PRE-PUBLICACIONES DEL SEMINARIO MATEMATICO "GARCIA DE GALDEANO"

2001

J.L. Arregui

N.º 34

seminario matemático
garcía de galdeano

BERNOULLI AND EULER NUMBERS, MOTZKIN PATHS AND NUMERICAL TRIANGLES

JOSÉ LUIS ARREGUI

ABSTRACT. If a function defined on the set of Motzkin paths or the subset of Dyck paths is recursive in a sense, then it can be integrated following a simple algorithm. In some particular instances we get not only Motzkin and Catalan numbers, but also the tangent and secant numbers (and then Bernoulli and Euler numbers) in a similar way. We make use of Calabi's treatment of $\zeta(2n)$, a relevant surjection from the symmetric group S_{n+1} onto the set of Motzkin paths of n steps, and Entringer's theorem on alternating permutations.

1. INTRODUCTION. NUMERICAL TRIANGLES GENERATED BY MATRICES.

Recall that the sequence of *tangent numbers* is defined as $(\tan^{(2n-1)}(0))$, with

$$\tan z = \sum_{n=1}^{\infty} \frac{\tan^{(2n-1)}(0)}{(2n-1)!} z^{2n-1} \quad (|z| < \pi/2)\,.$$

On the other hand, the sequence of *Bernoulli numbers* (B_n) is defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi)\,.$$

The first values of Bernoulli numbers are

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0, B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42},$$

and in general $B_{2n+1} = 0$ for all $n \in \mathbb{N}$, while the signs of $B(2n)$ alternate.

From the relation between the exponential, sine and cosine functions, it results that

$$\tan^{(2n-1)}(0) = |B_{2n}| \frac{4^n\,(4^n - 1)}{2n}\,.$$

Bernoulli numbers satisfy the recurrence relation

$$B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k\,.$$

2000 *Mathematics Subject Classification.* Primary 11B68, 11B75; Secondary 11M06.

Key words and phrases. Bernoulli and Euler numbers, tangent and secant numbers, triangles of numbers, Motzkin paths and numbers, Dyck paths, Catalan numbers, alternating permutations, Zeta function.

The author has been partially supported by Proyecto D.G.E.S. PB98-0146.

These numbers are also related to *Euler numbers* (E_n), given by $E_{2n+1} = 0$ and $E_{2n} = (-1)^n E_n^*$, where E_n^* are the *secant numbers* verifying

$$\sec z = \frac{1}{\cos z} = \sum_{n=0}^{\infty} \frac{E_n^*}{(2n)!} z^{2n} \quad (|z| < \pi/2).$$

Actually, if we consider (δ_n) the sequence such that

$$\frac{1}{2}(\sec z + \tan z) = \sum_{n=1}^{\infty} \delta_n z^{n-1} \quad (|z| < \pi/2)$$

it turns out that

$$\sum_{k=0}^{\infty} \frac{(-1)^{nk}}{(2k+1)^n} = \delta_n \frac{\pi^n}{2^n} \quad (n \in \mathbb{N}) \tag{1}$$

a result that goes back to Euler. For even values it follows the relation between Bernoulli numbers and the Riemann zeta function ζ,

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = |B_{2n}| \frac{(2\pi)^{2n}}{2(2n)!}.$$

Standard references for all of this are [9] and [10].

A much more recent way to relate Euler and Bernoulli numbers is their combinatorial interpretation, due to R. C. Entringer, in terms of alternating permutations. See the last section for the details.

E. Calabi, F. Beukers and J. A. C. Kolk used in [3] a remarkable change of variables to reduce the series in (1) to the volume of a polytope in $\mathbb{R}^n$, showing how it yields an elementary proof of (1). Very recently, N. D. Elkies has shown in [6] how the same idea (which seems to be commonly attributed to Calabi) gives another proof of Entringer's theorem. In this paper we follow Calabi's argument in a different way, showing that $\zeta(2n)$ can be expressed as a finite sum of addends indexed by Motzkin paths (related to Motzkin numbers) rather than natural numbers. In view of Entringer's result, two similar expressions of Bernoulli and Euler numbers in terms of Dyck paths (related to Catalan numbers) are shown.

The expressions lead to algorithms generating these numbers, concisely stated in terms of numerical triangles which are recursively defined in a very similar way to others that give Motzkin and Catalan numbers.

To this end, let us introduce the following definition:

Definition. Let $(A^{(n)})_{n\geq 1}$ be a sequence of matrices (with numerical entries), each one with n rows and $n+1$ columns. Let $t_{0,0} = 1$, $\bar{t}_1 = (t_{1,0}, t_{1,1}) = A^{(1)}$ and, for each $n > 1$,

$$\bar{t}_n = (t_{n,0}, t_{n,1}, t_{n,2}, \ldots, t_{n,n}) = \bar{t}_{n-1} A^{(n)} = A^{(1)} A^{(2)} \cdots A^{(n)}.$$

We obtain a numerical triangle

$$T \equiv (t_{n,m})_{0\leq m\leq n} \equiv \begin{array}{lllll} t_{0,0} & & & & \\ t_{1,0} & t_{1,1} & & & \\ t_{2,0} & t_{2,1} & t_{2,2} & & \\ t_{3,0} & t_{3,1} & t_{3,2} & t_{3,3} & \\ t_{4,0} & t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

and we name it the *triangle generated by* $(A^{(n)})$.

In particular, if $A = (a_{ij})_{1\le i,j}$ is an infinite matrix and $A^{(n)}$ is the submatrix formed by the first n rows and $n+1$ columns, the triangle obtained in this way is the *triangle generated by* A. For instance, $\bar{t}_3 = (t_{3,0}, t_{3,1}, t_{3,2}, t_{3,3}) = A^{(1)}A^{(2)}A^{(3)}$, that is

$$\bar{t}_3 = \begin{pmatrix} a_{1,1} & a_{1,2} \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix}.$$

Example 1. If $a_{ij} = 1$ for all $i, j \in \mathbb{N}$ then the triangle generated by $A = (a_{ij})$ is just

$$t_{n,m} = n! \qquad \text{for all } 0 \le m \le n.$$

Examples 2. Now let $a_{ij} = \begin{cases} 2 & \text{if i=j=1,} \\ 1 & \text{if } i = j-1, \\ 0 & \text{otherwise,} \end{cases}$

$$\text{and } A = (a_{ij}) = \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \multicolumn{5}{c}{\dots\dots\dots\dots} \end{pmatrix}.$$

The triangle generated by A is given by $t_{n,m} = 2^{n-m}$, that is

$$\begin{array}{llll} 1 \\ 2 & 1 \\ 4 & 2 & 1 \\ 8 & 4 & 2 & 1 \\ \multicolumn{4}{l}{\dots\dots\dots} \end{array}$$

If A begins

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \multicolumn{5}{c}{\dots\dots\dots\dots} \end{pmatrix}$$

and the rest is as before, the triangle generated also verifies $t_{n,m} = t_{n-1,m-1}$ if n, $m \ge 1$, but the first column gives the sequence of Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ...

As for

$$A = \begin{pmatrix} -1/2! & 1 & 0 & 0 & 0 & \cdots \\ -1/3! & 0 & 1 & 0 & 0 & \cdots \\ -1/4! & 0 & 0 & 1 & 0 & \cdots \\ -1/5! & 0 & 0 & 0 & 1 & \cdots \\ \multicolumn{6}{c}{\dots\dots\dots\dots\dots\dots} \end{pmatrix},$$

the mentioned recurrence for Bernoulli numbers shows that the first column of the triangle generated by A is just the sequence $(B_n/n!)$.

In general, If A is of this form then the sequence obtained (the first column in the triangle) is esentially the convolution of itself with the sequence entered (the first column in A) (see Ch. 7.5 in [9]).

Catalan and Motzkin numbers can be naturally introduced as important examples of this.

Example 3. For $a_{ij} = \begin{cases} 1 & \text{if } i \le j \le i+1, \\ 0 & \text{otherwise} \end{cases}$ we have that

$$A = (a_{ij}) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ \multicolumn{6}{c}{\dotfill} \end{pmatrix}$$

generates the *Pascal triangle*

$$\begin{array}{lllll} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ \multicolumn{5}{c}{\dotfill} \end{array}$$

where $t_{n,m} = \binom{n}{m}$ for all n and m. Each number plus the one next to its left in the triangle gives the number below.

Example 4. Take the matrix in the previous example (say the *Pascal matrix*) and put 1 in all the entries just below the diagonal, i. e.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & 1 & \cdots \\ \multicolumn{7}{c}{\dotfill} \end{pmatrix},$$

Then the triangle generated by A begins

$$\begin{array}{llllll} 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 4 & 5 & 3 & 1 \\ 9 & 12 & 9 & 4 & 1 \\ 21 & 30 & 25 & 14 & 5 & 1 \\ \multicolumn{6}{c}{\dotfill} \end{array}$$

This is the so-called *Motzkin triangle*. Each number in the triangle is the sum of the one above and its (one or two) contiguous ones.

The first column in the triangle gives the sequence of *Motzkin numbers* M_n.

Example 5. If we fill with 1 all below the diagonal in the Pascal matrix we get

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1 & \cdots \\ \multicolumn{6}{c}{\dotfill} \end{pmatrix},$$

which generates

$$\begin{array}{llllll}
1 \\
1 & 1 \\
2 & 2 & 1 \\
5 & 5 & 3 & 1 \\
14 & 14 & 9 & 4 & 1 \\
42 & 42 & 28 & 14 & 5 & 1 \\
\multicolumn{6}{l}{\dots\dots\dots\dots\dots\dots}
\end{array}$$

that is known as *Catalan triangle*. Note that (if we set $t_{n,-1} = 0$ for all n)

$$t_{n+1,m} = \sum_{k=m-1}^{n} t_{n,k} \,.$$

The first (second) column in the triangle gives the sequence of *Catalan numbers* C_n. It is well known that

$$t_{n,m} = \binom{2n-m}{n} \frac{m+1}{n+1} \quad (0 \le m \le n)\,,$$

and thus $C_n = \binom{2n}{n} \dfrac{1}{n+1} = \dfrac{(2n)!}{n!\,(n+1)!}$.

