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Restricted permutations

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Abstract

Restricted permutations are those constrained by having to avoid subsequences ordered in various prescribed ways. They have functioned as a convenient descriptor for several sets of permutations which arise naturally in combinatorics and computer science. We study the partial order on permutations that underlies the idea of restriction and which gives rise to sets of sequences closed under taking subsequences. In applications, the question of whether a closed set has a finite 'basis' is often considered. Several constructions that respect the finite basis property are given. A family of closed sets, called profile-closed sets, is introduced and used to solve some instances of the inverse problem: describing a closed set from its basis. Some enumeration results are also given. © 1999 Elsevier Science B.V. All rights reserved

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1. General setting

The study of permutations which are constrained by not having one or more subsequences ordered in various prescribed ways has been motivated both by its combinatorial difficulty and by its appearance in some data structuring problems in Computer Science. The fundamental relation that underpins this study is *involvement* which captures the idea of one sequence being ordered in the same way as a subsequence of another. Two numerical sequences $\pi = [p_1, p_2, \ldots, p_m]$ and $\rho = [r_1, r_2, \ldots, r_m]$ of the same length are said to be *order isomorphic* if, for all i, j, $p_i < p_j$ if and only if $r_i < r_j$. Order isomorphism is clearly an equivalence relation on sequences. Throughout this paper we shall consider only sequences of distinct elements. Every such sequence of length n is order isomorphic to a unique permutation of $1, 2, \ldots, n$ and, for this reason, most of our results are stated for permutations. Unless otherwise stated 'permutation' will always

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mean an arrangement of $1,2,\ldots,n$ for some n. Generally, sequences will be denoted by Greek letters and their elements by Roman letters.

If π and σ are sequences then π is said to be *involved* in σ if π is order isomorphic to a subsequence ρ of σ; we write $\pi \preccurlyeq \sigma$. For example, $[2,3,1,4] \preccurlyeq [6,3,5,7,2,4,1,8]$ because of the subsequence $[3,5,2,8]$ in the second permutation. For permutations on a small number of symbols it is often convenient to omit the brackets and commas and write $2314 \preccurlyeq 63572418$.

A map α from $\{1,\ldots,m\}$ to $\{1,\ldots,n\}$ is said to be *monotonic* if $\alpha(i)<\alpha(j)$ whenever $i<j$. Monotonic maps allow us to describe the terms 'subsequence' and 'order isomorphism' using functional composition (which we apply from left to right). Suppose that π and σ are permutations. A sequence of positive integers is order isomorphic to π if and only if it has the form $\pi\alpha$ where α is a monotonic map. Furthermore, a sequence is a subsequence of σ if and only if it has the form $\beta\sigma$ with β a monotonic map. In particular, $\pi \preccurlyeq \sigma$ if and only if there exist monotonic maps α, β such that $\pi\alpha = \beta\sigma$

A set $\mathscr{X}$ of permutations is said to be *closed* if, whenever $\sigma \in \mathscr{X}$ and $\pi \preccurlyeq \sigma$, then $\pi \in \mathscr{X}$. Closed sets are the principal object of study in this paper. Many natural sets of permutations are closed (some examples are given below) and a structure theory of closed sets would have many consequences. The beginnings of such a theory are given in Section 2 but much remains to be done.

The archetypal example of a closed set is the set of *stack sortable* permutations. A sequence is stack sortable if, when it is presented as input to a stack and subjected to an appropriate series of 'push' and 'pop' operations, the stack can produce the elements in ascending order. It is evident that if a sequence is stack sortable then so is any sequence order isomorphic to it and also any subsequence. In particular, if σ is a stack sortable permutation and $\pi \preccurlyeq \sigma$ then π is also stack sortable.

Stack sortable permutations were first studied in [8] where two results were proved which have continued to inspire the study of closed sets. The first is that a permutation is stack sortable if and only if it does not involve the permutation 231. The second is that the number of stack sortable permutations of length n is $\binom{2n}{n}/(n+1)$. The first of these results motivates the definition of the 'basis' of a closed set below and allows several combinatorial results in the literature to be described uniformly. We shall survey some of these below and give some new results in the next section. The second result has been generalised to a number of other closed sets and we shall present some further results in Section 3. At this point however it is convenient to introduce the terminology $\mathscr{X}_n$ to denote the subset of $\mathscr{X}$ whose permutations have length n.

