Journal of Combinatorial Theory, Series A 117 (2010) 568–582

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Mixed succession rules: The commutative case

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ARTICLE INFO

Article history:
Received 29 July 2008
Available online 3 December 2009
Communicated by Ronald L. Graham

Keywords:
Succession rule
ECO method
Generating tree
Rule operator
Riordan array

ABSTRACT

We begin a systematic study of the enumerative combinatorics of *mixed succession rules*, i.e. succession rules such that, in the associated generating tree, nodes are allowed to produce sons at several different levels according to different production rules. Here we deal with a specific case, namely that of two different production rules whose rule operators commute. In this situation, we are able to give a general formula expressing the sequence associated with the mixed succession rule in terms of the sequences associated with the component production rules. We end by providing examples illustrating our approach.

1. Introduction

Among the many methods that have been developed to enumerate combinatorial structures, the role of the *ECO method* has been growing in the last decade, thanks to its intrinsic simplicity and to the effectiveness of the combinatorial constructions it generates. The variety of problems in which the ECO method has shown its soundness ranges from enumerative and bijective combinatorics to random [2] and exhaustive [4] generation.

The roots of the ECO method can be traced back to [6], where the authors study Baxter permutations and introduce for the first time the concept of a *generating tree*. Successively, West [20] introduced the notion of a *succession rule* to give a formal description of generating trees in the context of permutation enumeration and Barcucci et al. [3] extended the technique of generating trees, finding a general way of constructing combinatorial objects which can be often described using such formal tools.

The *classical ECO method* (a detailed description of which can be found, for instance, in [3]) consists of a recursive construction for a class of objects by means of an operator which performs a "local

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0097-3165/$ – see front matter
doi:10.1016/j.jcta.2009.11.005

expansion" on the objects themselves. Typically, starting from an object *of size* n, an ECO construction allows to produce a set of new objects, *of size* $n+1$, in such a way that, iterating the construction, all the objects of the class are obtained precisely once. If the construction is sufficiently regular, it can be often described by means of a *succession rule*, which is a system of the form

$$\Omega: \quad \begin{cases} (a) \\ (k) \rightsquigarrow \big(e_1(k)\big)\big(e_2(k)\big)\cdots\big(e_k(k)\big) \end{cases}, \tag{1}$$

where k denotes a positive integer. This means that each object of the class is given a label (k). When performing the ECO construction, an object labelled (k) produces k new objects labelled, respectively, $(e_1(k)), (e_2(k)), \dots, (e_k(k))$. Moreover, the object of minimum size has label (a) (which is called the *axiom* of the succession rule). To have a graphical description of a succession rule, one usually draws its *generating tree*, that is the infinite, rooted, labelled tree whose root is labelled (a) (like the axiom) and such that each node labelled (k) has k sons, labelled $(e_1(k)), (e_2(k)), \dots, (e_k(k))$, respectively. It is evident from this definition that we have a notion of *level* on generating trees, by saying that the root lies at level 0, and a node lies at level n when its parent lies at level $n-1$.

We remark that, from the above definition, a node labelled (k) has precisely k sons. A succession rule having this property is said to be *consistent*. However, one can also consider succession rules in which the value of a label does not necessarily represent the number of its sons, and this will be frequently done in the sequel. Moreover, we would like to warn the reader that, even if we will sometimes give definitions using consistent succession rules (since this is the convention when working with the ECO method), we will constantly make use of succession rules which are not necessarily consistent.

From the enumerative point of view, the main information encoded in a generating tree (and thus in its associated succession rule) is given by the *level polynomial* $p_n(x) = \sum_k p_{n,k}x^k$, defined by setting $p_{n,k}$ equal to the number of nodes labelled k at level n, and by the *associated integer sequence* $(f_n)_{n\in\mathbf{N}}$, which is defined, in terms of the level polynomials, as $f_n = p_n(1)$, and represents the total number of nodes at level n. We point out that the infinite lower triangular array $(p_{n,k})_{n,k\in\mathbf{N}}$, also called *AGT matrix* [15], or *ECO matrix* [10], sometimes happens to be a *Riordan array* [18]. This means that every element $p_{n,k}$ can be expressed by using a pair of formal power series $(d(t), h(t))$ in such a way that it is precisely the coefficient of t^n of $d(t)h(t)^k$. In this case, many counting properties of the generating tree can be found in an algebraic way, by using the related theory.