The convolution property of Catalan numbers is

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \,.$$

If A is as in example 2 and its first column is the sequence (C_n), then the first column in the triangle is (C_n) again.

Both Motzkin and (much better known) Catalan numbers have a nice variety of realizations as the number of solutions to some combinatorial problem depending on n (see [5] and [8]). In this article we will briefly explore one of them, the numbers of Motzkin and Dyck paths of n steps, relating them to Bernoulli and Euler numbers via the symmetric groups of permutations.

For instance, we will show that the matrices

$$\begin{pmatrix}
2 & 6 & 0 & 0 & 0 & \cdots \\
2 & 6 & 12 & 0 & 0 & \cdots \\
2 & 6 & 12 & 20 & 0 & \cdots \\
2 & 6 & 12 & 20 & 30 & \cdots \\
\multicolumn{6}{c}{\dots\dots\dots\dots\dots\dots}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 4 & 0 & 0 & 0 & \cdots \\
1 & 4 & 9 & 0 & 0 & \cdots \\
1 & 4 & 9 & 16 & 0 & \cdots \\
1 & 4 & 9 & 16 & 25 & \cdots \\
\multicolumn{6}{c}{\dots\dots\dots\dots\dots\dots}
\end{pmatrix}$$

generate triangles where the first columns are, respectively, the sequences of tangent and secant numbers.

A very useful source of information on a huge number of sequences of integers is Sloanne's encyclopedia [11].

We start with Motzkin paths and numbers, then showing their relation with Calabi's idea.

2. Motzkin paths

A *Motzkin path* of n-th order (or n steps) is a finite sequence

$$\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \{-1, 0, 1\}^n \text{ such that } \sum_{j=1}^{n} \lambda_j = 0 \text{ and } \sum_{j=1}^{l} \lambda_j \geq 0 \text{ if } l < n.$$

The number of Motzkin paths of n steps is just the number M_n in example 4, so this is one of the realizations of Motzkin numbers (surely the most common by now). The term *path* comes after the usual visualitazion, in an obvious 1-1 correspondence, of any such sequence with a path that joins $(0,0)$ with $(n,0)$ in n steps –in the $\mathbb{Z} \times \mathbb{Z}$ lattice–, each one by summing the vector $(1, \lambda_j)$, with the condition that no intermediate point lies below the horizontal axis.

The *empty sequence* is considered as the only Motzkin path of 0 steps, to give $M_0 = 1$.

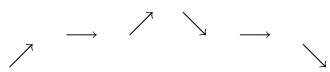
The only Motzkin path of 1 step is (0), and with 2 steps we have $(0,0)$ and $(1,-1)$.

The four Motzkin paths of 3 steps are

$(0,0,0)$ $\rightarrow\rightarrow\rightarrow$ $\nearrow\searrow\rightarrow$ $(1,-1,0)$

$(0,1,-1)$ $\rightarrow\nearrow\searrow$ $\nearrow\rightarrow\searrow$ $(1,0,-1)$

and a 6 step Motzkin path is $(1,0,1,-1,0,-1)$, visualized as



We will denote $\mathcal{M}_n$ the set of all Motzkin paths of n steps, being then M_n its cardinality. $\mathcal{M} = \cup_{n=1}^{\infty} \mathcal{M}_n$ is the set of all Motzkin paths of any positive order.

A usual way to obtain Motzkin numbers is as in example 4. Another recursive formula is

$$M_{n+1} = M_n + \sum_{k=0}^{n-1} M_k M_{n-k-1},$$

(which is the key to present them as in examples 2), and in the next section we will recover the formula that best relates them to Catalan numbers. These and further results can be seen in [1] and [2].

To fix notations, let $(\bar{\lambda}, \bar{\mu}) = (\lambda_1, \ldots, \lambda_{n_1}, \mu_1, \ldots, \mu_{n_2})$ if $\bar{\lambda} = (\lambda_1, \ldots, \lambda_{n_1})$ and $\bar{\mu} = (\mu_1, \ldots, \mu_{n_2})$. Here $\bar{\mu}$ could be the empty sequence, and $(\bar{\lambda}, \bar{\mu}) = \bar{\lambda}$.

The following proposition is then pictorially evident; it says that by removing flat steps, or by "flattening cusps" in a Motzkin path, we get another Motzkin path.

Proposition 2.1. *For any* $n \in \mathbb{N}$
(i) $(\bar{\lambda}, 0, \bar{\mu}) \in \mathcal{M}_{n+1}$ *if and only if* $(\bar{\lambda}, \bar{\mu}) \in \mathcal{M}_n$.
(ii) $(\bar{\lambda}, 1, -1, \bar{\mu}) \in \mathcal{M}_{n+1}$ *if and only if* $(\bar{\lambda}, 0, \bar{\mu}) \in \mathcal{M}_n$. □

For each $k \in \mathbb{N}$, let $-1_k = (\underbrace{-1,\dots,-1}_{k \text{ times}})$, so $(\bar{\lambda}, -1_k) = (\bar{\lambda}, \underbrace{-1,\dots,-1}_{k \text{ times}})$. We allow $k = 0$ by defining -1_0 as the empty sequence. In the same way, we define 1_k and 0_k for any $k \in \mathbb{N} \cup \{0\}$.

Now we define, for any $\bar{\mu} \in \{-1,0,1\}^k$, $\mathcal{M}_{n;\bar{\mu}} = \{(\bar{\lambda}, \bar{\mu}) \in \mathcal{M}_n\}$, the set of Motzkin paths of n steps that "end in $\bar{\mu}$", and $M_{n;\bar{\mu}}$ is its cardinality.

Note that $\mathcal{M}_{n;1}$ is empty, and then $\mathcal{M}_n = \mathcal{M}_{n;0} \cup \mathcal{M}_{n;-1}$. Moreover we have the following:

Proposition 2.2. *For every $n \in \mathbb{N}$*

$$\mathcal{M}_n = \mathcal{M}_{n;0} \cup \mathcal{M}_{n;1,-1} \cup \mathcal{M}_{n;0,-1} \cup \mathcal{M}_{n;1,-1_2} \cup \mathcal{M}_{n;0,-1_2} \cup \mathcal{M}_{n;1,-1_3} \cup \dots$$

(disjoint union), where the nonempty sets are exactly the first n sets in the list.

Proof. Note that $\mathcal{M}_{n;-1_k} = \mathcal{M}_{n;1,-1_k} \cup \mathcal{M}_{n;0,-1_k} \cup \mathcal{M}_{n;1,-1_{k+1}} \cup \dots$ is empty if $2k > n$, since for any $(\bar{\lambda}, -1_k) \in \mathcal{M}_n$ we have $0 = \lambda_1 + \dots + \lambda_{n-k} - k \le n - 2k$.

If $n = 2k$ then $(1_k, -1_k) \in \mathcal{M}_{n;1,-1_k}$, while the same argument as before shows that $\mathcal{M}_{n;0,-1_k}$ is empty.

If $n = 2k+1$ then $(1_k, 0, -1_k) \in \mathcal{M}_{n;0,-1_k}$ and $(0, 1_k, -1_k) \in \mathcal{M}_{n;1,-1_k}$.

Finally, for $n > 2k$ $(0_{n-2k}, 1_k, -1_k) \in \mathcal{M}_{n;1,-1_k}$ and $(0_{n-2k-1}, 1_k, 0, -1_k) \in \mathcal{M}_{n;0,-1_k}$. □

This decomposition of $\mathcal{M}_n$ allows us to prove the following "general" theorem, about the integration of certain functions defined on $\mathcal{M}$.

Theorem 2.3. *Let $f\colon \mathcal{M} \to \mathbb{C}$ a function such that:*
(I) If $(\bar{\lambda}, -1_k) \in \mathcal{M}_n$ then $f(\bar{\lambda}, 0, -1_k) = b_{n,k} f(\bar{\lambda}, -1_k)$ ($k \ge 0$, and $b_{n,0} \ne 0$ for all n),
(II) If $(\bar{\lambda}, 0, -1_{k-1}) \in \mathcal{M}_n$ then $f(\bar{\lambda}, 1, -1_k) = c_{n,k} f(\bar{\lambda}, 0, -1_{k-1})$ ($k \ge 1$).

Then the sequence $\left(b_{n,0} \sum_{\bar{\lambda} \in \mathcal{M}_n} f(\bar{\lambda})\right)_{n \ge 1}$ is the first column of the numerical triangle $f(0)T$, where T is the triangle generated by the sequence of matrices $(A^{(n)})$, $A^{(n)}$ being the $n \times (n+1)$ matrix

$$A^{(n)} = \begin{pmatrix} b_{n,0} & c_{n,1} & 0 & 0 & 0 & 0 & \cdots \\ b_{n,0} & 0 & b_{n,1} & 0 & 0 & 0 & \cdots \\ b_{n,0} & 0 & b_{n,1} & c_{n,2} & 0 & 0 & \cdots \\ b_{n,0} & 0 & b_{n,1} & 0 & b_{n,2} & 0 & \cdots \\ b_{n,0} & 0 & b_{n,1} & 0 & b_{n,2} & c_{n,3} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n,0} & 0 & b_{n,1} & 0 & b_{n,2} & 0 & \cdots \end{pmatrix}.$$

Proof. Let $(x_{n,1}, x_{n,2}, \dots, x_{n,n+1})$ be the vector given by

$$\Big(\sum_{\bar{\lambda} \in \mathcal{M}_{n+1;0}} f(\bar{\lambda}), \sum_{\bar{\lambda} \in \mathcal{M}_{n+1;1,-1}} f(\bar{\lambda}), \sum_{\bar{\lambda} \in \mathcal{M}_{n+1;0,-1}} f(\bar{\lambda}), \sum_{\bar{\lambda} \in \mathcal{M}_{n+1;1,-1_2}} f(\bar{\lambda}), \dots \Big),$$

i.e.

$$x_{n,2k+1} = \sum_{\bar{\lambda} \in \mathcal{M}_{n+1;0,-1_k}} f(\bar{\lambda}) \qquad (0 \le 2k \le n),$$

$$x_{n,2k} = \sum_{\bar{\lambda} \in \mathcal{M}_{n+1;1,-1_k}} f(\bar{\lambda}) \qquad (2 \le 2k \le n+1).$$