If $\mathscr{X}$ is closed let $\mathscr{X}^\star$ denote the set of permutations, minimal with respect to $\preccurlyeq$, that do not belong to $\mathscr{X}$. In turn, $\mathscr{X}^\star$ determines $\mathscr{X}$ as $\{\alpha|\beta \not\preccurlyeq \alpha \text{ for all } \beta \in \mathscr{X}^\star\}$. The set $\mathscr{X}^\star$ is called the *basis* of $\mathscr{X}$. In this terminology the set of stack sortable permutations has the basis $\{231\}$.

Many natural closed sets of permutations $\mathscr{X}^\star$ have a very simple basis. For example:

- If $\mathscr{X}$ is the set of permutations that can be sorted by a restricted input deque then $\mathscr{X}^\star = \{4231, 3241\}$ [8,11,15].

- If $\mathscr{X}$ is the set of permutations that can be expressed as the interleaving of two increasing subsequences then $\mathscr{X}^\star = \{321\}$ [8].
- If $\mathscr{X}$ is the set of permutations that can be expressed as the interleaving of an increasing subsequence and a decreasing subsequence then $\mathscr{X}^\star = \{3412, 2143\}$ (see [7,15]).
- If $\mathscr{X}$ is the set of permutations that can be obtained by a 'riffle' shuffle of a deck of cards $1, 2, \ldots, n$ then $\mathscr{X}^\star = \{321, 2143, 2413\}$ (see the proof of Proposition 3.4 below).
- If $\mathscr{X}$ is the set of all 'separable' permutations [6] then $\mathscr{X}^\star = \{3142, 2413\}$ (see also [13] where these permutations are considered in the context of 'bootstrap percolation').

However, there are also many closed sets whose basis is not simple to describe nor even finite; examples of closed sets with an infinite basis are given in [17,11]. The converse problem of describing the closed set defined by a given basis $\mathscr{B}$ has also attracted some study; we call this closed set $\mathscr{A}(\mathscr{B})$, the letter $\mathscr{A}$ recalling that $\mathscr{A}(\mathscr{B})$ is the set of permutations that avoid $\mathscr{B}$ ($\mathscr{A}(\{1\})$ is empty, $\mathscr{A}(\{21\})$ consists only of identity permutations, etc.). In [14] Simion and Schmidt gave complete descriptions of closed sets whose bases consist of sets of permutations of length 3. Bóna [2,3], West [18] and Stankova [15,16] have begun the study of bases comprising permutations of length 4 but this is still very incomplete.

Another theme running through the above works is enumeration: finding the number of permutations of each length in a closed set. We let $\mathscr{A}_n(\mathscr{B})$ be the set of permutations in $\mathscr{A}(\mathscr{B})$ of length n. Occasionally it is necessary to consider the permutations of length n of some set other than $\{1, 2, \ldots, n\}$ which avoid $\mathscr{B}$ but this set has the same size as $\mathscr{A}_n(\mathscr{B})$.

In all this work it is very useful to take advantage of some natural symmetries based on the following facts (which were first made explicit in [14]). If σ is any permutation on $\{1, 2, \ldots, n\}$, let $\bar{\sigma}$ and $\sigma^\star$, respectively, denote the permutations obtained from σ by replacing every element s_i by $n + 1 - s_i$ and reversing the elements of σ. Also, as usual, let σ^{-1} denote the permutation inverse of σ. Then

1. If $\pi \preccurlyeq \sigma$ then $\bar{\pi} \preccurlyeq \bar{\sigma}$
2. If $\pi \preccurlyeq \sigma$ then $\pi^\star \preccurlyeq \sigma^\star$
3. If $\pi \preccurlyeq \sigma$ then $\pi^{-1} \preccurlyeq \sigma^{-1}$

These 3 symmetries generate the dihedral group D of order 8. It acts in a natural way on sets of permutations. As a direct consequence of the definitions we have:

Lemma 1.1. *If λ is any element of the symmetry group D and $\mathscr{X}$ is any closed set of permutations with basis $\mathscr{X}^\star$ then $\lambda(\mathscr{X})$ is closed and has basis $\lambda(\mathscr{X}^\star)$. Furthermore, $|\mathscr{X}_n| = |\lambda(\mathscr{X})_n|$ for all n.*

