Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 38, 1871 - 1876

Fibonacci Identities as Binomial Sums

Mohammad K. Azarian

Department of Mathematics, University of Evansville

1800 Lincoln Avenue, Evansville, IN 47722, USA

E-mail: azarian@evansville.edu

Abstract

To facilitate rapid numerical calculations of identities pertaining to Fibonacci numbers, we present each identity as a binomial sum.

Mathematics Subject Classification: 05A10,11B39

Keywords: Fibonacci numbers, Fibonacci identities, Fibonacci sequence, Pascal's identity, Pascal's (Khayyām-Pascal's) triangle

1. Preliminaries

The most prominent linear homogeneous recurrence relation of order two with constant coefficients is the one that defines *Fibonacci numbers* (or Fibonacci sequence). It is defined recursively as

$$F_{n+2} = F_{n+1} + F_n, \quad \text{where} \quad F_0 = 0, \; F_1 = 1, \text{ and } n \geq 0.$$

It is well-known that the function

$$g(x) = \frac{x}{1-x-x^2} \tag{1}$$

generates Fibonacci sequence. Bicknell and Hoggatt in [2] stated that (1) can be verified by long division. But, since the method of long division is a long process, especially for a large n, the author in [1] used the method of generating functions to verify (1) which is quicker, regardless of the value of n, and obtained

$$F_n = \frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\right], \quad n \geq 0. \tag{2}$$

Finding the exact value of F_n from (2) requires multiple steps of busy and messy algebraic calculations which is not desirable. So, our goal in this note is to present F_n as a binomial sum for quick numerical calculations. Likewise, we use this binomial sum to write some well-known and fundamental identities concerning Fibonacci numbers as binomial sums as well.

It is known that Fibonacci numbers are the sum of the numbers along the rising diagonals of Pascal's (*Khayyam-Pascal's*) triangle, and if we write the elements of Pascal's triangle as binomial terms we have

$$S_1 = \binom{0}{0} = 1$$
$$S_2 = \binom{1}{0} = 1$$
$$S_3 = \binom{2}{0} + \binom{1}{1} = 2$$
$$S_4 = \binom{3}{0} + \binom{2}{1} = 3$$
$$S_5 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 5$$
$$S_6 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 8$$

.

.

.

$$S_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots, \quad n \geq 0.$$

Now, using *Pascal's identity* $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$, $(n \geq r \geq 0)$ we can easily verify that $S_{n+2} = S_{n+1} + S_n$, and hence the binomial sum S_n satisfies the Fibonacci relation. In practice, we need to know all terms in the binomial sum of S_n. By inspection we can see that for $n \geq 0$,

$$S_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots + \binom{n - \lfloor \frac{n}{2} \rfloor + 1}{\lfloor \frac{n}{2} \rfloor - 1} + \binom{n - \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i},$$

where $\lfloor n \rfloor$ represents the floor function. Again, if we use Pascal's identity, we can easily show that the above S_n does indeed satisfy the Fibonacci relation.

2. Identities

It is well-known that the left-hand side of each identity in Corollary 1 can be written as a (power of a) single Fibonacci number. For example, as early as 1876 Lucas has shown that $1 + \sum_{i=1}^{n} F_i = F_{n+2}$, $1 + \sum_{i=1}^{n} F_{2i} = F_{2n+1}$, and $\sum_{i=1}^{n} F_{2i-1} = F_{2n}$. One could use the principle of mathematical induction, combinatorial argument, or just simple algebra to verify the validity of these identities. These Fibonacci identities have been developed over the centuries by mathematicians and number enthusiasts alike, and their proofs can be found in various sources. So, as we stated earlier, the goal of this note is to write some of these fundamental identities as binomial sums for quick numerical calculations.

Theorem 1. If F_n is any Fibonacci number, then

$$F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + ... + \binom{n - \lfloor \frac{n}{2} \rfloor + 1}{\lfloor \frac{n}{2} \rfloor - 1} + \binom{n - \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor}$$
$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}, \quad n \geq 0.$$

Proof follows from our above discussion. Also, Theorem 1 can be proven by using the principle of mathematical induction or combinatorial methods.

As a direct consequence of Theorem 1 and the definition of Fibonacci numbers we obtain the following corollary.

Corollary 1. If n is any nonnegative integer, then

$$(i) \;\; 1 + \sum_{i=0}^{n} F_i = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i}$$

$$(ii) \;\; 1 + \sum_{i=0}^{n} F_{2i} = \sum_{i=0}^{n} \binom{2n-i}{i}$$

$$(iii) \;\; \sum_{i=0}^{n} F_{2i+1} = \sum_{i=0}^{\lfloor \frac{2n+1}{2} \rfloor} \binom{2n+1-i}{i}$$

$$(iv) \;\; 1 + 2\sum_{i=0}^{n} F_{3i+2} = \sum_{i=0}^{\lfloor \frac{3n+3}{2} \rfloor} \binom{3n+3-i}{i}$$

$$(v) \;\; 2\sum_{i=0}^{n} F_{3i+1} = \sum_{i=0}^{\lfloor \frac{3n+2}{2} \rfloor} \binom{3n+2-i}{i}$$

$$(vi) \;\; 1 + 2\sum_{i=0}^{n} F_{3i} = \sum_{i=0}^{\lfloor \frac{3n+1}{2} \rfloor} \binom{3n+1-i}{i}$$

$$(vii) \;\; 1 + \sum_{i=0}^{n} F_{4i} = \left[\sum_{i=0}^{n} \binom{2n-i}{i} \right]^2$$

$$(viii) \;\; \sum_{i=0}^{n+1} \binom{n+1}{i} F_{n+1-i} = \sum_{i=0}^{\lfloor \frac{2n+1}{2} \rfloor} \binom{2n+1-i}{i}$$

$$(ix) \;\; 1 + F_{n+1}F_{n+2} + 2\sum_{i=0}^{n} F_i F_{i+1} = \sum_{i=0}^{n+1} \binom{2n+2-i}{i}$$

$$(x)\quad 2+(n+1)F_{n+2}-\sum_{i=0}^{n} iF_i=\sum_{i=0}^{\lfloor\frac{n+3}{2}\rfloor}\binom{n+3-i}{i}$$

$$(xi)\quad F_nF_{n+3}^2-F_{n+2}^3=(-1)^{n+1}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-i}{i}$$

$$(xii)\quad F_{n+3}F_n^2-F_{n+1}^3=(-1)^{n+1}\sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+1-i}{i}$$

3. Conclusion

We presented just a few widely-known Fibonacci identities as binomial sums. We are hoping that this article would motivate the curious reader to write her/his favorite Fibonacci identities as binomial sums too. The author himself will be working on writing some other Fibonacci and Lucas identities as binomial sums as well.

References

[1] Mohammad K. Azarian, *The Generating Function for the Fibonacci Sequence*, Missouri Journal of Mathematical Sciences, Vol. 2, No. 2, Spring 1990, pp. 78-79.

[2] Marjorie Bicknel and Verner E. Hoggatt, *Fibonacci's Problem Book,* The Fibonacci Association, 1974.

[3] Verner E. Hoggatt and Gerald E. Bergum, *A Problem of Fermat and the Fibonacci Sequence,* The Fibonacci Quarterly, Vol. 15, No. 4, Oct. 1977, pp. 323-330.

[4] K. Subba Rao, *Some Properties of Fibonacci Numbers*, The American Mathematical Monthly, Vol. 60, No. 10, Dec. 1953, pp. 680-684.

[5] Neil. J. Sloan, http://oeis.org/A000045&A007318.

Received: April, 2012