet series given by an arithmetic function has an Euler product expansion by how we have defined an Euler product. However, since $D(s,\mu)$ does not have meromorphic continuation, $D(s,\mu)$ is *not* an L-function.

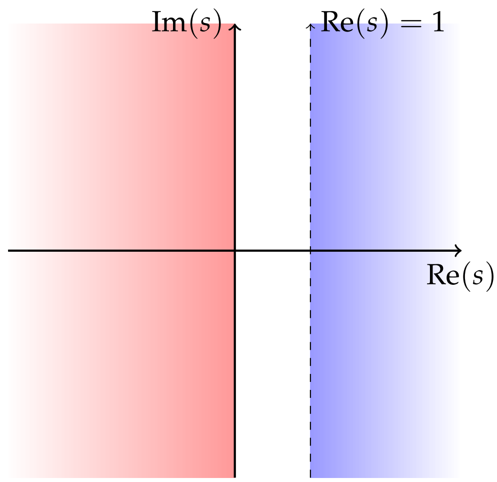

Figure 5.2 The domain of $D(s, \mu)$.

5.5.2 Euler's Totient Function

Theorem 5.9. *Let ϕ denote Euler's Totient function. Then,*

$$D(s, \phi) = \frac{\sin\left(\frac{\pi(s-1)}{2}\right)\Gamma(2-s)D(1-s,\phi)}{2\pi\sin(\pi s/2)\Gamma(1-s)}.$$

Proof. Recall Proposition 2.3. We then see that

$$D(s, \phi) = \frac{\zeta(s-1)}{\zeta(s)}$$

can be rewritten according to Theorem 5.4 to give us

$$\frac{\pi^{(s-1)-1}2^{s-1}\sin\left(\frac{\pi(s-1)}{2}\right)\Gamma(1-(s-1))\zeta(1-(s-1))}{\pi^{s-1}2^{s}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)}$$

which is equal to

$$\frac{\sin\left(\frac{\pi(s-1)}{2}\right)\Gamma(2-s)\zeta(2-s)}{2\pi\sin(\pi s/2)\Gamma(1-s)\zeta(1-s)}.$$

We can rewrite this once more using the first proposition to get

$$\frac{\sin\left(\frac{\pi(s-1)}{2}\right)\Gamma(2-s)D(1-s,\phi)}{2\pi\sin(\pi s/2)\Gamma(1-s)}.$$

Therefore, we can say that

$$D(s,\phi) = \frac{\sin\left(\frac{\pi(s-1)}{2}\right)\Gamma(2-s)D(1-s,\phi)}{2\pi\sin(\pi s/2)\Gamma(1-s)}.$$

□

As $D(s,\phi)$ converges for Re(s)21, we then have that $D(s,\phi)$ is defined for Re$(s) > 2$ and Re$(s) < -1$. Figure 5.3 provides a visual for this domain.

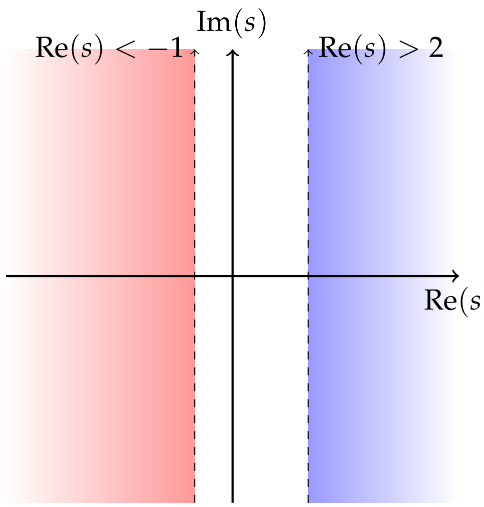

Figure 5.3 The domain of $D(s,\phi)$.

While we have now shown that $D(s,\phi)$ has a functional equation, $D(s,\phi)$ does not have meromorphic continuation. Therefore, $D(s,\phi)$ is not an L-function.

5.5.3 The Sum of k-th Powers of Divisors

Since the sum of divisors function is a special case of the sum of k-th powers of divisors, we have combined them to one section.

Theorem 5.10. *Let σ_k denote the sum of k-th powers of divisors function. Then,*

$$D(s,\sigma_k) = \pi^{2s-(2+k)}2^{2s-k}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\sin\left(\frac{\pi(s-k)}{2}\right)\Gamma(k+1-s)D(1-s,\sigma_k).$$

Proof. Recall Proposition 2.5. We then see that

$$D(s,\sigma_k) = \zeta(s)\zeta(s-k)$$

can be rewritten according to Theorem 5.4 to give us

$$\pi^{2s-(2+k)}2^{2s-k}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\sin\left(\frac{\pi(s-k)}{2}\right)\Gamma(k+1-s)\zeta(k+1-s).$$

We can rewrite this using the first proposition to get

$$\pi^{2s-(2+k)}2^{2s-k}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\sin\left(\frac{\pi(s-k)}{2}\right)\Gamma(k+1-s)D(1-s,\sigma_k).$$

Therefore, we can say that

$$D(s,\sigma_k) = \pi^{2s-(2+k)}2^{2s-k}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\sin\left(\frac{\pi(s-k)}{2}\right)\Gamma(k+1-s)D(1-s,\sigma_k).$$

□

As $D(s,\sigma_k)$ converges for $\mathrm{Re}(s) > 1$, we then have that $D(s,\sigma_k)$ is defined for $\mathrm{Re}(s) > k$ and $\mathrm{Re}(s) < 1-k$. Figure 5.4 provides a visual for this domain.