As an example of the power of this lemma consider the problem of finding $\mathscr{A}(\sigma)$ when σ has length 4. Although there are 24 such problems they fall into 7 symmetry classes under the action of D. According to the lemma $|\mathscr{A}_n(\sigma)| = |\mathscr{A}_n(\tau)|$ whenever

σ and τ are equivalent under D. Mysteriously, this equation sometimes holds when σ and τ are not equivalent. Some reasons for this are given in [15,16,18,19] which use generating trees but these do not furnish a complete explanation. There are other equalities of this sort also [14,19]; again, generating trees explain these in some cases.

In Section 2 of the paper we give some constructions and results for combining closed sets. We follow this with a discussion of a large family of closed sets each of which has a finite basis and use them to solve a problem on riffle shuffles. In Section 3 we consider closed sets where the basis consists of a permutation of length 3 and a permutation of length 4. West [19] has reported on some enumeration results for this problem, omitting most of the proofs. Our results confirm his (and are available electronically [1]). Here we give only some new results obtained by the elementary structure theory in Section 2 but they suggest how effective a more general structure theory would be.

2. Some finitely based sets

2.1. Constructions

There are several ways in which one or more closed sets can give rise to another closed set. This subsection reviews some constructions which respect the finite basis property.

Theorem 2.1. *Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are closed sets. Then $\mathscr{X}\cap\mathscr{Y}$ and $\mathscr{X}\cup\mathscr{Y}$ are also closed. Moreover, if $\mathscr{X}$ and $\mathscr{Y}$ each have a finite basis then both $\mathscr{X}\cap\mathscr{Y}$ and $\mathscr{X}\cup\mathscr{Y}$ have a finite basis.*

Proof. That $\mathscr{X}\cap\mathscr{Y}$ and $\mathscr{X}\cup\mathscr{Y}$ are closed follows directly from the definitions. Now suppose that $\mathscr{X}=\mathscr{A}(S)$ and $\mathscr{Y}=\mathscr{A}(T)$ for finite sets S and T. Since, obviously, $\mathscr{X}\cap\mathscr{Y}=\mathscr{A}(S\cup T)$ it follows that $\mathscr{X}\cap\mathscr{Y}$ has a finite basis.

Finally consider a permutation α in the basis of $\mathscr{X}\cup\mathscr{Y}$. Such a permutation belongs neither to $\mathscr{X}$ nor to $\mathscr{Y}$ and so has subsequences σ and τ which are order isomorphic to permutations in S and T, respectively. However, α is minimal and so no proper subsequence also has this property. Thus, α must be the union of σ and τ and so has bounded length. Therefore there are only finitely many possibilities for α. □

Theorem 2.2. *Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are closed sets each with a finite basis. Let $[\mathscr{X},\mathscr{Y}]$ be the set of all permutations which are concatenations $\sigma\tau$ where σ is order isomorphic to a permutation in $\mathscr{X}$ and τ is order isomorphic to a permutation in $\mathscr{Y}$. Then $[\mathscr{X},\mathscr{Y}]$ is closed. Moreover if $\mathscr{X}$ and $\mathscr{Y}$ are each finitely based then so is $[\mathscr{X},\mathscr{Y}]$.*

Proof. It is evident that $[\mathscr{X},\mathscr{Y}]$ is closed. Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are each finitely based and that α is a permutation in the basis of $[\mathscr{X},\mathscr{Y}]$. Let $\alpha=\sigma\tau k$ where k is the last symbol of α and where (since α is minimal with respect to not belonging to $[\mathscr{X},\mathscr{Y}]$)

we may presume that σ and τ are order isomorphic to permutations of $\mathscr{X}$ and $\mathscr{Y}$, respectively. Among all such decompositions for α choose the one with σ of maximal length. Then, if t is the first symbol of τ, σt is not order isomorphic to a permutation in $\mathscr{X}$ and τk is not order isomorphic to a permutation in $\mathscr{Y}$.