We claim that this vector equals $f(0)A^{(1)}A^{(2)}\cdots A^{(n)}$, as we see by induction on n: for $n=1$ it is $(f(0,0),f(1,-1))=(b_{1,0}f(0),c_{1,1}f(0))=f(0)A^{(1)}$. Assuming it true for $n-1$, note first that

$$\begin{aligned} x_{n,1} &= \sum_{\bar\lambda\in\mathcal{M}_{n+1;0}} f(\bar\lambda) = \sum_{(\bar\lambda,0)\in\mathcal{M}_{n+1}} f(\bar\lambda,0) = \sum_{\bar\lambda\in\mathcal{M}_n} f(\bar\lambda,0) \\ &= \sum_{\bar\lambda\in\mathcal{M}_n} b_{n,0}f(\bar\lambda) = b_{n,0}\sum_{\bar\lambda\in\mathcal{M}_n} f(\bar\lambda) = b_{n,0}\sum_{j=1}^{n} x_{n-1,j} \\ &= f(0)\,A^{(1)}\cdots A^{(n-1)}\begin{pmatrix} b_{n,0}\\ \cdots \\ b_{n,0}\end{pmatrix}. \end{aligned}$$

For $1\le k\le n/2$ we have

$$\begin{aligned} x_{n,2k+1} &= \sum_{\bar\lambda\in\mathcal{M}_{n+1;0,-1_k}} f(\bar\lambda) = \sum_{(\bar\lambda,-1_k)\in\mathcal{M}_n} f(\bar\lambda,0,-1_k) \\ &= \sum_{(\bar\lambda,-1_k)\in\mathcal{M}_n} b_{n,k}f(\bar\lambda,-1_k) = b_{n,k}\sum_{\bar\lambda\in\mathcal{M}_{n;-1_k}} f(\bar\lambda) \\ &= b_{n,k}\Big(\sum_{\bar\lambda\in\mathcal{M}_{n;1,-1_k}} f(\bar\lambda) + \sum_{\bar\lambda\in\mathcal{M}_{n;0,-1_k}} f(\bar\lambda) + \sum_{\bar\lambda\in\mathcal{M}_{n;1,-1_{k+1}}} f(\bar\lambda) + \cdots\Big) \\ &= b_{n,k}\sum_{j=2k}^{n} x_{n-1,j} = f(0)A^{(1)}\cdots A^{(n-1)}\begin{pmatrix} 0\\ \cdots \\ 0 \\ b_{n,k} \\ \cdots \\ b_{n,k}\end{pmatrix}\begin{matrix} \\ \\ \\ \leftarrow 2k \\ \\ \\ \end{matrix}, \end{aligned}$$

and finally, for $1\le k\le \dfrac{n+1}{2}$,

$$\begin{aligned} x_{n,2k} &= \sum_{\bar\lambda\in\mathcal{M}_{n+1;1,-1_k}} f(\bar\lambda) = \sum_{(\bar\lambda,0,-1_{k-1})\in\mathcal{M}_n} f(\bar\lambda,1,-1_k) \\ &= \sum_{(\bar\lambda,0,-1_{k-1})\in\mathcal{M}_n} c_{n,k}f(\bar\lambda,0,-1_{k-1}) = c_{n,k}\sum_{\bar\lambda\in\mathcal{M}_{n;0,-1_{k-1}}} f(\bar\lambda) \\ &= c_{n,k}\,x_{n-1,2k-1} = f(0)A^{(1)}\cdots A^{(n-1)}\begin{pmatrix} 0\\ \cdots \\ 0 \\ c_{n,k} \\ 0 \\ \cdots \\ 0\end{pmatrix}\begin{matrix} \\ \\ \\ \leftarrow 2k-1 \\ \\ \\ \\ \end{matrix}, \end{aligned}$$

so our claim is true. □

Our first application of this theorem is counting Motzkin paths.

Corollary 2.4. *The sequence of Motzkin numbers* (M_n) *is the first column of the triangle generated by* $A = (a_{ij})$*, with*

$$a_{ij} = \begin{cases} 1 & \text{if } i = j-1, \\ 1 & \text{if } j \text{ is odd and } i > j-1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Apply Theorem 2.3 to $f = 1$. □

Note that

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 1 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

and the triangle (which is not the Motzkin triangle of example 4) begins

$$\begin{array}{cccccc} 1 \\ 1 & 1 \\ 2 & 1 & 1 \\ 4 & 2 & 2 & 1 \\ 9 & 4 & 5 & 2 & 1 \\ 21 & 9 & 12 & 5 & 3 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Once the $(n-1)$-th row is given, we get the number $t_{n,m}$ as in Catalan triangle for even m, but $t_{n,m} = t_{n-1,m-1}$ if m is odd.

Following the proof of Theorem 2.3, note that the n-th row is just

$$(M_{n+1;0}, M_{n+1;1,-1}, M_{n+1;0,-1}, M_{n+1;1,-1_2}, \cdots).$$

For instance, the third row is $(4, 2, 2, 1)$, the cardinalities of

$$\begin{aligned} \mathcal{M}_{4;0} &= \{(0,0,0,0), (1,-1,0,0), (1,0,-1,0), (0,1,-1,0)\}, \\ \mathcal{M}_{4;1,-1} &= \{(0,0,1,-1), (1,-1,1,-1)\}, \\ \mathcal{M}_{4;0,-1} &= \{(1,0,0,-1), (0,1,0,-1)\}, \\ \mathcal{M}_{4;1,-1_2} &= \{(1,1,-1,-1)\}, \end{aligned}$$

which sum up $9 = M_4 = M_{5;0} = t_{4,0}$.

3. Dyck paths and Catalan numbers.

There is antoher easy decomposition of Motzkin paths that relates Motzkin and Catalan numbers, by counting how many $\lambda_j = 0$ there are in each $\bar{\lambda}$. For each $n \in \mathbb{N}$ and $k = 0, 1, \dots, n$, let

$$\mathcal{D}_{n,k} = \{\bar{\lambda} \in \mathcal{M};\ \bar{\lambda} \text{ has } k \text{ 0's}\}$$

and let $D_{n,k}$ its cardinality. $D_{n,k}$ is then the number of Motzkin paths of n steps, k of which are flat.

Since any Motzkin path has the same number of 1's and -1's, the number of flat steps is of the same parity as n. Hence $D_{n,k} = 0$ if n is even and k is odd or vice-versa.

It is also obvious that $D_{n,n} = 1$ for each n.

$\mathcal{D}_{n,0}$ is the set of the so-called *Dyck paths*, i.e. Motzkin paths with no flat steps. It is empty for any odd n, and then

$$\mathcal{D}_0 = \cup_{n=1}^{\infty} \mathcal{D}_{2n,0}$$

is the set of Dyck paths of any order. It is well known that

$$D_{2n,0} = C_n$$

for all n, where C_n is the Catalan number of example 5. See for instance [4]. We will find out a simple proof of this.

With the same notations as in the previous section, we clearly have

$$\mathcal{D}_{2n,0} = \mathcal{D}_{2n,0;1,-1} \cup \mathcal{D}_{2n,0;1,-1_2} \cup \cdots \cup \mathcal{D}_{2n,0;1,-1_n}$$

(disjoint union). Denote the cardinality of each set by $D_{2n,0;1,-1_k} > 0$.

Moreover, note that

$$(\bar{\lambda}, -1_k) \in \mathcal{D}_{2n,0;-1_k} \text{ if and only if } (\bar{\lambda}, 1, -1_{k+1}) \in \mathcal{D}_{2n+2,0;1,-1_{k+1}}\,,$$

and

$$\mathcal{D}_{2n,0;-1_k} = \mathcal{D}_{2n,0;1,-1_k} \cup \mathcal{D}_{2n,0;1,-1_{k+1}} \cup \cdots \cup \mathcal{D}_{2n,0;1,-1_n}\,.$$

Similarly to Theorem 2.3, we can formulate a "general" theorem:

Theorem 3.1. *Let $f\colon \mathcal{D}_0 \to \mathbb{C}$ a function such that:*
If $(\bar{\lambda}, -1_k) \in \mathcal{D}_{2n,0}$ then $f(\bar{\lambda}, 1, -1_{k+1}) = a_{n,k} f(\bar{\lambda}, -1_k)$ ($k \geq 0$, and $a_{n,0} \neq 0$ for all n).

Then the sequence $\big(a_{n,0} \sum_{\bar{\lambda} \in \mathcal{D}_{2n,0}} f(\bar{\lambda})\big)_{n \geq 1}$ is the first column of the numerical triangle $f(1,-1)T$, where T is the triangle generated by the sequence of matrices $(A^{(n)})$, $A^{(n)}$ being the $n \times (n+1)$ matrix

$$A^{(n)} = \begin{pmatrix} a_{n,0} & a_{n,1} & 0 & 0 & 0 & \cdots & 0 \\ a_{n,0} & a_{n,1} & a_{n,2} & 0 & 0 & \cdots & 0 \\ a_{n,0} & a_{n,1} & a_{n,2} & a_{n,3} & 0 & \cdots & 0 \\ a_{n,0} & a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,0} & a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \cdots & a_{n,n} \end{pmatrix}.$$

Proof. The proof is similar to that of Theorem 2.3. We have to prove that

$$\Big(\sum_{\bar\lambda\in\mathcal{D}_{2n+2,0;1,-1}} f(\bar\lambda), \sum_{\bar\lambda\in\mathcal{D}_{2n+2,0;1,-1_2}} f(\bar\lambda), \ \dots, \sum_{\bar\lambda\in\mathcal{D}_{2n+2,0;1,-1_{n+1}}} f(\bar\lambda) \ \Big)$$

equals $f(1,-1)A^{(1)}A^{(2)}\cdots A^{(n)}$, the first component being $a_{n,0}\sum_{\bar\lambda\in\mathcal{D}_{2n,0}} f(\bar\lambda)$. □

