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Duality for classical orthogonal polynomials

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In memory of Joaquin Bustoz

Abstract

Some aspects of duality for the classical orthogonal polynomials are explained. Duality deals with the similarity of these functions as functions of the orthogonality variable and of the degree of the polynomials.

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1. Introduction

The classical orthogonal polynomials are Jacobi, Laguerre and Hermite polynomials, which are orthogonal with respect to the beta, gamma and normal distributions, respectively. They can be given in many forms. The most useful one here is as a hypergeometric series.

Jacobi polynomials are

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right), \tag{1.1}$$

where

$$(a)_k = \begin{cases} a(a+1)\cdots(a+k-1), & k=1,2,\ldots,\\ 1, & k=0, \end{cases} \tag{1.2}$$

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and

$$ {}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};t\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{t^k}{k!}. \tag{1.3} $$

Laguerre polynomials are

$$ L_n^{\alpha}(x)=\frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix};x\right). \tag{1.4} $$

Hermite polynomials are

$$ H_n(x)=(2x)^n\,{}_2F_0\left(\begin{matrix}-n/2,-(n-1)/2\\ -\end{matrix};-\frac{1}{x^2}\right). \tag{1.5} $$

The following identities are known:

$$ \sum_{n=0}^{\infty}L_n^{\alpha}(x)r^n=(1-r)^{-\alpha-1}\exp(-xr/(1-r)),\quad |r|<1, \tag{1.6} $$

$$ L_n^{\beta}(x)=\sum_{k=0}^{n}\frac{(\beta-\alpha)_{n-k}}{(n-k)!}L_k^{\alpha}(x), \tag{1.7} $$

$$ x^{\beta}\frac{L_n^{\beta}(x)}{L_n^{\beta}(0)}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}\int_0^x(x-y)^{\beta-\alpha-1}y^{\alpha}\frac{L_n^{\alpha}(y)}{L_n^{\alpha}(0)}\,\mathrm{d}y, \tag{1.8} $$

$$ P_n^{(a,b)}(x)=\sum_{k=0}^{n}g(n,k)P_k^{(\alpha,\beta)}(x), \tag{1.9} $$

with

$$ \begin{aligned} g(n,k)=&\frac{\Gamma(n+\alpha+1)\Gamma(k+\alpha+\beta+1)\Gamma(n+k+a+b+1)}{\Gamma(k+\alpha+1)\Gamma(2k+\alpha+\beta+1)\Gamma(n+a+b+1)(n-k)!}\\ &\cdot{}_3F_2\left(\begin{matrix}k-n,n+k+a+b+1,k+\alpha+1\\ k+a+1,2k+\alpha+\beta+2\end{matrix};1\right), \end{aligned} \tag{1.10} $$

$$ (1-x)^{\alpha+\mu}\frac{P_n^{(\alpha+\mu,\beta-\mu)}(x)}{P_n^{(\alpha+\mu,\beta-\mu)}(1)}=\frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)}\cdot\int_x^1(1-y)^{\alpha}\frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)}(y-x)^{\mu-1}\,\mathrm{d}y, \tag{1.11} $$

$$ \begin{aligned} (1+x)^{n+\alpha+\beta}P_n^{(\alpha-\mu,\beta)}(x)=&\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1-\mu)\Gamma(\mu)}\\ &\cdot\int_{-1}^{x}(1+y)^{n+\alpha+\beta-\mu}P_n^{(\alpha,\beta)}(y)(x-y)^{\mu-1}\,\mathrm{d}y, \end{aligned} \tag{1.12} $$

$$\frac{(1-x)^{\alpha+\mu}}{(1+x)^{n+\alpha+1}}\frac{P_n^{(\alpha+\mu,\beta)}(x)}{P_n^{(\alpha+\mu,\beta)}(1)}=\frac{2^\mu\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)}\cdot\int_x^1\frac{(1-y)^\alpha}{(1+y)^{n+\alpha+\mu+1}}\frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)}(y-x)^{\mu-1}\,\mathrm{d}y. \tag{1.13}$$

All of these formulas hold for the obvious conditions on the variables and parameters. Integrals must exist and that is the only unstated restriction. See [2] for a derivation of the Jacobi polynomial formulas. One just uses the series representation and the evaluation of a beta integral along with one of the standard transformations of a ${}_2F_1$ to obtain the integral identities. The generating function (1.6) can be found in many books, say in [5]; the sum (1.7) is an immediate consequence of (1.6). Series (1.9) has also been derived a number of times. See [1] for one way to obtain this identity.