It follows that σt has a subsequence $\sigma' t$ order isomorphic to a permutation in the basis S of $\mathscr{X}$ and τk has a subsequence $\tau' k$ order isomorphic to a permutation in the basis T of $\mathscr{Y}$. But then the subsequence $\sigma' t\tau' k$ (or $\sigma'\tau' k$ if t is a symbol of τ') of α cannot be order isomorphic to a permutation of $[\mathscr{X},\mathscr{Y}]$ and, by minimality of α, must be α itself. Since $\sigma' t$ and $\tau' k$ are bounded in length (since S and T are finite), the length of α is also bounded. □

Noonan [9,10] and Bóna [4,5] have investigated classes of permutations which are allowed to involve a finite set permutations (particularly 123 or 132) but only a limited number of times. In general, given a finite list $\alpha_1,\dots,\alpha_k$ of permutations and a list $m_1,\dots,m_k$ of non-negative integers, we let

$$\mathscr{Y}(\alpha_1,\dots,\alpha_k,m_1,\dots,m_k)$$

denote the set of permutations σ which involve each α_i at most m_i times (i.e. σ has at most m_i subsequences order isomorphic to α_i), and we let

$$\mathscr{Z}(\alpha_1,\dots,\alpha_k,m_1,\dots,m_k)$$

denote the set of permutations σ which involve each α_i exactly m_i times. Notice that

$$\mathscr{Y}(\alpha_1,\dots,\alpha_k,m_1,\dots,m_k)=\bigcup_{p_i\leqslant m_i}\mathscr{Z}(\alpha_1,\dots,\alpha_k,p_1,\dots,p_k).$$

It is conjectured in [10] that all $\mathscr{Z}$-type sets have P-recursive enumeration sequences. By inclusion–exclusion this is equivalent to all $\mathscr{Y}$-type sets having P-recursive enumeration sequences. The following theorem shows that this conjecture would follow from the (apparently weaker) conjecture of Gessel that all finitely based closed sets have P-recursive enumeration sequences.

Theorem 2.3. *$\mathscr{Y}(\alpha_1,\dots,\alpha_k,m_1,\dots,m_k)$ is closed and finitely based.*

Proof. Obviously, $\mathscr{Y}(\alpha_1,\dots,\alpha_k,m_1,\dots,m_k)$ is closed. Let π be a basis permutation. Then $\pi\notin\mathscr{Y}(\alpha_1,\dots,\alpha_k,m_1,\dots,m_k)$ and so, for some i, π has at least m_i+1 subsequences order isomorphic to α_i. But the union of these m_i+1 subsequences is a subsequence of π also not in $\mathscr{Y}(\alpha_1,\dots,\alpha_k,m_1,\dots,m_k)$; therefore, as π is minimal, this union must be π itself. Thus $|\pi|$ is bounded. □

All the theorems of this subsection are, in principle, constructive. For example, the method of the last theorem gives that $\mathscr{Y}(123,1)$ (see [9]) has basis $\{1234,1243,1324,2134,14523,34125,351624,356124,451623,456123\}$.

2.2. Profile classes

If A and B are sets or sequences we write $A<B$ to denote that $a<b$ for all $a\in A, b\in B$. As a first use of this notation we define the *profile* of a permutation. If ρ and π are permutations then ρ is said to have profile $\pi=[p_1\dots p_m]$ if ρ has a partition into segments $\rho=\rho_1\dots\rho_m$ where m is minimal subject to

1. each ρ_i is a non-empty sequence of increasing consecutive symbols,
2. $\rho_i<\rho_j$ if and only if $p_i<p_j$

For example, 34597812 has profile 2431 because of its segments 345, 9, 78, 12. Clearly, a permutation determines its profile uniquely. Not every permutation can be a profile however; to be a profile the permutation must not contain any segment $t, t+1$.

Lemma 2.4. *If π is a valid profile and has length m then the number of permutations of length n which have profile π is $\binom{n-1}{m-1}$.*

Proof. If ρ is a permutation with profile π (by way of a decomposition $\rho=\rho_1\dots\rho_m$) then ρ is determined by the lengths of the ρ_i, i.e. by an ordered set of m positive integers whose sum is n. Since every such composition of n can arise in this way and there are $\binom{n-1}{m-1}$ such compositions the result follows. □

We define a set Σ of permutations to be *profile-closed* if all its members are valid profiles and, whenever β is a valid profile with $\beta\preccurlyeq\alpha\in\Sigma$, then $\beta\in\Sigma$. The *profile closure* of a set of profiles is defined to be the smallest profile-closed set containing it. As an example, the profile closure of $\{2431\}$ is the profile-closed set $\{2431, 132, 321, 21, 1\}$.