Take in particular $f=1$. Then $a_{n,k}=1$ for all $0\le k\le n$, and A is just the matrix of example 5. Theorem 3.1 says in this case that

$$C_n = \sum_{\bar\lambda\in\mathcal{D}_{2n,0}} 1 = D_{2n,0}\,.$$

As in "our" Motzkin triangle, we also get that the numbers of the n-th row in the Catalan triangle are the numbers of Dyck paths of $2n$ steps finishing in -1_k, for $k=0,\dots,n$. For instance, the third row is

$$(5,5,3,1) = (D_{6,0}, D_{6,0;-1}, D_{6,0;-1_2}, D_{6,0;-1_3})\,,$$

the cardinalities of (listed backwards)

$$\begin{aligned}
\mathcal{D}_{6,0;-1_3} &= \{(1,1,1,-1,-1,-,1)\}\,,\\
\mathcal{D}_{6,0;-1_2} &= \mathcal{D}_{6,0;-1_3}\cup\{(1,-1,1,1,-1,-1),(1,1,-1,1,-1,-1)\}\,,\\
\mathcal{D}_{6,0;-1} &= \mathcal{D}_{6,0;-1_2}\cup\{(1,-1,1,-1,1,-1),(1,1,-1,-1,1,-1)\}\,,\\
\mathcal{D}_{6,0} &= \mathcal{D}_{6,0;-1}\,.
\end{aligned}$$

Theorem 3.2. *Let*

$$A(x) = \begin{pmatrix} x & 1/x & 0 & 0 & 0 & 0 & \cdots\\ x & 0 & x & 0 & 0 & 0 & \cdots\\ x & 0 & x & 1/x & 0 & 0 & \cdots\\ x & 0 & x & 0 & x & 0 & \cdots\\ x & 0 & x & 0 & x & 1/x & \cdots\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

and let $T(x)$ the triangle generated by $A(x)$. Then its first column is given by the polynomials

$$\sum_{k=0}^{n} D_{n,k}\, x^k\,.$$

Proof. Let $z(\bar\lambda)$ the function that counts the zeroes in each $\bar\lambda\in\mathcal{M}$, and let $f(\bar\lambda)=x^{z(\bar\lambda)}$. Note that

$$\begin{aligned}
f(\bar\lambda,1,-1_k) &= \frac{1}{x} f(\bar\lambda,0,-1_{k-1}) \text{ and}\\
f(\bar\lambda,0,-1_k) &= x f(\bar\lambda,-1_k)\,,
\end{aligned}$$

so Teorem 2.3 applies, and the matrices $A^{(n)}$ for each x are the corresponding submatrices of $A(x)$. According to this, the first element of the n-th row is

$$\sum_{\bar\lambda\in\mathcal{M}_n} x^{z(\bar\lambda)} = \sum_{k=0}^{n} D_{n,k}\, x^k\,.$$

□

Some remarks are in order: of course, if $x = 1$ it results the matrix of Corollary 2.4, what is just saying that $M_n = \sum_{k=0}^{n} D_{n,k}$, obvious since $(\mathcal{D}_{n,k})_{0\le k\le n}$ is a partition of $\mathcal{M}_n$.

On the other hand, note that from each Dyck path of $\mathcal{D}_{2m-2k,0}$ we get, by placing $2k$ 0's in all the $\binom{2m}{2k}$ possible ways, this number of different paths in $\mathcal{D}_{2m,2k}$, and any element in this set is obtained in this way from a unique path in $\mathcal{D}_{2m-2k,0}$. It follows that

$$D_{2m,2k} = \binom{2m}{2k} C_{m-k} = \binom{2m}{2(m-k)} C_{m-k}\,.$$

Analogously

$$D_{2m+1,2k+1} = \binom{2m+1}{2k+1} C_{m-k} = \binom{2m+1}{2(m-k)} C_{m-k}\,.$$

These facts show the well-known formula

$$M_n = \sum_{k\ge 0} \binom{n}{2k} C_k\,.$$

4. Calabi's idea and Motzkin paths.

Let us briefly explain Calabi's argument reducing the computation of $\zeta(2n)$ to the volume of a polytope in $\mathbb{R}^{2n}$:

For any natural number $n \ge 2$, by writing

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^n} + \sum_{k=1}^{\infty} \frac{1}{(2k)^n}$$

we get

$$\zeta(n)(1-2^{-n}) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^n}.$$

Since $\dfrac{1}{2k+1} = \displaystyle\int_0^1 x^{2k}dx$, then

$$\zeta(n) = \frac{2^n}{2^n-1} \sum_{k=0}^{\infty} \int_{(0,1)^n} (x_1x_2\cdots x_n)^{2k} dm(\bar{x}),$$

where of course $\bar{x} = (x_1, \ldots, x_n)$ and m stands for the Lebesgue measure. Then

$$\begin{aligned}\zeta(n) &= \frac{2^n}{2^n-1} \int_{(0,1)^n} \sum_{k=0}^{\infty} (x_1x_2\cdots x_n)^{2k} dm(\bar{x}) \\ &= \frac{2^n}{2^n-1} \int_{(0,1)^n} \frac{1}{1-x_1^2x_2^2\cdots x_n^2} dm(\bar{x})\,.\end{aligned}$$

Try now the change of variables

$$(x_1, x_2, \ldots, x_n) = \varphi(\bar{u}) = \Big(\frac{\sin u_1}{\cos u_2}, \frac{\sin u_2}{\cos u_3}, \ldots, \frac{\sin u_{n-1}}{\cos u_n}, \frac{\sin u_n}{\cos u_1}\Big)\,.$$

Note that if $0 < u, v < \pi/2$ then $\dfrac{\sin u}{\cos v} = \dfrac{\sin u}{\sin(\pi/2 - v)} > 0$, and it is < 1 if and only if $u < \pi/2 - v$. Then, if we define

$$A_n = \{\bar{u} > 0;\ u_1 + u_2 < \pi/2, u_2 + u_3 < \pi/2, \ldots, u_{n-1} + u_n < \pi/2, u_n + u_1 < \pi/2\}$$

we have that $\varphi(A_n) \subseteq (0,1)^n$. It is not difficult to see that $\varphi(A_n)$ is actually $(0,1)^n$ (moreover φ is injective on A_n), and hence this change of variables gives

$$\int_{(0,1)^n} \frac{1}{1 - x_1^2 x_2^2 \cdots x_n^2} dm(\bar{x}) = \int_{A_n} \frac{\det(J\varphi(\bar{u}))}{1 - \tan^2 u_1 \cdots \tan^2 u_n} dm(\bar{u}).$$

Now, it is readily checked that

$$\det(J\varphi(\bar{u})) = 1 + (-1)^{n+1} \tan^2 u_1 \cdots \tan^2 u_n,$$

and thus, if n is even, it results

$$\int_{(0,1)^n} \frac{1}{1 - x_1^2 x_2^2 \cdots x_n^2} dm(\bar{x}) = \int_{A_n} dm = m(A_n).$$

Take instead

$$E_n = \{0 < \bar{x} \in \mathbb{R}^n; x_1 + x_2 < 1, x_2 + x_3 < 1, \ldots, x_{n-1} + x_n < 1, x_n + x_1 < 1\}.$$

Since $m(A_n) = \dfrac{\pi^n}{2^n} m(E_n)$, we have shown that

$$\zeta(2n) = \frac{\pi^{2n}}{4^n - 1} m(E_{2n}) \tag{2}$$

for all $n \in \mathbb{N}$. Now let $x_{2j} = y_{2j}$ and $x_{2j-1} = 1 - y_{2j-1}$ for $j = 1, \ldots, n$. This new change of variables gives

$$\zeta(2n) = \frac{\pi^{2n}}{4^n - 1} m(F_{2n}),$$

where $F_{2n} = \{0 < \bar{y} < 1;\ y_2 < y_1,\ y_2 < y_3, \ldots, y_{2n} < y_{2n-1},\ y_{2n} < y_1\}$.

For $n = 1$, we have $F_2 = \{(a, b) \in \mathbb{R}^2;\ 0 < b < a < 1\}$, so $m(F_2) = 1/2$ and $\zeta(2) = \pi^2/6$.

We take our deviation here. Changing the coordinates order, we can re-write

$$F_{2n} = \{(\bar{t}, \bar{x}) \in (0,1)^n \times (0,1)^n;\ t_1 < x_1,\ t_1 < x_2,\ t_2 < x_2,\ t_2 < x_3, \ldots, t_n < x_n,\ t_n < x_1\}.$$

Let $a \wedge b = \min\{a, b\}$ for all real numbers a and b. For each $\bar{x} \in (0,1)^n$ the corresponding section of F_{2n} is

$$\begin{aligned} F_{\bar{x}} &= \{\bar{t};\ (\bar{t}, \bar{x}) \in F_{2n}\} \\ &= (0, x_1 \wedge x_2) \times (0, x_2 \wedge x_3) \times \cdots \times (0, x_{n-1} \wedge x_n) \times (0, x_n \wedge x_1), \end{aligned}$$

so

$$m(F_{2n}) = \int_{(0,1)^n} m(F_{\bar{x}}) dm(\bar{x}) = \int_{(0,1)^n} (x_1 \wedge x_2) \cdots (x_{n-1} \wedge x_n)(x_n \wedge x_1) dm(\bar{x}),$$

and we have obtained the following expression of $\zeta(2n)$ for each natural n:

$$\zeta(2n) = \frac{\pi^{2n}}{4^n - 1} \int_{(0,1)^n} \xi(\bar{x}) dm(\bar{x}), \tag{3}$$

where $\xi(\bar{x}) = (x_1 \wedge x_2)(x_2 \wedge x_3) \cdots (x_{n-1} \wedge x_n)(x_n \wedge x_1)$.

For $n = 2$ we have $\xi(a, b) = (a \wedge b)^2$, and

$$\int_{(0,1)^2} (a \wedge b)^2 dm(a, b) = 2 \int_{\{0<b<a<1\}} (a \wedge b)^2 dm(a, b) = 2 \int_{\{0<b<a<1\}} b^2 dm(a, b)$$
$$= 2 \int_0^1 \int_0^a b^2 db da = \frac{2}{3} \int_0^1 a^3 da = \frac{1}{6}\,,$$

so (3) says that $\zeta(4) = \frac{\pi^4}{90}$.

For $n = 3$ we can also write

$$m(F_6) = 6 \int_{\{0<x_1<x_2<x_3<1\}} \xi(\bar{x}) dm(\bar{x}),$$

since any coordinate is compared to each other in $(x_1 \wedge x_2)(x_2 \wedge x_3)(x_3 \wedge x_1)$. This makes it easy to see that $m(F_6) = 1/15$ and thus $\zeta(6) = \pi^6/945$.