Despite its wide range of applicability, there are many combinatorial constructions which cannot be naturally described by using the classical ECO method (and classical succession rules) as exploited above. For instance, in [5] a generalization of the method is considered, allowing succession rules in which the labels are pairs of integers (rather than integers). Applications of this generalized method to the enumeration of pattern avoiding permutations are shown in the cited paper.

A strong limitation in the possibility of describing a combinatorial construction by means of a generating tree lies in the fact that, if the level of a node is n, then the level of all its sons is $n+1$. Combinatorially, this means that a (classical) ECO construction performed on an object of a given size produces objects of the successive size. However, it may well happen that a combinatorial construction, having all the reasonable features to be called ECO, does not behave in the standard way with respect to the notion of size. More precisely, starting from an object of size n, we can construct new objects whose sizes are greater than n (but not necessarily equal to $n+1$). The formalization of these concepts leads to the notion of what can be called a *mixed succession rule*. Roughly speaking, the idea is to consider a set of (possibly different) succession rules acting on the objects of a class and producing sons at different levels. To be more formal, we introduce here the simplest instance of this general situation, by considering two succession rules producing their sons at the two successive levels. These will be called *doubled mixed succession rules*. Given two succession rules Ω as in (1) and

$$\Sigma: \quad \begin{cases} (b) \\ (k) \rightsquigarrow \big(d_1(k)\big)\big(d_2(k)\big)\cdots\big(d_k(k)\big) \end{cases},$$

we define the *doubled mixed succession rule associated with the pair* (Ω, Σ) *with axiom* (c) to be the succession rule $(c)\Omega^{+1}\Sigma^{+2}$, defined by

$$(c)\Omega^{+1}\Sigma^{+2}: \quad \begin{cases} (c) \\ (k) \overset{+1}{\rightsquigarrow} \big(e_1(k)\big)\big(e_2(k)\big)\cdots\big(e_k(k)\big) \\ \quad\;\; \overset{+2}{\rightsquigarrow} \big(d_1(k)\big)\big(d_2(k)\big)\cdots\big(d_k(k)\big) \end{cases}.$$

The generating tree associated with $(c)\Omega^{+1}\Sigma^{+2}$ has the property that each node labelled (k) lying at level n produces two sets of sons, the first set being $(e_1(k)), (e_2(k)), \dots, (e_k(k))$ at level $n+1$ and the second one being $(d_1(k)), (d_2(k)), \dots, (d_k(k))$ at level $n+2$ (thus producing a total of $2k$ sons).

To justify our interest in this kind of notion, we remark that instances of (general) mixed succession rules have occasionally surfaced in some previous works; to cite only one example, in [13] vexillary involutions are enumerated by making use of a specific mixed succession rule. The first systematic treatment of mixed succession rules has been undertaken in [9], where the special case $\Sigma = \Omega$ has been examined in great detail. The present paper represents the first attempt to tackle the general case, aiming at developing a general theory of mixed succession rules. Our main goal is to express the sequence associated with a mixed succession rule $(c)\Omega^{+1}\Sigma^{+2}$ in terms of the sequences associated with Ω and Σ, possibly changing the axioms. For this reason, in Section 3 we study some enumerative properties of what we have called *production rules*, which are succession rules without the axiom. The problem of studying mixed rules in its full generality seems quite difficult; it is somehow related to the theory of power series in several *noncommuting* variables. Here we deal only with a special case, namely when the two rule operators (see Section 2) of Ω and Σ commute. In this situation, we are able to find a general formula for the sequence associated with $(c)\Omega^{+1}\Sigma^{+2}$.

We remark that this problem has also been considered from the point of view of Riordan arrays [1]. Each Riordan array determines a specific sequence $(a_k)_{k\in\mathbf{N}}$, called the *A-sequence* of the array, which allows to provide a recursive description of the elements of the array.