Theorem 2.5. *If Σ is a profile-closed set of permutations then $P(\Sigma)$, the set of permutations whose profile lies in Σ, is closed. Furthermore, if Σ is finite then $P(\Sigma)$ has a finite basis.*

Proof. It follows from the definitions that, if ρ has profile π and $\lambda\preccurlyeq\rho$, then λ has profile μ where $\mu\preccurlyeq\pi$. This proves the first part. For the second part let β be a permutation on $1,2,\dots,m$ in the basis of $P(\Sigma)$. Suppose that β has two adjacent consecutive symbols $t, t+1$; then, β and $\beta-t$ have the same profile. However, $\beta-t$ is order isomorphic to a permutation in $P(\Sigma)$ and so its profile lies in Σ. Thus $\beta\in P(\Sigma)$ which is impossible. Hence no two adjacent symbols of β can be consecutive.

The permutation $\beta-m$ can have at most two adjacent consecutive symbols (which, in β, were separated by m) and so $\beta-m$ has length at most 1 more than the length of its profile. But $\beta-m\in P(\Sigma)$ and so its profile lies in Σ. Therefore the length of β is bounded and the proof is complete. □

We shall appeal to these results in the next section. They may be generalised in several ways. We can, of course, consider profiles based on decreasing segments rather than increasing segments. More interestingly, we can consider profiles where segments are allowed to be both increasing and decreasing; a similar finite basis result can be proved. We can also consider permutations with a profile where one or more of the increasing segments is of bounded length. In particular, in the next section we require, at one point, profiles where one of the segments has length 0 or 1; we shall show this by a superscript ^{1}; so, for example, permutations with the (generalised) profile 13^12 would be structured as $[1,2,\dots,k,n,k+1,\dots,n-1]$ for some k.

2.3. *Riffle shuffles*

We have already mentioned, in Section 1, the closed set of permutations obtained by a standard riffle shuffle of a deck of n cards. These riffle shuffle permutations are, of course, just merges of cards $1,2,\dots,m$ (for some m) and cards $m+1,\dots,n$. More generally we wish to consider S_r the set of r-shuffles which are defined by cutting a deck into r sections and interleaving these sections in any way. The inverse of an r-shuffle π is, by definition, an ordering of the deck of cards from which the r-shuffle π could restore the deck to its original order.

Lemma 2.6. *A permutation π of length n is an r-shuffle if and only if there exist partitions $\bigcup_{k=1}^{r} A_k$ and $\bigcup_{k=1}^{r} I_k$ of $\{1,\dots,n\}$ such that*

1. *$I_k < I_{k+1}$ for all k,*
2. *$\pi(A_k) = I_k$ for all k,*
3. *$\pi|_{A_k}$ is monotonic increasing for all k.*

Proof. An r-shuffle begins by dividing $\{1,\dots,n\}$ into segments $I_1,\dots,I_r$ satisfying property 1. When the segments are interleaved each set I_k is distributed, without disturbing its order, into a set of positions A_k of the resulting permutation π and therefore conditions 2 and 3 hold. The converse is clear. $\square$

An immediate consequence of this lemma is a corresponding characterisation of the inverses of shuffles.

Lemma 2.7. *A permutation π of length n is the inverse of a t-shuffle if and only if there exist partitions $\bigcup_{k=1}^{t} B_k$ and $\bigcup_{k=1}^{t} J_k$ of $\{1,\dots,n\}$ such that*

1. *$J_k < J_{k+1}$ for all k,*
2. *$\pi(J_k) = B_k$ for all k,*
3. *$\pi|_{J_k}$ is monotonic increasing for all k.*

Notice that π is the inverse of a t-shuffle if and only if π has at most $t-1$ *descents* (positions i where $\pi_i > \pi_{i+1}$). The number $S_t(n)$ of permutations of this type is the classical Simon Newcomb's problem (see p. 213ff of [12]). Also notice that $S_t^{-1} = [\mathscr{I},\mathscr{I},\dots]$ where $\mathscr{I}$ is the set of all identity permutations and so S_t^{-1} and S_t are finitely based by Theorem 2.2 and Lemma 1.1.