For $n > 3$ it does not suffice to consider the set $\{0 < x_1 < \cdots < x_n < 1\}$.

Let S_n be the group of permutations of the set $\{1, 2, \ldots, n\}$, for each $n \in \mathbb{N}$.

Note that, in the integral in (3), we can ignore all points in $(0, 1)^n$ with two equal coordinates, since $m(H) = 0$ for any hyperplane H. Besides, if we fix j and integrate in $\{\bar{x};\ x_j < x_i \text{ for all } i \neq j,\ x_i \neq x_k \text{ for all } i, k\}$ then the result is independent from j, by symmetry. Choose $j = n$. For each $\bar{x}$ in the corresponding set there exists a unique permutation $\sigma \in S_{n-1}$ such that

$$0 < x_n < x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n-1)} < 1.$$

If E_σ is the set of such points for any given σ, we have

$$(4) \qquad \int_{(0,1)^n} \xi(\bar{x}) dm(\bar{x}) = n \sum_{\sigma \in S_{n-1}} \int_{E_\sigma} \xi(\bar{x}) dm(\bar{x}) = n \sum_{\sigma^{-1} \in S_{n-1}} \int_{E_{\sigma^{-1}}} \xi(\bar{x}) dm(\bar{x})\,.$$

Let $E = E_{\mathrm{id}} = \{\bar{x};\ 0 < x_n < x_1 < x_2 < \cdots < x_{n-1} < 1\}$.

The change of variables $\bar{x} \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}, x_n)$ maps E onto $E_{\sigma^{-1}}$, and then

$$\int_{E_{\sigma^{-1}}} \xi(\bar{x}) dm(\bar{x}) = \int_E \xi(x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}, x_n) dm(\bar{x})$$
$$= \int_E x_n^2 (x_{\sigma(1)} \wedge x_{\sigma(2)}) \cdots (x_{\sigma(n-2)} \wedge x_{\sigma(n-1)}) dm(\bar{x})\,.$$

In order to handle this integral we consider the following map:

Definition. Let

$$\Phi\colon\ \cup_{n=2}^{\infty} S_n \longrightarrow \cup_{n=1}^{\infty} \{-1, 0, 1\}^n$$

defined as follows: for any $n \in \mathbb{N}$ and $\sigma \in S_{n+1}$, $\Phi(\sigma)$ is the family $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \{-1, 0, 1\}^n$ such that, in the list

$$0, \sigma(1), \sigma(2), \ldots, \sigma(n), \sigma(n+1), 0$$

$\lambda_j + 1$ is the number of neighbours of j that are greater than j.

For example, if $\sigma \in S_5$ is the given by $0, 2, 1, 4, 5, 3, 0$ then $\Phi(\sigma) = (1, -1, 0, 0)$: $\lambda_1 + 1 = 2$ because 2 and 4 (the neighbours of 1) are both > 1; $\lambda_2 + 1 = 0$ since $0, 1 < 2$, and $\lambda_3 + 1 = \lambda_4 + 1 = 1$, for both 3 and 4 have 5 as a neighbour.

It is easy that $\sum_{j=1}^{n} \lambda_j = 0$ for all $\bar{\lambda} = \Phi(\sigma)$: consider all the couples of neighbours; except for the first and the last (containing 0) each one adds +1 to $\sum_{j=1}^{n} \lambda_j - n$, and there are n such couples.

On the other hand it is clear that $\lambda_1 > -1$ and $\lambda_n < 1$, as in Motzkin paths.

Definition. For each $\bar{\lambda} \in \cup_{n=1}^{\infty}\{-1, 0, 1\}^n$, $\nu(\bar{\lambda})$ is the number of permutations σ such that $\Phi(\sigma) = \bar{\lambda}$.

Note that $\nu(\bar{\lambda}) > 0$ if and only if $\bar{\lambda}$ is in the image of Φ.

Let $e_j = (0, 0, \dots, 0, \underset{j}{1}, 0, \dots)$, with $\bar{\lambda} \pm e_j = (\lambda_1, \lambda_2, \dots, \lambda_j \pm 1, \lambda_{j+1}, \dots)$.

Lemma 4.1. *For each* $\bar{\lambda} \in \{-1, 0, 1\}^n$ *we have:*
(i) $\nu(\bar{\lambda}, 0) = 2\nu(\bar{\lambda})$ *;*
(ii) $\nu(\bar{\lambda}, -1) = 2 \sum_{\lambda_j = 0} \nu(\bar{\lambda} - e_j) + \sum_{\lambda_j = 1} \nu(\bar{\lambda} - e_j)\,.$

Proof. Let $\sigma \in S_{n+1}$ given by $0, \sigma(1), \sigma(2), \dots, \sigma(n), \sigma(n+1), 0$. From σ we obtain $n+2$ permutations $\tau \in S_{n+2}$, by placing $n+2$ between any two neighbours. These τ are different each other, and the $(n+2)!$ permutations of S_{n+2} arise in this way by taking the $(n+1)!$ permutations of S_{n+1}.

If τ is so-derived from σ, with $\Phi(\sigma) = \bar{\lambda}$ and $\Phi(\tau) = \bar{\mu}$, then or $\lambda_j = \mu_j$ for $j = 1, \dots, n$ with $\mu_{n+1} = 0$ (this happens in two cases, setting $n+2$ next to $n+1$) or $\mu_j = \lambda_j + 1$ for exactly one j, $\lambda_j = \mu_j$ for the rest and $\mu_{n+1} = -1$.

If $\lambda_j = -1$ then $\mu_j = 0$ for each τ that puts $n+2$ next to j, and if $\lambda_j = 0$ then $\mu_j = 1$ only if τ sets $n+2$ between j and its least neighbour. □

Theorem 4.2. *The image of* Φ *is* $\mathcal{M}$*, the set of Motzkin paths.*

Proof. It is easy for $n = 1$: $\mathcal{M}_1 = \{(0)\}$, $S_2 = \{(0, 1, 2, 0), (0, 2, 1, 0)\}$ and $\nu(0) = 2$. Assume that it is true for n.

If $\bar{\lambda} \in \Phi(S_{n+2})$ and $\bar{\lambda} = (\bar{\mu}, 0)$, by Lemma 4.1 $\nu(\bar{\mu}) \neq 0$, hence $\bar{\mu} \in \mathcal{M}_n$ and then $\bar{\lambda} \in \mathcal{M}_{n+1}$ (by Lemma 2.1). If $\bar{\lambda} = (\bar{\mu}, -1)$ Lemma 4.1 says that, for some j, $\bar{\mu} - e_j$ has preimages in S_{n+1}, so $\bar{\mu} - e_j \in \mathcal{M}_n$ and this easily implies that $\bar{\lambda} \in \mathcal{M}_{n+1}$ (a simple drawing may help).

Conversely, if $\bar{\lambda} \in \mathcal{M}_{n+1}$ and $\bar{\lambda} = (\bar{\mu}, 0)$ then $\mu \in \mathcal{M}_n$, and thus $2\nu(\bar{\mu}) = \nu(\bar{\lambda}) > 0$, so $\bar{\lambda} \in \Phi(S_{n+2})$.

If $\bar{\lambda} \in \mathcal{M}_{n+1}$ with $\bar{\lambda} = (\bar{\mu}, -1)$, by the definition of Motzkin paths we have $\sum_{j=1}^{n} \mu_j = 1$ and $\sum_{j=1}^{k} \mu_j \geq 0$ for $1 \leq k$. Let l the maximum such that $\sum_{j=l}^{n} \mu_j = 1$. As $\sum_{j<l} \mu_j = 0$ it can't be $\mu_l = -1$, so we can take $\bar{\mu} - e_l$; Would it be $\sum_{j=l}^{k} \mu_j = -1$ for some $k > l$, it should be $\sum_{j=k+1}^{n} \mu_j = 2$, in contradiction with our choice of l. It follows that $\bar{\mu} - e_l \in \mathcal{M}_n$, and hence $\nu(\bar{\mu} - e_l) > 0$. Lemma 4.1 gives now that $\nu(\bar{\mu}, -1) = \nu(\bar{\lambda}) > 0$, and hence $\lambda \in \Phi(S_{n+2})$. □

Let's get back to formula (4), where $n \geq 3$, and the integral

$$\int_E x_n^2 (x_{\sigma(1)} \wedge x_{\sigma(2)}) \cdots (x_{\sigma(n-2)} \wedge x_{\sigma(n-1)}) dm(\bar{x})\,.$$

If $\bar\lambda = \Phi(\sigma) \in \mathcal{M}_{n-2}$ then this integral is the same as

$$\int_E x_n^2\, x_1^{\lambda_1+1} \cdots x_{n-2}^{\lambda_{n-2}+1} dm(\bar x)$$
$$= \int_0^1 \int_0^{x_{n-1}} x_{n-2}^{\lambda_{n-2}+1} \cdots \int_0^{x_2} x_1^{\lambda_1+1} \int_0^{x_1} x_n^2\, dx_n\, dx_1 \cdots dx_{n-2}\, dx_{n-1}$$
$$= \frac{1}{6n} \cdot \frac{1}{\lambda_1+5} \cdot \frac{1}{\lambda_1+\lambda_2+7} \cdot \cdots \cdot \frac{1}{\lambda_1+\cdots+\lambda_{n-2}+(2n-1)}\,.$$

Definition. For each $\bar\lambda \in \{-1,0,1\}^n$,

$$\rho(\bar\lambda) = (\lambda_1+5)(\lambda_1+\lambda_2+7)\cdots(\lambda_1+\cdots+\lambda_n+2n+3)\,.$$

In this expression, each j-th factor ($j>1$) is the previous one plus 3 (if $\lambda_j = 1$), plus 2 (if $\lambda_j = 0$) or plus 1 (if $\lambda_j = -1$). If $\bar\lambda \in \mathcal{M}$ then $\sum_j \lambda_j = 0$, so the last factor is $2n+3$.