Other orthogonal polynomials will be used to help explain some of these formulas. Meixner polynomials are

$$M_n(x;\beta,c)={}_2F_1\left(\begin{matrix}-n,-x\\ \beta\end{matrix};1-\frac{1}{c}\right). \tag{1.14}$$

Hahn polynomials are

$$Q_n(x;\alpha,\beta,N)={}_3F_2\left(\begin{matrix}-n,n+\alpha+\beta+1,-x\\ \alpha+1,-N\end{matrix};1\right), \tag{1.15}$$

when $x, n=0,1,\dots,N$, and dual Hahn polynomials are

$$R_n(\lambda(x);\alpha,\beta,N)={}_3F_2\left(\begin{matrix}-n,-x,x+\alpha+\beta+1\\ \alpha+1,-N\end{matrix};1\right), \tag{1.16}$$

when $\lambda(x)=x(x+\alpha+\beta+1)$ and $x,n=0,1,\dots,N$.

The series in (1.16) are considered as sums from $k=0$, to $k=n$, and since $n\leqslant N$ this avoids the problem of dividing by the zero caused by $(-N)_k$ when $k>N$. The restriction that $x=0,1,\dots,N$ is made to keep the series from restarting when $k>N$, as it would in certain identities.

See [4] for some of the basic formulas for all of these polynomials.

2. Laguerre and Meixner polynomials

Identities (1.7) and (1.8) share a common property. In (1.7) the sum is on $k=0,1,\dots,n$, rather than $k=0,1,\dots$. This is obvious since the left-hand side is a polynomial of degree n, and the right-hand side is a sum of polynomials of degree k, so the sum must stop at $k=n$. The integral in (1.8) is on $0\leqslant y\leqslant x$ rather than $0\leqslant y<\infty$, and it is not obvious that this must happen, so it should be explained. One explanation is that this is what the formula is, but as we will see when we get to Jacobi polynomials, this is not a completely satisfactory explanation. A better explanation comes from Meixner polynomials.

Meixner polynomials satisfy the following generating function.

$$\sum_{n=0}^\infty\frac{(\beta)_n}{n!}M_n(x;\beta,c)r^n=(1-r)^{-\beta-x}\left(1-\frac{r}{c}\right)^x. \tag{2.1}$$

To derive (2.1), use (1.14), reverse the order of summation, and sum the inner series by the binomial theorem. From (2.1), it follows immediately that

$$\frac{(\beta)_n}{n!}M_n(x;\beta,c)=\sum_{k=0}^{n}\frac{(\beta-\alpha)_{n-k}}{(n-k)!}\frac{(\alpha)_k}{k!}M_k(x;\alpha,c). \tag{2.2}$$

One can obtain (1.7) as a limiting result from (2.2), using

$$L_n^{\beta}(x)=\frac{(\beta+1)_n}{n!}\lim_{c\to 1} M_n\left(\frac{x}{(1-c)};\beta+1,c\right). \tag{2.3}$$

For (2.2), just as for (1.7), it is clear that the sum in (2.2) stops when $k=n$.

It is not as obvious how to obtain (1.8) as a limit from (2.2), but it is possible. First rewrite (2.2) as

$$\frac{(\beta+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,-m\\ \beta+1\end{matrix};\frac{c-1}{c}\right)=\sum_{k=0}^{n}\frac{(\beta-\alpha)_{n-k}}{(n-k)!}\frac{(\alpha+1)_k}{k!}\,{}_2F_1\left(\begin{matrix}-k,-m\\ \alpha+1\end{matrix};\frac{c-1}{c}\right). \tag{2.4}$$

Next replace n by $x/(1-c)$, and on the right consider k as $y/(1-c)$, so y runs through a uniformly distributed discrete set of points on $[0,x]$.