The main result of this subsection is a structure theorem for the closed set $S_r \cap S_t^{-1}$.

Theorem 2.8. *Let Σ be the profile closure of the single permutation*

$$[1, r+1, 2r+1, \ldots, (t-1)r+1, 2, r+2, \ldots, (t-1)r+2, 3, \ldots, n]$$

Then $P(\Sigma) = S_r \cap S_t^{-1}$.

Proof. Suppose that $\pi \in S_r \cap S_t^{-1}$. Let $\{A_i\}_{k=1}^r, \{I_k\}_{k=1}^r, \{B_k\}_{k=1}^t, \{J_k\}_{k=1}^t$ be the sets defined and guaranteed by the previous two lemmas. Let $C_{hk} = A_h \cap J_k$ and $D_{hk} = I_h \cap B_k$.

Since $C_{hk}, C_{h+1,k} \subseteq J_k$, $\pi|_{J_k}$ is monotonic increasing, and

$$\pi(C_{hk}) = D_{hk} = I_h \cap B_k < I_{h+1} \cap B_k = D_{h+1,k} = \pi(C_{h+1,k})$$

we have $C_{hk} < C_{h+1,k}$. Furthermore, $C_{rk} = A_r \cap J_k < A_1 \cap J_{k+1} = C_{1,k+1}$. Therefore,

$$C_{11} < C_{21} < \cdots < C_{r1} < C_{12} < C_{22} < \cdots .$$

Also, by a similar argument,

$$D_{11} < D_{12} < \cdots < D_{1s} < D_{21} < D_{22} < \cdots .$$

It follows that the profile of π is in the set Σ. This proves one half of the theorem. The converse can be proved by reversing the foregoing argument. □

In principle, this theorem allows the enumeration problem to be solved for any fixed $S_r \cap S_t^{-1}$. We illustrate this for the standard riffle shuffles (2-shuffles) in the next lemma.

Lemma 2.9. *The number of riffle shuffles of a deck of n cards which can be restored by a riffle shuffle is* $\binom{n+1}{3} + 1$.

Proof. According to Theorem 2.8 $S_2 \cap S_2^{-1} = P(\Sigma)$ where $\Sigma = \{1324, 213, 132, 21, 1\}$ is the profile closure of 1324. Therefore, by Lemma 2.4,

$$|(S_2 \cap S_2^{-1})_n| = \binom{n-1}{3} + 2\binom{n-1}{2} + \binom{n-1}{1} + \binom{n-1}{0}$$
$$= \binom{n+1}{3} + 1. \qquad \square$$

3. Closed sets with a basis of two permutations of lengths 3 and 4

In this section we consider all closed sets which have a basis of two permutations, α of length 3, β of length 4. Of the $144 = 3! \times 4!$ pairs of such permutations we may immediately reduce to a complete set of pairs inequivalent under the symmetry group D. There are 30 such pairs but 12 of them are degenerate in the sense that $\alpha \preccurlyeq \beta$ and therefore $\{\alpha, \beta\}$ is not a basis of a closed class. For the remaining 18 pairs Table 1 gives the values of $a_n = |\mathscr{A}_n(\alpha, \beta)|$ or a recurrence relation they satisfy. Every pair α, β with $\alpha \not\preccurlyeq \beta$ is equivalent to one of these pairs.

Table 1

	α, β	$a_n = \lvert\mathscr{A}_n(\alpha,\beta)\rvert$
1	123, 4321	0 for $n \geqslant 7$
2	321, 2134	$n + \binom{n}{3} + \binom{n+1}{4}$
3	321, 1324	$1 + \binom{n}{2} + \binom{n+2}{5}$
4	132, 4321	$1 + \binom{n+1}{3} + 2\binom{n}{4}$
5	123, 4213	$3 \times 2^{n-1} - \binom{n+1}{2} - 1$
6	123, 3412	$2^{n+1} - 2n - 1 - \binom{n+1}{3}$
7	132, 4312	$(n-1)2^{n-2} + 1$
8	132, 4231	$(n-1)2^{n-2} + 1$
9	132, 3214	$a_n = 4a_{n-1} - 5a_{n-2} + 3a_{n-3}$
10	123, 3214	$a_n = 3a_{n-1} - a_{n-2}$
11	132, 1234	$a_n = 3a_{n-1} - a_{n-2}$
12	132, 4213	$a_n = 3a_{n-1} - a_{n-2}$
13	132, 4123	$a_n = 3a_{n-1} - a_{n-2}$
14	132, 3124	$a_n = 3a_{n-1} - a_{n-2}$
15	123, 2143	$a_n = 3a_{n-1} - a_{n-2}$
16	123, 3142	$a_n = 3a_{n-1} - a_{n-2}$
17	132, 2134	$a_n = 3a_{n-1} - a_{n-2}$
18	132, 3412	$a_n = 3a_{n-1} - a_{n-2}$