Summarizing, we have that

$$\int_{(0,1)^n} \xi(\bar x) dm(\bar x) = \frac{1}{6} \sum_{\sigma \in S_{n-1}} \frac{1}{\rho(\Phi(\sigma))}$$

and then

$$\int_{(0,1)^n} \xi(\bar x) dm(\bar x) = \frac{1}{6} \sum_{\bar\lambda \in \mathcal{M}_{n-2}} \frac{\nu(\bar\lambda)}{\rho(\bar\lambda)}\,, \tag{5}$$

which along with (2) gives:

Proposition 4.3. *For any $n \geq 3$,*

$$\zeta(2n) = \frac{\pi^{2n}}{6(4^n-1)} \sum_{\bar\lambda \in \mathcal{M}_{n-2}} \frac{\nu(\bar\lambda)}{\rho(\bar\lambda)}\,. \tag{6}$$

Stop for a moment and compute $\zeta(6)$ and $\zeta(8)$:

$\mathcal{M}_1 = \{(0)\}$, with $\nu(0) = 2$ and $\rho(0) = 5$. Then

$$\zeta(6) = \frac{\pi^6}{6(4^3-1)} \cdot \frac{2}{5} = \frac{\pi^6}{945}\,.$$

$\mathcal{M}_2 = \{(0,0),(1,-1)\}$. From Lemma 4.1 $\nu(0,0) = 2\nu(0) = 4$ and $\nu(1,-1) = \nu(0) = 2$. Besides $\rho(0,0) = 5\cdot 7$ and $\rho(1,-1) = 6 \cdot 7$, whence

$$\zeta(8) = \frac{\pi^8}{6(4^4-1)} \cdot \Big(\frac{4}{5\cdot 7} + \frac{2}{6\cdot 7}\Big) = \frac{\pi^8}{9450}\,.$$

Proposition 4.4. *For any $\bar\lambda \in \cup_{n=1}^{\infty}\{-1,0,1\}^n$ we have that*
(i) $\nu(\bar\lambda,0,-1_k) = 2(k+1)\nu(\bar\lambda,-1_k)$ for each $k \geq 0$, and
(ii) $\nu(\bar\lambda,1,-1_k) = k(k+1)\nu(\bar\lambda,-1_{k-1}) = \dfrac{k+1}{2}\nu(\bar\lambda,0,-1_{k-1})$ for each $k \geq 1$.

Proof. The proof follows by induction on k, using Lemma 4.1.
(i) If $k=0$ the result is just (i) in Lemma 4.1. If $k>0$ and we know that it is true for $k-1$, by (ii) in the lemma we can write

$$\begin{aligned}
&\nu(\bar\lambda,0,-1_k)=\nu(\bar\lambda,0,-1_{k-1},-1)\\
&\quad=2\nu(\bar\lambda,-1,-1_{k-1})+2\sum_{\lambda_j=0}\nu(\bar\lambda-e_j,0,-1_{k-1})+\sum_{\lambda_j=1}\nu(\bar\lambda-e_j,0,-1_{k-1})\\
&\quad=2\nu(\bar\lambda,-1_k)+2k\big(2\sum_{\lambda_j=0}\nu(\bar\lambda-e_j,-1_{k-1})+\sum_{\lambda_j=1}\nu(\bar\lambda-e_j,-1_{k-1})\big)\\
&\quad=2\nu(\bar\lambda,-1_k)+2k\nu(\bar\lambda,-1_{k-1},-1)\\
&\quad=2(k+1)\nu(\bar\lambda,-1_k).
\end{aligned}$$

(ii) If $k=1$ then

$$\begin{aligned}
\nu(\bar\lambda,1,-1)&=2\sum_{\lambda_j=0}\nu(\bar\lambda-e_j,1)+\sum_{\lambda_j=1}\nu(\bar\lambda-e_j,1)+\nu(\bar\lambda,0)\\
&=\nu(\bar\lambda,0)=2\nu(\bar\lambda)
\end{aligned}$$

since $\nu(\bar\mu,1)=0$ for any $\bar\mu$.

If $k>1$ and the result is true for $k-1$, then

$$\begin{aligned}
&\nu(\bar\lambda,1,-1_k)=\nu(\bar\lambda,1,-1_{k-1},-1)\\
&\quad=2\sum_{\lambda_j=0}\nu(\bar\lambda-e_j,1,-1_{k-1})+\sum_{\lambda_j=1}\nu(\bar\lambda-e_j,1,-1_{k-1})+\nu(\bar\lambda,0,-1_{k-1})\\
&\quad=(k-1)k\big(2\sum_{\lambda_j=0}\nu(\bar\lambda-e_j,-1_{k-2})+\sum_{\lambda_j=1}\nu(\bar\lambda-e_j,-1_{k-2})\big)\\
&\qquad+2k\nu(\bar\lambda,-1_{k-1})\quad\text{(by (i))}\\
&\quad=(k-1)k\,\nu(\bar\lambda,-1_{k-2},-1)+2k\,\nu(\bar\lambda,-1_{k-1})\\
&\quad=(k+1)k\,\nu(\bar\lambda,-1_{k-1}).
\end{aligned}$$

The rest follows from (i): $\nu(\bar\lambda,0,-1_{k-1})=2k\nu(\bar\lambda,-1_{k-1})$, so

$$\nu(\bar\lambda,1,-1_k)=(k+1)k\frac{1}{2k}\nu(\bar\lambda,0,-1_{k-1})=\frac{k+1}{2}\nu(\bar\lambda,0,-1_{k-1}).$$

□

Corollary 4.5. *Let $A=(a_{ij})$ the matrix given by*

$$a_{ij}=\begin{cases}2j & \text{if } j \text{ is odd and } i\geq j-1\,,\\ \dfrac{(j+2)}{4} & \text{if } j \text{ is even and } i=j-1\,,\\ 0 & \text{otherwise,}\end{cases}$$

i.e.

$$A=\begin{pmatrix}2&1&0&0&0&0&\cdots\\2&0&4&0&0&0&\cdots\\2&0&4&3/2&0&0&\cdots\\2&0&4&0&6&0&\cdots\\2&0&4&0&6&2&\cdots\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\end{pmatrix}.$$

Then, if $T = (t_{n,m})$ is the triangle generated by A, for all $n \in \mathbb{N}$

$$t_{n,0} = (n+1)!\,.$$

Proof. Proposition 4.4 says that we can apply Theorem 2.3 to $f = \nu$, and conditions (I) and (II) there are satisfied with $b_{n,k}$ and $c_{n,k}$ independent from n. It results that the corresponding $A^{(n)}$ matrices are the submatrices of A as stated, and since $f(0) = 2 = b_{2,0}$ we get

$$t_{n,0} = \sum_{\bar\lambda \in \mathcal{M}_n} \nu(\bar\lambda)\,,$$

which is just the cardinality of S_{n+1}, that is $(n+1)!$. □

Proposition 4.6. *(i) If $k \geq 0$ and $(\bar\lambda, -1_k) \in \mathcal{M}_n$ then*

$$\rho(\bar\lambda, 0, -1_k) = \rho(\bar\lambda)(2n+5-k)(2n+6-k)\cdots(2n+4)(2n+5), \text{ and}$$
$$\rho(\bar\lambda, -1_k) = \rho(\bar\lambda)(2n+4-k)(2n+5-k)\cdots(2n+2)(2n+3)$$
$$(\text{just } \rho(\bar\lambda) \text{ if } k = 0)$$

(ii) If $k \geq 1$ and $(\bar\lambda, 0, -1_k) \in \mathcal{M}_n$, then

$$\rho(\bar\lambda, 1, -1_k) = \rho(\bar\lambda)(2n+5-k)(2n+6-k)\cdots(2n+4)(2n+5).$$

Proof. If $(\bar\lambda, 0, -1_k) \in \mathcal{M}_{n+1}$ then $\rho(\bar\lambda, 0, -1_k) = \rho(\bar\lambda)\alpha_1\alpha_2\cdots\alpha_{k+1}$, with $\alpha_{k+1} = 2n+5$ and $\alpha_{j+1} - \alpha_j = 1$ for each j, so

$$\rho(\bar\lambda, 0, -1_k) = \rho(\bar\lambda)(2n+5-k)(2n+6-k)\cdots(2n+4)(2n+5).$$

The rest is similar. □

Theorem 4.7. *For each $n \geq 3$, let*

$$a_{ij}^{(n)} = \begin{cases} \dfrac{(j+1)(4n-j+1)}{8} & \text{if } j \text{ is odd and } i \leq j-1\,, \\ \dfrac{(j+2)(4n-j)}{32} & \text{if } j \text{ is even and } i = j-1\,, \\ 0 & \text{otherwise,} \end{cases}$$

and let $A^{(n)} = (a_{ij}^{(n)})_{1\leq i\leq n-2,\ 1\leq j\leq n-1}$.

Then

$$\zeta(2n) = \frac{\pi^{2n}}{(2n)!} \cdot \frac{4^{n-1}}{4^n - 1}\, b_n\,,$$

where b_n is the first component of $A^{(3)}A^{(4)}\cdots A^{(n)}$.

Therefore

$$B_{2n} = (-1)^{n+1}\frac{b_n}{2(4^n-1)}\,, \text{ and}$$

$$\tan^{(2n-1)}(0) = \frac{4^{n-1}}{n}\, b_n\,.$$

Remark. Note that the matrices $A^{(n)}$ are as follows:

$$A^{(3)} = \begin{pmatrix} 3 & 5/4 \end{pmatrix}, \qquad A^{(4)} = \begin{pmatrix} 4 & 7/4 & 0 \\ 4 & 0 & 7 \end{pmatrix},$$

$$A^{(5)} = \begin{pmatrix} 5 & 9/4 & 0 & 0 \\ 5 & 0 & 9 & 0 \\ 5 & 0 & 9 & 12/4 \end{pmatrix}, \qquad A^{(6)} = \begin{pmatrix} 6 & 11/4 & 0 & 0 & 0 \\ 6 & 0 & 11 & 0 & 0 \\ 6 & 0 & 11 & 15/4 & 0 \\ 6 & 0 & 11 & 0 & 15 \end{pmatrix},$$

$$A^{(7)} = \begin{pmatrix} 7 & 13/4 & 0 & 0 & 0 & 0 \\ 7 & 0 & 13 & 0 & 0 & 0 \\ 7 & 0 & 13 & 18/4 & 0 & 0 \\ 7 & 0 & 13 & 0 & 18 & 0 \\ 7 & 0 & 13 & 0 & 18 & 22/4 \end{pmatrix}$$

and so on.