Use the limit

$$\lim_{n\to\infty}\frac{(\beta+1)_n}{n!}n^{-\beta}=\frac{1}{\Gamma(\beta+1)}. \tag{2.5}$$

Multiply (2.4) by $(1-c)^{\beta}$ and let $c\to 1$, or $n\to\infty$. The left-hand side of (2.4) is

$$\frac{y^{\beta}}{\Gamma(\beta+1)}\,{}_1F_1\left(\begin{matrix}-m\\ \beta+1\end{matrix};y\right). \tag{2.6}$$

On the right-hand side we have $1/\Gamma(\beta-\alpha)\Gamma(\alpha+1)$ times

$$\begin{aligned}&\lim_{c\to 1}(1-c)^{\beta}\sum_{x}\left(\frac{y-x}{1-c}\right)^{\beta-\alpha-1}\left(\frac{x}{1-c}\right)^{\alpha}{}_2F\left(\begin{matrix}-x/(1-c),-m\\ \alpha+1\end{matrix};\frac{c-1}{c}\right)\\ &=\int_0^y (y-x)^{\beta-\alpha-1}x^{\alpha}_1F_1\left(\begin{matrix}-m\\ \alpha+1\end{matrix};x\right)\mathrm{d}x.\end{aligned} \tag{2.7}$$

Formula (1.8) then follows from (2.6) and (2.7).

Laguerre polynomials satisfy the differential equation

$$xy''+(\alpha+1-x)y'+ny=0. \tag{2.8}$$

The three-term recurrence relation for Laguerre polynomials is

$$-xL_n^{\alpha}(x)=(n+1)L_{n+1}^{\alpha}(x)-(2n+\alpha+1)L_n^{\alpha}(x)+(n+\alpha)L_{n-1}^{\alpha}(x). \tag{2.9}$$

These two equations are dual to each other. One interesting question is whether one can show that the kernel in the integral representation (1.8) is supported on $0\leqslant x\leqslant y$ using the differential equation (2.9). The interest in this question is more important in the case of Jacobi polynomials, since there are more formulas for Jacobi polynomials with the integration kernel supported on a smaller set and not all of them currently have explanations similar to the one above.

3. Jacobi polynomials

Jacobi polynomials satisfy the differential equation

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0. \tag{3.1}$$

All sets of polynomials which are orthogonal with respect to a positive measure on the real line satisfy

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \tag{3.2}$$

a_n, b_n, c_n real, $a_{n-1}c_n > 0$, $n = 1, 2, \ldots$.

The positivity of $a_{n-1}\ c_n$ holds for $n = 1, 2, \ldots,$ when the measure has infinitely many points of support. When there are only finitely many mass points as the full set of support, the positivity holds for finitely many values of n.

For Hahn polynomials, one has

$$Q_n(x; a, b, N) = \sum_{k=0}^{n} C(k, n) Q_k(x; \alpha, \beta, N), \tag{3.3}$$

$n = 0, 1, \ldots, N$. This can be derived directly and as above, get (1.9) from it, or derived by integrating (1.9) with respect to a beta distribution.

To obtain (1.11) with the kernel supported on $[x, 1]$ from another formula which just involves polynomials written as the sum of other polynomials it suffices to argue as in Section 2. Recall that the argument involved interchanging n and x in (2.2). When this is done for Hahn polynomials, we get dual Hahn polynomials (1.16) and the polynomial variable is $\lambda(x) = x(x+\alpha+\beta+1)$. To have an expansion

$$R_n(\mu(x); a, b, N) = \sum_{k=0}^{n} a(k, n) R_k(\lambda(x); \alpha, \beta, N), \tag{3.4}$$

with $\mu(x) = x(x+a+b+1)$, $\lambda(x) = x(x+\alpha+\beta+1)$, we need to have $a+b = \alpha+\beta$. Otherwise, the two sides are not polynomials in the same variable and the sum will go from $k=0$ to $k=N$ and the identity will only hold for $x = 0, 1, \ldots, N$.

To find the coefficients in (3.4) when $a+b=\alpha+\beta$, it is easier to work them out directly than to search the literature. An equivalent way of writing (2.2) is

$$\frac{(a)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, -x\\ a\end{matrix}; t\right) = \sum_{k=0}^{n} \frac{(a-\alpha)_{n-k}}{(n-k)!}\frac{(\alpha)_k}{k!}\,{}_2F_1\left(\begin{matrix}-k, -x\\ \alpha\end{matrix}; t\right). \tag{3.5}$$

Multiply both sides of (3.5) by $t^{b-1}(1-t)^{c-b-1}$ and integrate on [0, 1]. The result is

$$\frac{(a)_n}{n!}\,{}_3F_2\left(\begin{matrix}-n, -x, b\\ a, c\end{matrix}; 1\right) = \sum_{k=0}^{n} \frac{(a-\alpha)_{n-k}}{(n-k)!}\frac{(\alpha)_k}{k!}\,{}_3F_2\left(\begin{matrix}-k, -x, b\\ \alpha, c\end{matrix}; 1\right). \tag{3.6}$$

Since both sides of (3.6) are polynomials in x, we can relax the restriction on b and c needed for integration.