When the A-sequence contains integer numbers only, it is related to the succession rule of an associated generating tree (if any) as shown in [15]. However, it can happen that the A-sequence has a complicated expression, whereas the A-matrix (as defined in [1]) is simple. This may correspond to an ECO construction in which elements of size n produce objects of different greater sizes. However, even if the Riordan array approach is useful from a computational point of view, we deem that the present method has at least two major advantages: it provides a more general theoretical framework in which this kind of problems can be properly described, and it makes much easier to determine combinatorial interpretations for the resulting numerical sequences.

In closing this introduction, we recall some notations we will frequently use in the next pages. The sets of natural and real numbers will be denoted by $\mathbf{N}$ and $\mathbf{R}$, respectively. The following linear operators on the vector space of one-variable polynomials will be often considered: $\mathbf{x}$ (respectively, $\mathbf{t}$) is the operator of multiplication by x (respectively, t), D is the usual derivative operator, and T is the *factorial derivative operator*, which is, by definition, the linear operator mapping x^n into $1 + x + \cdots + x^{n-1} = \sum_{i=0}^{n-1} x^i$ (for $n \geqslant 1$) and 1 into 0. The operator T has been considered in [11], where the name "factorial derivative operator" has been chosen since T plays the same role for factorial succession rules as D plays for differential ones.

2. Preliminaries on rule operators

Given a succession rule Ω as in (1), we can associate with it a linear operator on the vector space of one-variable polynomials $\mathbf{R}[x]$, to be denoted $L = L_\Omega$ (the subscript will be omitted when it is clear from the context). To define such an operator, we use the canonical basis $(x^n)_{n\in\mathbf{N}}$:

$$\begin{aligned} L &: \mathbf{R}[x] \longrightarrow \mathbf{R}[x] \\ &: 1 \longmapsto x^a \\ &: x^k \longmapsto x^{e_1(k)} + \cdots + x^{e_k(k)}, \quad \text{if } k \text{ appears in } \Omega, \\ &: x^h \longmapsto hx^h, \quad \text{otherwise.} \end{aligned}$$

Each enumerative property of Ω can be suitably translated into some property of L. For instance, if $(f_n)_{n\in\mathbf{N}}$ is the sequence associated with Ω, then $f_n = [L^{n+1}(1)]_{x=1}$, where we have used square brackets to denote the operator of evaluation at a specific value. Moreover, if $p_n(x) = \sum_k p_{n,k}x^k$ is the n-th level polynomial of Ω, then $L_\Omega(p_n(x)) = p_{n+1}(x)$. The linear operator L is called the *rule operator* associated with Ω. We refer to [8,10,11] for the main properties and some applications of this notion.

Examples.

1. A rule for Bell numbers with the associated rule operator:

$$\begin{cases} (1) \\ (k) \rightsquigarrow (k)^{k-1}(k+1) \end{cases},$$

$$L(1) = x; \qquad L\bigl(x^k\bigr) = (\mathbf{x}D + \mathbf{x} - 1)\bigl(x^k\bigr) = (k-1)x^k + x^{k+1}, \quad k \geqslant 1. \tag{2}$$

   Here $k-1$ is the number of blocks of a set partitions in a well-known (ECO) generation of them.

2. A rule for Catalan numbers with the associated rule operator:

$$\begin{cases} (1) \\ (k) \rightsquigarrow (2)(3)\cdots(k)(k+1) \end{cases},$$

$$L(1) = x; \qquad L\bigl(x^k\bigr) = \bigl(\mathbf{x}^2 T\bigr)\bigl(x^k\bigr) = \sum_{i=2}^{k+1} x^i, \quad k \geqslant 1. \tag{3}$$

   This rule can be obtained by generating Dyck paths by adding a peak in each integer point of the last sequence of descending steps of each Dyck path, and k is the number of such integer points [3].