Proof. Write formula (6) as

$$\zeta(2n) = \frac{\pi^{2n}}{6(4^n - 1)} \sum_{\bar\lambda \in \mathcal{M}_{n-2}} f(\bar\lambda),$$

with $f(\bar\lambda) = \frac{\nu(\bar\lambda)}{\rho(\bar\lambda)}$ for each $\bar\lambda \in \mathcal{M}$.

From Propositions 4.4 and 4.6 it follows that

$$f(\bar\lambda, 0, -1_k) = \frac{2(k+1)(2n+4-k)}{(2n+4)(2n+5)} f(\bar\lambda, -1_k) \text{ if } (\bar\lambda, -1_k) \in \mathcal{M}_n, \text{ and}$$

$$f(\bar\lambda, 1, -1_k) = \frac{(k+1)(2n+4-k)}{2(2n+4)(2n+5)} f(\bar\lambda, 0, -1_{k-1}) \text{ if } (\bar\lambda, 0, -1_{k-1}) \in \mathcal{M}_n.$$

Hence f satisfies conditions I and II in Theorem 2.3, with

$$b_{n,k} = \frac{2(k+1)(2n+4-k)}{(2n+4)(2n+5)} \text{ and } c_{n,k} = \frac{1}{4} b_{n,k}.$$

Note that $f(0) = \dfrac{2}{5}$, and then Theorem 2.3 gives, for any $n \geq 3$,

$$\frac{5}{2n+1} \sum_{\bar\lambda \in \mathcal{M}_{n-2}} f(\bar\lambda) = \langle \tilde{A}^{(3)} \tilde{A}^{(4)} \cdots \tilde{A}^{(n)}, e_1 \rangle$$

where $\tilde{A}^{(k+2)}$ is the k-th matrix in the statement of Theorem 2.3 and $\langle \cdot, \cdot \rangle$ is the usual scalar product.

Let now $A^{(n)} = \dfrac{2n(2n+1)}{4} \tilde{A}^{(n)}$. It is easily checked that $A^{(n)}$ are the matrices in the statement, and we have now

$$\sum_{\bar\lambda \in \mathcal{M}_{n-2}} f(\bar\lambda) = \frac{6 \cdot 4^{n-1}}{(2n)!} \langle A^{(3)} A^{(4)} \cdots A^{(n)}, e_1 \rangle,$$

so

$$\zeta(2n) = \pi^{2n} \frac{4^{n-1}}{4^n - 1} \frac{1}{(2n)!} \langle A^{(3)} A^{(4)} \cdots A^{(n)}, e_1 \rangle.$$

□

5. Entringer's theorem and Dyck paths.

Now we recall Entringer's theorem about alternating permutations, named also *zig-zag* or *up and down* permutations.

A permutation $\sigma \in S_n$ is said *alternating* if it is such that

$$\sigma(j) < \sigma(j+1) \text{ if and only if } \sigma(j+1) > \sigma(j+2)$$

for $j = 1, \ldots, n-2$, i.e. if $\sigma(j)$ is not a number between $\sigma(j-1)$ and $\sigma(j+1)$ for $j = 2, \ldots, n-1$.

Let α_n the number of alternating permutations in S_n. Let $\tau \in S_n$ given by $\tau(j) = n+1-j$, i.e.

$$\tau \equiv n, n-1, n-2, \ldots, 2, 1.$$

Then $\sigma \mapsto \tau \circ \sigma$ defines a bijection between alternating permutations σ such that $\sigma(1) < \sigma(2)$ and those such that $\sigma(1) > \sigma(2)$, and thus the number of any of these is $\alpha_n/2$.

Let then $\beta_n = \alpha_n/2$, with $\beta_0 = \beta_1 = 1$. Entringer (see [7]) proved that these numbers give a combinatorial interpretation of tangent and secant numbers, namely

$$\sec z = \beta_0 + \beta_2 \frac{z^2}{2}! + \beta_4 \frac{z^4}{4!} + \cdots \quad \text{and}$$
$$\tan z = \beta_1 z + \beta_3 \frac{z^3}{3}! + \beta_5 \frac{z^5}{5!} + \cdots$$

for each $z \in \mathbb{C}$ with $|z| < \pi/2$. This means that

$$\beta(2n+1) = \tan^{(2n+1)}(0) \text{ and}$$
$$\beta(2n) = \sec^{(2n)}(0)$$

for all $n \geq 0$.

Independently from Calabi's argument, R. Stanley obtained in [12] the tangent part by considering the polytopes in (1) and, as we mentioned in the introduction, N. D. Elkies ([6]) has derived the result for both the secant and tangent numbers starting from Calabi's idea.

Proposition 5.1. *Given $n \in \mathbb{N}$ and $\sigma \in S_{2n+1}$, $\Phi(\sigma)$ is a Dyck path if and only if σ is alternating and $\sigma(1) > \sigma(2)$.*

Proof. $\Phi(\sigma) \in \mathcal{D}_{2n,0}$ means that $\lambda_j = \pm 1$ for all j, so in

$$0, \sigma(1), \sigma(2), \sigma(3), \ldots, \sigma(2n), \sigma(2n+1), 0$$

or two or none of the neighbours of each j are $> j$. This is exactly as saying that σ is alternating and $\sigma(1) > \sigma(2)$ (as a consequence, $\sigma(2n) < \sigma(2n+1)$). □

Corollary 5.2. *For each $n \in \mathbb{N}$*

$$\tan^{(2n+1)}(0) = \sum_{\bar{\lambda} \in \mathcal{D}_{2n,0}} \nu(\bar{\lambda})\,. \qquad \square$$

Theorem 5.3. *Let $A = (a_{ij})$ the matrix given by*

$$a_{ij} = \begin{cases} j(j+1) & \text{if } i \geq j-1, \\ 0 & \text{if } i < j-1, \end{cases}$$

i.e.

$$A = \begin{pmatrix} 2 & 6 & 0 & 0 & 0 & \cdots \\ 2 & 6 & 12 & 0 & 0 & \cdots \\ 2 & 6 & 12 & 20 & 0 & \cdots \\ 2 & 6 & 12 & 20 & 30 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

The first column of the triangle generated by A is then the sequence of tangent numbers $\big(\tan^{(2n-1)}(0)\big)$.

Proof. By Proposition 4.4, for each $k \geq 0$

$$\nu(\bar{\lambda}, 1, -1_{k+1}) = (k+1)(k+2)\nu(1, -1_k)$$

whenever $(\bar{\lambda}, -1_k) \in \mathcal{D}_0$. We can use Theorem 3.1 with $f = \nu$, and this (together with Corollary 5.2) gives the result. □

Note that A is the product of the matrix in example 5 and (on the right) the diagonal infinite matrix $\big(j(j+1)\delta_{ij}\big)$.

The triangle begins

$$\begin{array}{ccccc} 1 \\ 2 & 6 \\ 16 & 48 & 72 \\ 272 & 816 & 1440 & 1440 \\ 7936 & 23808 & 44352 & 57600 & 43200 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

and, if $t_{n,-1} = 0$ for all n, it is given by

$$t_{n+1,m} = (m+1)(m+2) \sum_{k=m-1}^{n} t_{n,k}\,.$$

We can also use Corollary 5.2 in combination with Theorem 2.3:

Theorem 5.4. *Let $T(x)$ be the triangle generated by the infinite matrix*

$$A(x) = \begin{pmatrix} 2x & 1/x & 0 & 0 & 0 & 0 & \cdots \\ 2x & 0 & 4x & 0 & 0 & 0 & \cdots \\ 2x & 0 & 4x & 3/2x & 0 & 0 & \cdots \\ 2x & 0 & 4x & 0 & 6x & 0 & \cdots \\ 2x & 0 & 4x & 0 & 6x & 2/x & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

(so that $A(1)$ is the matrix of Corollary 4.5). Then the first element of the n-th row of $T(x)$ is a polynomial $P_n(x)$ of degree n such that $P_n(1) = (n+1)!$ and

$$P_n(0) = \tan^{(n+1)}(0).$$

Proof. Let $f\colon \mathcal{M} \to \mathbb{R}$ given by $f(\bar\lambda) = \nu(\bar\lambda) x^{z(\bar\lambda)}$, with $z(\bar\lambda)$ the number of 0's in $\bar\lambda$, as in Theorem 3.2. By Proposition 4.4 we have

$$f(\bar\lambda, 0, -1_k) = 2(k+1)\, x\, f(\bar\lambda, -1_k), \quad \text{and}$$
$$f(\bar\lambda, 1, -1_k) = \frac{k+1}{2x}\, f(\bar\lambda, 0, -1_{k-1}),$$

and by Theorem 2.3 the first element in the n-th row of $T(x)$ is

$$\sum_{\bar\lambda \in \mathcal{M}_n} f(\bar\lambda) = \sum_{\bar\lambda \in \mathcal{M}_n} \nu(\bar\lambda) x^{z(\bar\lambda)} = \sum_{k=0}^{n} \Big(\sum_{\bar\lambda \in \mathcal{D}_{n,k}} \nu(\bar\lambda) \Big)\, x^k =: P_n(x)\,.$$

If n is odd then $P_n(0) = 0$ since $\mathcal{D}_{n,0}$ is empty.

If n is even, we have seen in Corollary 5.2 that $P_n(0) = \tan^{(n+1)}(0)$. This holds for $n = 0$ too, since $P_0 = 1$.

Finally, $P_n(1) = (n+1)!$ as in Corollary 4.5. □

Remark. Of course, the matrix $A(x)$ is not defined for $x = 0$, but we certainly have that

$$\lim_{x\to 0} P_n(x) = \tan^{(n+1)}(0).$$

If x is positive and small enough then $\tan^{(n+1)}(0)$ will equal the floor function applied to $P_n(x)$. If we fix n and want to find $\tan^{(n+1)}(0)$ it suffices to take $x = 1/(n+1)!$: we have seen that

$$P_n(x) - \tan^{(n+1)}(0) = \sum_{k=1}^{n} \Big(\sum_{\bar\lambda \in \mathcal{D}_{n,k}} \nu(\bar\lambda) \Big) x^k\,,$$

so if $0 < x < 1$ and n is even

$$\begin{aligned} 0 < P_n(x) - \tan^{(n+1)}(0) &< x \sum_{k=1}^{n} \Big(\sum_{\bar\lambda \in \mathcal{D}_{n,k}} \nu(\bar\lambda) \Big) \\ &< x\, P_n(1) = x\,(n+1)!\,, \end{aligned}$$

and for $x = \dfrac{1}{(n+1)!}$ we have $0 < P_n(x) - \tan^{(n+1)}(0) < 1$.