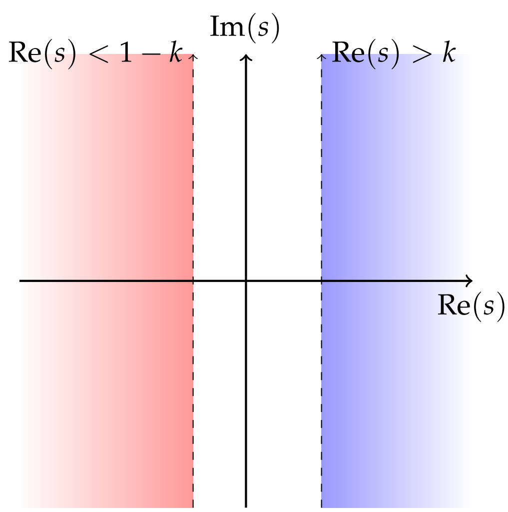

Figure 5.4 The domain of $D(s,\sigma_k)$.

Theorem 5.11. *Let σ_k be the sum of k-th powers of divisors function. Then, $D(s, \sigma_k)$ is an L-function, which we denote $L(s, \sigma_k)$.*

This theorem follows immediately from Theorems 5.7 and 5.10 and the fact that the sum of k-th powers of divisors is an arithmetic function.

Chapter 6

Applications and Further Research

So far in this thesis, we have been able to show the for the sequences $\{a_n\}$, where $a_n = 1$ and $a_n = \sigma_k(n)$, we get that that their respective Dirichlet series yield L-functions. This chapter will focus on the potential extensions that one can take after reading this research.

6.1 Modular Forms Associated to a Given L-function

In the beginning of Chapter 5, we mentioned that L-functions are related to modular forms. For example, the Jacobi Theta Function in Section 5.2 is a weight $1/2$ modular form. However, the functions found for the other Dirichlet series are not well studied. A potential extension of this project is to figure out if they are modular forms, and if so what is their weight and what congruence subgroup are they over.

6.2 Wiles's Theorem

A way in which the information presented in this thesis has been used previously is in the following theorem.

Theorem 6.1 (Wiles's Theorem)**.** *Every elliptic curve is modular.*

This theorem gives us that every elliptic curve has a modular form. The modular form given to an elliptic curve has a q-series whose coefficients follow the same rules as those present in Section 2.4. Then, we would get that the Dirichlet series is related to the modular form through Theorem 5.1. Further study would then show that the Dirichlet series associated to the elliptic curve is an L-function, whose Euler product expansion would look like

$$L(s,E) = \left(\prod_{p\in S} \frac{1}{1-a_p p^{-s}}\right)\left(\prod_{p\notin S} \frac{1}{1-a_p p^{-s}+p^{1-2s}}\right),$$

where S denotes the set of bad primes.

6.3 The Birch Swinnerton-Dyer Conjecture

The L-function associated to an elliptic curve is a highly studied object, because it is related to one the Clay Institute's Millennium Prize Problems, the Birch Swinnerton-Dyer Conjecture. Some other things that are of relevance to this problem are that the points in $E(\mathbb{Q})$ form a group under the operation we described in Section 1.1, and the Taylor expansion of $L(s,E)$ at $s=1$. The algebraic rank of an elliptic curve , E, refers to the rank of the group $E(\mathbb{Q})$. The analytic rank of E is the value of the exponent in the first nonzero term in the Taylor expansion at $s=1$.

Conjecture 6.1 (The weak Swinnerton-Dyer Conjecture)**.** *For any elliptic curve E defined over $\mathbb{Q}$, the algebraic rank of E and the analytic rank of E are equal.*

This conjecture has proved to hold for ranks 0 and 1, and other computational data suggests that there does not exist elliptic curves which have any other rank. For more information, the reader should consider Ash and Gross (2012) as great resource to begin exploration.

6.4 The Langlands Program

The Langlands Program is a web of conjectures that use other conjectures currently present in the program to be able to create further conjectures. The Birch Swinnerton-Dyer Conjecture is just one of many conjectures that they assume to be true in order to keep developing the field of L-functions. The Langlands Program is a resource for find other problems one might find interesting in this area of mathematics.

Appendix A

Code for Computations

All code in this appendix is written for SAGE.

A.1 Counting Points

```
#Function: Numpts
#Goal: This function should give the number of points for the
# curve y^2 = x^3+Ax+B for a finite field with prime size.
#Input: p = the size of the finite field (prime numbers only)
# A = the A term in our curve
# B = the B term in our curve
def Numpts(p,A,B):
if (-16*(4*A^3+27*B^2))%p == 0:
     pnts = 0;
    else:
     pnts = 1;
    for i in range(0,p):
        for j in range(0,p):
            if (i^2%p) == ((j^3+A*j+B)%p):
                pnts+=1;
    return(pnts);
```

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