3. A rule for Motzkin numbers with the associated rule operator:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (k) \rightsquigarrow (1)(2)\cdots(k-1)(k+1) \end{cases},$$

$$L(1) = x; \qquad L\bigl(x^k\bigr) = (\mathbf{x}T + \mathbf{x} - \mathbf{1})\bigl(x^k\bigr) = \sum_{i=1}^{k-1} x^i + x^{k+1}, \quad k \geqslant 2. \tag{4}$$

   An interpretation for this rule can also be found in [3].

3. Production rules

With the expression *production rule* we will mean here a succession rule without its axiom. Hence the generic form of a production rule is

$$(k) \rightsquigarrow \bigl(e_1(k)\bigr)\bigl(e_2(k)\bigr)\cdots\bigl(e_k(k)\bigr). \tag{5}$$

Clearly, in the same way a succession rule determines a unique numerical sequence, a production rule defines a family of sequences $(f_n^{(a)})_{n\in\mathbf{N}}$, depending on the axiom (a) which we choose for the rule (5).

From now on, given a succession rule Ω as in (1), we will denote with L_a the associated rule operator, (a) being the axiom of Ω. Using this terminology, given a production rule as in (5), the family of operators $(L_a)_{a\in\mathbf{N}}$ will be called the *family of rule operators* associated with the production rule. In order to obtain formulas relating the various sequences associated with the same production rule, we start by observing the following facts.

1. For any $a, b \in \mathbf{N}$, $L_a(x^k) = L_b(x^k)$, when $k \neq 0$, and $L_a(1) = x^a$, $L_b(1) = x^b$. Thus in what follows, for $k \neq 0$, we will simply write $L(x^k)$, without the axiom, and we will speak of *the* rule operator associated with the production rule whenever we restrict to the subspace $x\mathbf{R}[x]$ (the subspace spanned by the positive powers of x).

2. The n-th term of the numerical sequence of the family with axiom (b), that is $f_n^{(b)}$, can be computed using the following formula:

$$f_n^{(b)} = \left[L^n\left(x^b\right)\right]_{x=1}.$$

In the same way, to compute $f_n^{(b+1)}$ we get:

$$f_n^{(b+1)} = \left[L^n\left(x^{b+1}\right)\right]_{x=1} = \left[L^n\mathbf{x}\left(x^b\right)\right]_{x=1}.$$

Then, if one knows the *Pincherle derivative* of L, which is the operator $L' = L\mathbf{x} - \mathbf{x}L$ [16], it should be possible to express the operator $L^n\mathbf{x}$ as a linear combination of monomials of the kind $\mathbf{x}^\alpha L^\beta$. This should allow, at least in principle, to obtain an expression for $f_n^{(b+1)}$ in terms of known quantities (namely $f_m^{(b+1)}$, with $m < n$, and $f_k^{(a)}$, with $a \leqslant b$).

3.1. Some examples

(1) *A Bell-like production rule.* Consider the following production rule:

$$\omega\colon \quad (k) \rightsquigarrow (k)^{k-1}(k+1),$$

which is related to Bell numbers. The rule operator associated with ω is $L = \mathbf{x}D + \mathbf{x} - \mathbf{1}$ and its Pincherle derivative is $L' = \mathbf{x}$ (on $x\mathbf{R}[x]$).

Proposition 3.1. *For any $n \geqslant 1$, we have*:

$$L^n\mathbf{x} = \sum_{k=0}^{n-1}\binom{n-1}{k}\mathbf{x}L^{k+1} + L^{n-1}\mathbf{x}.$$

If $(f_n^{(b)})_{n\in\mathbf{N}}$ is the sequence associated with L with axiom (b), it is

$$f_n^{(b+1)} = \sum_{k=0}^{n-1}\binom{n-1}{k} f_{k+1}^{(b)} + f_{n-1}^{(b+1)}.$$

In terms of generating functions, we get

$$f^{(b)}(x) = e^{e^x+(b-1)x-1}.$$

(2) *A Catalan-like production rule.* Consider the production rule:

$$\omega\colon \quad (k) \rightsquigarrow (2)(3)\cdots(k)(k+1),$$

which is related to Catalan numbers. It is known that the generating function of ω with axiom (b) is $(C(x))^b$ ($C(x)$ being the generating function of Catalan numbers). We use our approach to rediscover it.
We first observe that $L = \mathbf{x}^2T$ is the rule operator associated with ω. Denoting $c_2 : x\mathbf{R}[x] \longrightarrow x\mathbf{R}[x]$ the linear operator defined on the canonical basis by setting $c_2(x^n) = x^2$, for $n > 0$, we have that $L' = c_2$.