Take $b=x+\alpha+\beta+1$ and $c=-N$, getting

$$\frac{(\alpha+\mu)_n}{n!}R_n(\lambda(x);\alpha+\mu,\beta-\mu,N)=\sum_{k=0}^{n}\frac{(\mu)_{n-k}}{(n-k)!}\frac{(\alpha)_k}{k!}R_k(\lambda(x);\alpha,\beta,N). \tag{3.7}$$

To derive (1.11) from (3.7), we can argue as above. Let $x=m$, and take the same type of limit with $n=xN$, $k=yN$, and then let N go to infinity.

Formula (1.11) is useful, but formula (1.13) is not very useful as it stands. What is useful is an integral with a nonnegative kernel which does not depend on n, i.e.

$$\frac{P_n^{(\alpha+\mu,\beta)}(x)}{P_n^{(\alpha+\mu,\beta)}(1)}=\int_{-1}^{1}K_\mu(x,y)\frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)}\,\mathrm{d}y \tag{3.8}$$

with $K_\mu(x,y)\geqslant 0$, $-1\leqslant x,\ y\leqslant 1$, with the possibility that sets of measure zero can be removed from this square.

Here is one way to show that (3.8) follows from (1.13). Let

$$f(x)=\sum_{n=0}^{\infty}a_nP_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1), \tag{3.9}$$

and

$$g(x)=\sum_{k=0}^{\infty}a_nP_n^{(\alpha+\mu,\beta)}(x)/P_n^{(\alpha+\mu,\beta+\mu)}(1). \tag{3.10}$$

By orthogonality

$$a_n=\frac{P_n^{(\alpha,\beta)}(1)\int_{-1}^{1}f(t)P_n^{(\alpha,\beta)}(t)(1-t)^\alpha(1-t)^\beta\,\mathrm{d}t}{\int_{-1}^{1}[P_n^{(\alpha,\beta)}(t)]^2(1-t)^\alpha(1+t)^\beta\mathrm{d}t}. \tag{3.11}$$

Use (1.13) and (3.11) in (3.10) to get

$$g(x)=\int_{-1}^{1}\int_{x}^{1}f(z)K(x,t,z)\,\mathrm{d}y(1-z)^\alpha(1+z)^\beta\,\mathrm{d}z, \tag{3.12}$$

with

$$K(x,t,z)=\sum_{n=0}^{\infty}G_n(x,t)P_n^{(\alpha,\beta)}(t)P_n^{(\alpha,\beta)}(z)\Bigg/\int_{-1}^{1}[P_n^{(\alpha,\beta)}(x)]^2(1-x)^\alpha(1+x)^\beta\,\mathrm{d}x, \tag{3.13}$$

and

$$G_n(x,t)=\frac{2^\mu\Gamma(\alpha+\mu+1)(1-t)^\alpha(1+x)^{n+\alpha+1}(t-x)^{\mu-1}}{\Gamma(\alpha+1)\Gamma(\mu)(1-x)^{\alpha+\mu}(1+t)^{n+\alpha+\mu+1}}. \tag{3.14}$$

In [3] it was shown that

$$\sum_{n=0}^{\infty}r^nP_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)\Bigg/\int_{-1}^{1}[P_n^{(\alpha,\beta)}(x)]^2(1-x)^\alpha(1+x)^\beta\,\mathrm{d}x>0, \tag{3.15}$$

when $-1 \leqslant x, y \leqslant 1$, $0 \leqslant r < 1$. Thus,

$$g(x) = \int_{-1}^{1} f(z) K_\mu(x, z)(1-z)^\alpha (1+z)^\beta \, dz, \tag{3.16}$$

with $K_\mu(x, z) \geqslant 0$ when $\mu > 0$. When $a_k = 0$, $k \neq n$, $a_n = 1$, we have (3.8).

I do not know how to find the coefficients in the extension of (3.7) when $a + b \neq \alpha + \beta$ in a nice enough form which allows us to obtain either (1.12) or (1.13) from a polynomial sum identity.

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