Next we use Φ to prove the analogue of Theorem 5.3 for the secant (Euler) numbers.

If $\sigma \in S_{n+1}$, let $(\sigma, n+1) \in S_{n+1}$ the extension of σ, given by $\sigma(1), \sigma(2), \ldots, \sigma(n), n+1$.

Definition. For any $\bar\lambda \in \{-1, 0, 1\}^n$, let $\eta(\bar\lambda)$ be the number of permutations $\sigma \in S_n$ such that $\Phi(\sigma, n+1) = \bar\lambda$.

Note that $\eta(\bar\lambda) = 0$ if $\bar\lambda$ is not a Motzkin path, and that

$$\Phi(\sigma, n+1) = \begin{cases} (\Phi(\sigma), 0) & \text{if } n = \sigma(n)\,, \\ (\Phi(\sigma) + e_{\sigma(n)}, -1) & \text{if } n \neq \sigma(n)\,. \end{cases} \tag{7}$$

Similarly to Lemma 4.1 we have the following:

Lemma 5.5. *For each* $\bar\lambda \in \{-1, 0, 1\}^n$ *we have:*
(i) $\eta(\bar\lambda, 0) = \eta(\bar\lambda)$,

(ii) $\eta(\bar\lambda,-1) = 2\sum_{\lambda_j=0}\eta(\bar\lambda-e_j) + \sum_{\lambda_j=1}\eta(\bar\lambda-e_j)\,.$

Proof. (i) follows readily from (7).

(ii) Let $\Phi(\sigma,n+1)=\bar\lambda\in\mathcal{M}_n$. If we change the position of $n+1$ (in n different ways) we obtain a new $\tau\in S_{n+1}$. Let $\bar\mu=\Phi(\tau)\in\mathcal{M}_n$.

If $\lambda_{\sigma(n)}=0$, interchanging places of $n+1$ and $\sigma(n)$ we get $\bar\mu=\bar\lambda$, and $\Phi(\tau,n+2)=\bar\mu+e_{\sigma(n)}$.

If $\lambda_j=0$ with $j\neq\sigma(n)$, setting $n+1$ between j and its least neighbour we get $\mu_j=\lambda_j+1=1$, $\mu_{\sigma(n)}=\lambda_{\sigma(n)}-1$ and $\mu_i=\lambda_i$ for any other i. Then $\Phi(\tau,n+2)=\bar\lambda+e_j$.

If $\lambda_j=-1$, setting $n+1$ next to j (in two ways) we get $\mu_j=\lambda_j+1=0$ and $\mu_{\sigma(n)}=\lambda_{\sigma(n)}-1$, with $\Phi(\tau,n+2)=\bar\lambda+e_j$ too.

Conversely, $\Phi(\tau,n+2)=(\bar\lambda,-1)$ if and only if τ is obtained as before from some $\sigma\in S_n$, with $\Phi(\sigma,n+1)=\bar\lambda-e_j$ for some j, and there are two such σ if $\lambda_j=0$ and only one if $\lambda_j=-1$. □

Note that the lemma implies that $\eta(\bar\lambda)$ is positive for any Motzkin path λ.

Proposition 5.6. *For any* $\bar\lambda$*,*
(i) $\eta(\bar\lambda,0,-1_k)=(2k+1)\eta(\bar\lambda,-1_k)$ *for each* $k\geq 0$*, and*
(ii) $\eta(\bar\lambda,1,-1_k)=k^2\eta(\bar\lambda,-1_{k-1})$ *for each* $k\geq 1$.

Proof. It is inductively seen from the previous lemma, just the same as Proposition 4.4 followed from Lemma 4.1. □

Theorem 5.7. *Let* $A=(a_{ij})$ *the matrix given by*

$$a_{ij}=\begin{cases} j^2 & \text{if } i\geq j-1,\\ 0 & \text{if } i<j-1,\end{cases}$$

i.e.

$$A=\begin{pmatrix} 1 & 4 & 0 & 0 & 0 & \cdots\\ 1 & 4 & 9 & 0 & 0 & \cdots\\ 1 & 4 & 9 & 16 & 0 & \cdots\\ 1 & 4 & 9 & 16 & 25 & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

The first column of the triangle generated by A *is then the sequence of secant numbers* $\big(\sec^{(2n)}(0)\big)$.

Proof. Note that

$$\sec^{(2n)}(0)=\beta_{2n}=\sum_{\bar\lambda\in\mathcal{D}_{2n,0}}\eta(\bar\lambda)\,,$$

since $\Phi(\sigma,2n+1)\in\mathcal{D}_{2n,0}$ if and only if $\sigma\in S_{2n}$ is alternating and $\sigma(2)>\sigma(1)$, as we saw in Proposition 5.1.

The result follows then by the previous proposition and Theorem 3.1, taking as f the restriction of η to the set of Dyck paths. □

The new triangle begins

$$\begin{array}{ccccc} 1 \\ 1 & 4 \\ 5 & 20 & 36 \\ 61 & 244 & 504 & 576 \\ 1385 & 5540 & 11916 & 17280 & 14400 \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \end{array}$$

according to the recurrence (if $t_{n,-1} = 0$ for all n)

$$t_{n+1,m} = (m+1)^2 \sum_{k=m-1}^{n} t_{n,k}\,.$$

Note that $t_{n,n} = n!^2$. Following the proof of Theorem 3.1 we infer that $\eta(1_n, -1_n) = n!^2$ for all $n \in \mathbb{N}$. This is easily seen by the definition of η.

Finally we show the analogue of Theorem 5.4:

Theorem 5.8. *Let $T(x)$ be the triangle generated by the infinite matrix*

$$A(x) = \begin{pmatrix} x & 1/x & 0 & 0 & 0 & 0 & \cdots \\ x & 0 & 3x & 0 & 0 & 0 & \cdots \\ x & 0 & 3x & 4/3x & 0 & 0 & \cdots \\ x & 0 & 3x & 0 & 5x & 0 & \cdots \\ x & 0 & 3x & 0 & 5x & 9/5x & \cdots \\ \multicolumn{7}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \end{pmatrix}$$

i.e.

$$a_{ij}(x) = \begin{cases} (2j-1)\,x & \text{if } j \text{ is odd and } i \geq j-1\,, \\ \dfrac{(j/2)^2}{(j-1)x} & \text{if } j \text{ is even and } i = j-1\,. \end{cases}$$

Then the first element of the n-th row of $T(x)$ is a polynomial $P_n(x)$ of degree n such that $P_n(1) = n!$ and

$$P_n(0) = \sec^{(n)}(0).$$

Proof. Theorem 2.3 applies to $f(\bar\lambda) = \eta(\bar\lambda)x^{z(\bar\lambda)}$ (with $z(\bar\lambda)$ as in Theorem 5.4), since the recurrence seen for η easily implies

$$\eta(\bar\lambda, 1, -1_k) = \frac{k^2}{2k-1}\eta(\bar\lambda, 0, -1_{k-1})\,.$$

We get the matrix of the statement, and the first component of $\bar t_n(x)$ is

$$P_n(x) := \sum_{\bar\lambda \in \mathcal{M}_n} \eta(\bar\lambda)x^{z(\bar\lambda)} = \sum_{k=0}^{n} \Big(\sum_{\bar\lambda \in \mathcal{D}_{n,k}} \eta(\bar\lambda)\Big)\, x^k\,.$$

Then

$$P_n(1) = \sum_{k=0}^{n} \sum_{\bar\lambda \in \mathcal{D}_{n,k}} \eta(\bar\lambda) = \sum_{\bar\lambda \in \mathcal{M}_n} \eta(\bar\lambda) = n!\,, \text{ and}$$

$$P_n(0) = \sum_{\bar\lambda \in \mathcal{D}_{n,0}} \eta(\bar\lambda) = \sec^{(n)}(0)$$

as we have seen in the proof of Theorem 5.7. □

The same remark that follows Theorem 5.4 makes sense here.

References

[1] M. Aigner, *Motzkin numbers*, Europ. J. Combinatorics, 19 (1998), 663–675.
[2] E. Barcucci, R. Pinzani and R. Sprugnoli, *The Motzkin family*, Pure Math. and App., Ser. A, 2 (1991), 249–279.
[3] F. Beukers, J. A. C. Kolk and E. Calabi, *Sums of generalized harmonic series and volumes*, Nieuw Arch. Wisk. (4), 11 (1993), 217–224.
[4] E. Deutsch, *Dyck path enumeration*, Discrete Math., 204 (1999), 167–224.
[5] R. Donaghey and L. W. Shapiro, *Motzkin numbers*, J. Combin. Th. Ser. A, 23 (1977), 291–202.
[6] N. D. Elkies, *On the sums* $\sum_{k=-\infty}^{\infty}(4k+1)^{-n}$, preprint, avalaible in the World Wide Web at arXiv:math.CA/0101168 v2(2001).
[7] R. C. Entringer, *A combinatorial interpretation of the Euler and Bernoulli numbers*, Nieuw Arch. Wisk. (3), 14 (1966), 241–246.
[8] M. Gardner, *Catalan numbers: An integer sequence that materializes in unexpected places*, Sci. Amer., 234(1976), 120–125.
[9] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994.
[10] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford Science Publications, 1979.
[11] N. J. A. Sloanne, An On-line Version of the Encyclopedia of Integer Sequences, World Wide Web (www.research.att.com/~njas/sequences/Seis.html).
[12] R. Stanley, *Two poset polytopes*, Discrete and Computational Geometry, 1 (1986), 9–23.

José Luis Arregui Casaus
Departamento de Matemáticas
Facultad de Ciencias, Universidad de Zaragoza
50009 Zaragoza, Spain
E-mail address: arregui@posta.unizar.es