Proposition 3.2. *For every $n \in \mathbf{N}$, it holds*:

$$L^n\mathbf{x} = \mathbf{x}L^n + \sum_{i=0}^{n-1} L^i c_2 L^{n-1-i},$$

whence, for every $b \in \mathbf{N}$:

$$f_n^{(b+1)} = f_n^{(b)} + \sum_{i=0}^{n-1} f_i^{(2)} f_{n-1-i}^{(b)}.$$

As a consequence, the generating function of $(f_n^{(b)})_{n\in\mathbf{N}}$ is given by $(C(x))^b$, where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

(3) *A Motzkin-like production rule.* Consider the production rule:

$$\omega\colon \quad (k) \rightsquigarrow (1)(2)\cdots(k-1)(k+1),$$

which is related to Motzkin numbers. The rule operator associated with ω is $L = \mathbf{x}T + \mathbf{x} - 1$ (on $x\mathbf{R}[x]$). Denoting $c_1 : x\mathbf{R}[x] \longrightarrow x\mathbf{R}[x]$ the linear operator defined on the canonical basis by setting $c_1(x^n) = x$, for every $n > 0$, we have that $L' = c_1$.

Proposition 3.3. *For every $n \in \mathbf{N}$, it holds:*

$$L^n\mathbf{x} = \mathbf{x}L^n + \sum_{i=0}^{n-1} L^i c_1 L^{n-1-i},$$

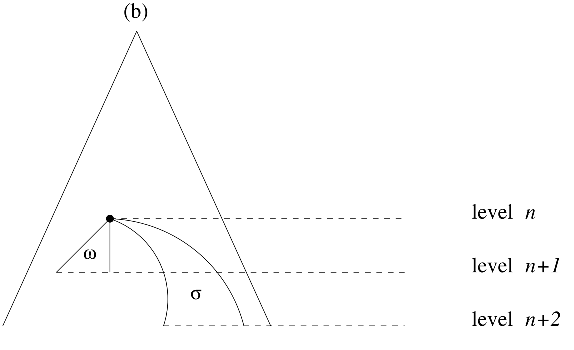
whence, for every $b \in \mathbf{N}$:

$$f_n^{(b+1)} = f_n^{(b)} + \sum_{i=0}^{n-1} f_i^{(1)} f_{n-1-i}^{(b)}.$$

Thus the generating function of $(f_n^{(b)})_{n\in\mathbf{N}}$ satisfies $f^{(b)}(x) = f^{(b-1)}(x)(xM(x)+1)$, where $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$ is the generating function of Motzkin numbers, and so $f^{(b)}(x) = M(x)\cdot(xM(x)+1)^{b-1}$.

4. Commuting rule operators

In what follows we will deal with the simplest case of a mixed succession rule, namely the case of a *doubled rule*: this means that, in the associated generating tree, each node at level n produces a set of sons at level $n+1$, according to a production rule ω, and another set of sons at level $n+2$, according to another production rule σ. If (b) is the axiom of the doubled rule, such a generating tree can be synthetically represented as follows:



Let's start by fixing some notations. First of all, L and M will be the rule operators associated with ω and σ, respectively. Using production rules, our doubled mixed succession rule will be denoted $(b)\omega^{+1}\sigma^{+2}$ ((b) being the axiom), whereas in terms of rule operators it will be $(b)L^{+1}M^{+2}$. Moreover, we will denote $p_n(x)$ the level polynomials of $(b)\omega^{+1}\sigma^{+2}$. Finally, $f^{(b)}(x,t)$ will be the bivariate generating function of the generating tree, where t keeps track of the level and x keeps track of the label.