The table contains essentially the same information as the data given by West in [19] which he obtained with a generating tree approach; he gave detailed proofs for cases 5, 10, and 15 only. Rather than give the detailed proofs here we invite the reader to consult [1] where proofs which avoid the use of generating trees are given. Instead, we shall look at those entries in the table where the theory of the previous section allows us to derive structure theorems for $\mathscr{A}_n(\alpha,\beta)$ from which the corresponding enumeration result follows easily.

The following three propositions treat those cases where the enumeration formula is a polynomial. Since the proof strategies are all similar we give details for the last only.

Proposition 3.1. *If $\alpha = 321$ and $\beta = 2134$ then $\mathscr{A}(\alpha,\beta)$ is the set of permutations whose profiles are in the profile closure of $14627^1 35^1$. Moreover,*

$$|\mathscr{A}_n(\alpha,\beta)| = n + \binom{n}{3} + \binom{n+1}{4}.$$

Proposition 3.2. *If $\alpha = 321$ and $\beta = 1324$ then $\mathscr{A}(\alpha,\beta)$ is the set of permutations whose profiles are in the profile closure of 21354 and 351624. Moreover,*

$$|\mathscr{A}_n(\alpha,\beta)| = 1 + \binom{n}{2} + \binom{n+2}{5}.$$

Proposition 3.3. *If* $\alpha = 132$ *and* $\beta = 4321$ *then* $\mathscr{A}(\alpha, \beta)$ *is the set of permutations whose profiles are in the profile closure of* 32415 *and* 42135. *Moreover,*

$$|\mathscr{A}_n(\alpha, \beta)| = 1 + \binom{n+1}{3} + 2\binom{n}{4}.$$

Proof. Note first that the profile closure of 32415 and 42135 is the set of profiles

$$P = \{32415, 42135, 3214, 3241, 4213, 213, 321, 21, 1\}.$$

It is straightforward to verify that any permutation whose profile is in P must avoid both 132 and 4321. To prove that any permutation of length n which avoids both 132 and 4321 has profile in P we argue by induction on n. Let σ' be the permutation obtained by removing n from σ. By induction, the profile of σ' is one of the profiles in P. We shall consider the different possibilities for the profile of σ' and verify that when n is inserted into such a permutation to produce a permutation that avoids both 132 and 4321 then the result has a profile that is also in P.

1. $\sigma' = \gamma_3\gamma_2\gamma_4\gamma_1\gamma_5$. This is the case that σ' has profile 32415; each γ_i is an increasing sequence of consecutive symbols and the subscripts indicate the relative values of symbols in different γ_i. Notice that n cannot be inserted in the interior of any γ_i since that would introduce a subsequence order isomorphic to 132 (this observation applies to all the cases). Also n cannot be inserted before γ_3 since that would introduce a subsequence order isomorphic to 4321. Nor can n be inserted anywhere between γ_3 and γ_5 for that would produce a subsequence order isomorphic to 132. So the only valid place where n can be inserted is after γ_5 and then the result also has profile 32415
2. $\sigma' = \gamma_4\gamma_2\gamma_1\gamma_3\gamma_5$. Again we need only consider insertion points for n which fall between γ-strings and, just as above, the only possible place is at the end of σ' giving a permutation also of profile 42135.
3. $\sigma' = \gamma_3\gamma_2\gamma_1\gamma_4$. The argument is exactly the same.
4. $\sigma' = \gamma_3\gamma_2\gamma_4\gamma_1$. To avoid introducing a subsequence order isomorphic to 4321 or 132 the only possible places to insert n are between γ_4 and γ_1, or after γ_1. The former yields a permutation with profile 3214 and the latter yields a permutation with profile 32415.
5. $\sigma' = \gamma_4\gamma_2\gamma_1\gamma_3$. Here the valid insertion points are between γ_4 and γ_2 which gives the profile 4213, and after γ_3 giving the profile 42135.