Proposition 4.1. *Denoting by* $^{[-1]}$ *the compositional inverse, we have*:

$$f^{(b)}(x,t) = \big(\mathbf{1} - \mathbf{t}L - \mathbf{t}^2 M\big)^{[-1]}\big(x^b\big).$$

Proof. Since each node at level n can be generated either by a node at level $n-1$ (according to ω) or by a node at level $n-2$ (according to σ), we have the following expression for $p_n(x)$:

$$p_n(x) = L\big(p_{n-1}(x)\big) + M\big(p_{n-2}(x)\big).$$

If we set, by convention, $p_i(x) = 0$, for $i < 0$, then the above expression is meaningful when $n \geqslant 1$ (recall that, under our assumptions, $p_0(x) = x^b$). In order to translate the above recursion into generating functions, we multiply by t^n both sides of the above equality and sum up for $n \geqslant 1$, thus obtaining:

$$\sum_{n\geqslant 1} p_n(x)t^n = \sum_{n\geqslant 1} L\big(p_{n-1}(x)\big)t^n + \sum_{n\geqslant 1} M\big(p_{n-2}(x)\big)t^n,$$

whence, using linearity:

$$\sum_{n\geqslant 1} p_n(x)t^n = L\Bigg(\sum_{n\geqslant 1} p_{n-1}(x)t^n\Bigg) + M\Bigg(\sum_{n\geqslant 1} p_{n-2}(x)t^n\Bigg).$$

Since $f^{(b)}(x,t) = \sum_{n\geqslant 0} p_n(x)t^n$, we will then get $f^{(b)}(x,t) - x^b = \mathbf{t}L(f^{(b)}(x,t)) + \mathbf{t}^2 M(f^{(b)}(x,t))$, whence

$$f^{(b)}(x,t) = \big(\mathbf{1} - \mathbf{t}L - \mathbf{t}^2 M\big)^{[-1]}\big(x^b\big). \qquad \square$$

Expressing the operator $(\mathbf{1} - \mathbf{t}L - \mathbf{t}^2 M)^{[-1]}$ using power series, we get

$$f^{(b)}(x,t) = \sum_{n\geqslant 0} \mathbf{t}^n (L + \mathbf{t}M)^n \big(x^b\big). \tag{6}$$

Therefore, if we want to know the sequence associated with the doubled rule, we need to find an expression for $(L + \mathbf{t}M)^n$. In general, this is a nontrivial problem, since the linear operators L and M usually *do not commute*. Thus, in the rest of the paper, we will assume the following hypothesis:

L and M commute, i.e. $LM = ML$.

We can now prove the following, crucial result.

Theorem 4.1. *Denoting by* $\mu_r^{(s)}(x) = \sum_i \mu_{r,i}^{(s)} x^i$ *the r-th level polynomial of the generating tree of σ with axiom (s) and by $(l_n^{(a)})_{n\in\mathbf{N}}$ the numerical sequence associated with ω with axiom (a), if $(f_n^{(b)})_{n\in\mathbf{N}}$ is the sequence determined by $(b)\omega^{+1}\sigma^{+2}$, we have*:

$$f_n^{(b)} = \sum_{k\geqslant 0} \binom{n-k}{k} \sum_i \mu_{k,i}^{(b)} l_{n-2k}^{(i)}. \tag{7}$$

Proof. Since L and M commute, we get immediately:

$$(L + \mathbf{t}M)^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{t}^k L^{n-k} M^k.$$

As a consequence, equality (6) can be rewritten as:

$$\begin{aligned} f^{(b)}(x,t) &= \sum_{n\geqslant 0} \mathbf{t}^n \sum_{k=0}^{n} \mathbf{t}^k \binom{n}{k} L^{n-k} M^k\bigl(x^b\bigr) \\ &= \sum_{n\geqslant 0} \biggl(\sum_{k\geqslant 0} \binom{n-k}{k} L^{n-2k} M^k\bigl(x^b\bigr) \biggr) \mathbf{t}^n. \end{aligned}$$

From the above expression we immediately deduce that $p_n(x) = \sum_{k\geqslant 0} \binom{n-k}{k} L^{n-2k} M^k(x^b)$, and so the n-th term of the sequence associated with the doubled rule, which is $f_n^{(b)} = p_n(1)$, can be computed as follows:

$$\begin{aligned} p_n(1) &= \biggl[\sum_{k\geqslant 0} \binom{n-k}{k} L^{n-2k} M^k\bigl(x^b\bigr) \biggr]_{x=1} \\ &= \sum_{k\geqslant 0} \binom{n-k}{k} \bigl[L^{n-2k}\bigl(\mu_k^{(b)}(x)\bigr)\bigr]_{x=1} \\ &= \sum_{k\geqslant 0} \binom{n-k}{k} \sum_i \mu_{k,i}^{(b)} l_{n-2k}^{(i)}, \end{aligned}$$

and this is precisely our thesis. □

Formula (7) expresses the sequence associated with a doubled mixed succession rule when the related rule operators commute. It involves:

- the distribution of the labels of σ with axiom (b) inside its generating tree (i.e. the coefficients $\mu_{k,i}^{(b)}$);
- the sequences associated with ω (i.e. the coefficients $l_{n-2k}^{(i)}$).

5. Examples

We close by giving two applications of formula (7). The first case is somehow trivial (but leads to interesting enumerative results), since we deal with the identity operator, which does not raise any problem concerning commutativity. In the second case we determine some pairs of commuting rule operators. To this aim, the easiest way is perhaps to fix a rule operator L and then find the general form of the rule operators commuting with it.

5.1. The identity operator

As it is obvious, the identity operator $\mathbf{1}$ commutes with any linear operator. Therefore, if L, M are any rule operators, we can consider the two doubled mixed succession rules $(b)L^{+1}\mathbf{1}^{+2}$ and $(b)\mathbf{1}^{+1}M^{+2}$.

Consider first $(b)L^{+1}\mathbf{1}^{+2}$. To apply Theorem 4.1, observe that $(l_n^{(s)})_{n\in\mathbf{N}}$ is the sequence determined by L with axiom (s), whereas $\mu_r^{(s)}(x)$ is the r-th level polynomial of the succession rule determined by the identity operator $\mathbf{1}$ with axiom (s), and so it is trivially $\mu_r^{(s)}(x) = x^s$, whence

$$\mu_{r,i}^{(s)} = \begin{cases} 1, & i = s, \\ 0, & i \neq s. \end{cases}$$

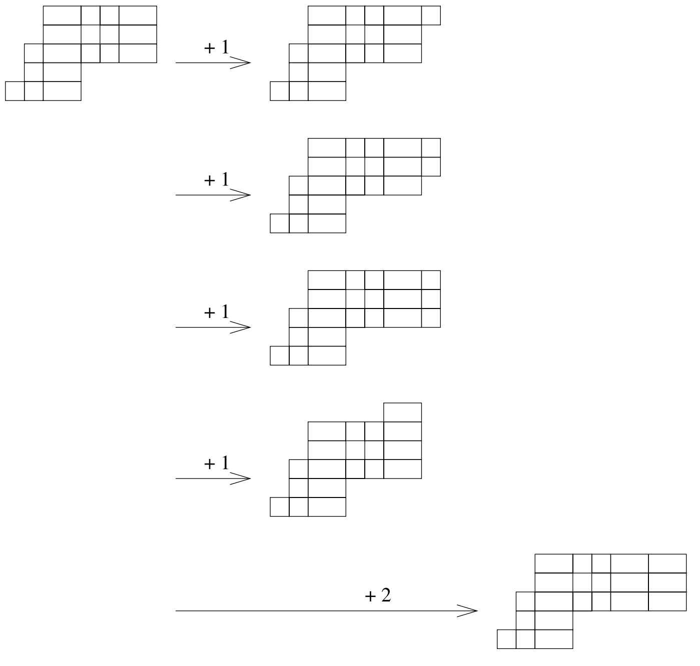


Fig. 1. Our ECO construction performed on a polyomino of semilength 13 and such that the rightmost column has 3 cells.