For σ' of the form $\gamma_2\gamma_1\gamma_3, \gamma_3\gamma_2\gamma_1, \gamma_2\gamma_1, \gamma_1$ the argument is similar.

Finally, we apply Lemma 2.4 to each of the profiles in P. This shows that

$$\begin{aligned}|\mathscr{A}_n(\alpha, \beta)| &= 2\binom{n-1}{4} + 3\binom{n-1}{3} + 2\binom{n-1}{2} + \binom{n-1}{1} + \binom{n-1}{0}\\ &= 1 + \binom{n+1}{3} + 2\binom{n}{4}. \qquad \square\end{aligned}$$

We end with a result that demonstrates that profile-closed classes are useful even when the enumeration formula is not a polynomial.

Proposition 3.4. 1. $\mathscr{A}(321,2143)=\mathscr{A}(321,2143,3142)\cup\mathscr{A}(321,2143,2413)$,
2. $\mathscr{A}(321,2143,3142)=S_2^{-1}$,
3. $\mathscr{A}(321,2143,3142)^{-1}=\mathscr{A}(321,2143,2413)=S_2$,
4. $|\mathscr{A}_n(321,2143,3142)|=2^n-n$
5. $|\mathscr{A}_n(321,2143)|=2^{n+1}-2n-1-\binom{n+1}{3}$.

Proof. For part 1 we can confirm, by case checking, that any permutation τ which involves 3142 and 2413 necessarily involves 321 or 2143; only a finite number of cases have to be checked since we may presume that τ is a minimal permutation involving 3142 and 2413 (and so of length at most 8). This implies that a permutation which avoids 321 and 2143 must avoid at least one of 3142 and 2413.

For part 2 it is easy to see that the right-hand side set is contained in the left-hand side set. Now, let $\sigma\in\mathscr{A}(321,2143,3142)$ and write

$$\sigma=[1,2,\dots,m,a_1,a_2,\dots,a_r,m+1,b_1,\dots,b_s],$$

where $m\geqslant 0$ and $r\geqslant 1$. Since every $a_i>m+1$ and σ avoids 321 $a_1<a_2<\cdots<a_r$. Moreover, $b_1,\dots,b_s$ must also be increasing since, if $b_i>b_{i+1}$ then the subsequence $[a_1,m+1,b_i,b_{i+1}]$ is either order isomorphic to 4132 which involves 321 if $a_1>b_i$, or order isomorphic to 3142 if $b_i>a_1>b_{i+1}$, or order isomorphic to 2143 if $b_{i+1}>a_i$. Thus, $\sigma=\gamma\delta$ where γ,δ are increasing and so $\sigma\in S_2^{-1}$.

Part 3 is true because the permutation inverse of 3142 is 2413.

For part 4 a permutation $\sigma=\gamma\delta$ (with γ,δ increasing) of $\mathscr{A}_n(321,2143,3142)$ is defined once the subset of values in γ is determined. However, although there are 2^n such subsets, $n+1$ of them (those of the form $\{1,2,\dots,i\}$) all give the same permutation σ and so there are 2^n-n such permutations.

Finally, to prove part 5 we use

$$\begin{aligned}&|\mathscr{A}_n(321,2143,3142)\cup\mathscr{A}_n(321,2143,2413)|\\&\quad=|\mathscr{A}_n(321,2143,3142)|+|\mathscr{A}_n(321,2143,2413)|\\&\qquad-|\mathscr{A}_n(321,2143,3142)\cap\mathscr{A}_n(321,2143,2413)|.\end{aligned}$$

However, by Theorem 2.8, $\mathscr{A}(321,2143,3142)\cap\mathscr{A}(321,2143,2413)=S_2\cap S_2^{-1}$ and so, by Lemma 2.9, this means that

$$|\mathscr{A}_n(321,2143,3142)\cap\mathscr{A}_n(321,2143,2413)|=\binom{n+1}{3}+1.$$

The result now follows using part 4. $\square$

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