Thus, denoting by $(f_n^{(b)})_{n\in\mathbf{N}}$ the sequence determined by the doubled mixed rule $(b)L^{+1}\mathbf{1}^{+2}$, we get:

$$f_n^{(b)} = \sum_{k\geqslant 0} \binom{n-k}{k} l_{n-2k}^{(b)}.$$

Moreover, if $f^{(b)}(x)$ and $l^{(b)}(x)$ are the two generating functions of the above sequences, standard arguments leads to the following expression:

$$f^{(b)}(x) = \frac{1}{1-x^2} \cdot l^{(b)}\left(\frac{x}{1-x^2}\right).$$

Examples.

1. If L is the rule operator of Catalan numbers described in (3), the sequence determined by $(1)L^{+1}\mathbf{1}^{+2}$ is sequence A105864 in [17], which has no significant combinatorial interpretation. Define a *1–2 column parallelogram polyomino* to be a parallelogram polyomino whose cells can be either monominoes or dominoes, such that
   (i) each column is entirely made of either monominoes or dominoes;
   (ii) given a set of consecutive columns starting at the same height, the leftmost one must be made of monominoes.

   Such a class of polyominoes can be constructed as follows, according to the semilength. Given a polyomino $\mathcal{P}$ of semilength n whose rightmost column has $k-1$ cells, we construct new polyominoes as follows (see Fig. 1): either add a new rightmost column made of monominoes ending at the same height of the rightmost column of $\mathcal{P}$, or add a new cell on the top of the rightmost column of $\mathcal{P}$ (such a new cell will be a monomino or a domino according to the type of the column), or add a new rightmost column made of dominoes, starting and ending at the same height as the rightmost column of $\mathcal{P}$.

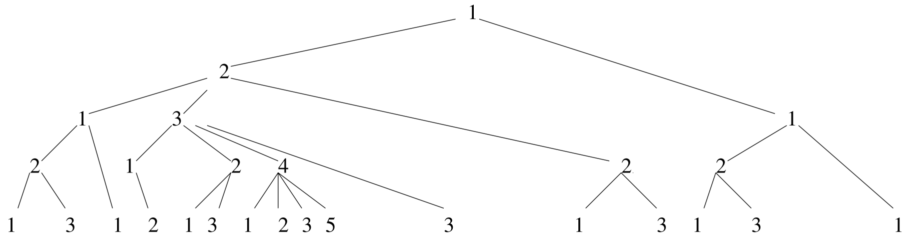


Fig. 2. The generating tree of Ω.

In the first two cases polyominoes of semilength $n+1$ are produced, whereas in the third case a polyomino of semilength $n+2$ comes out. Now the reader can check that the above construction can be encoded by the following succession rule:

$$\begin{cases} (1) \\ (k) \overset{+1}{\rightsquigarrow} (2)(3)\cdots(k)(k+1) \\ \quad\ \overset{+2}{\rightsquigarrow} (k) \end{cases},$$

that is precisely the mixed succession rule $(1)L^{+1}\mathbf{1}^{+2}$.

2. Taking for L the rule operator of Motzkin numbers recalled in (4) (and choosing again (1) as axiom), we get sequence A128720 of [17]. One of the given combinatorial interpretations for such a sequence is the following: it counts the number of *2-generalized Motzkin paths*, i.e. paths in the first quadrant from $(0,0)$ to $(n,0)$ using steps $U=(1,1)$, $D=(1,-1)$, $h=(1,0)$, and $H=(2,0)$. Various kinds of generalized Motzkin paths have been extensively studied in the literature, see for instance [19]. The mixed succession rule arising in this case is the following:

$$\Omega: \quad \begin{cases} (1) \\ (k) \overset{+1}{\rightsquigarrow} (1)(2)\cdots(k-1)(k+1) \\ \quad\ \overset{+2}{\rightsquigarrow} (k) \end{cases}.$$

It is interesting to notice that Ω (whose generating tree is depicted in Fig. 2) indeed describes an ECO construction for the above class of paths. Leaving the details to the interested reader, we quickly justify this claim: given a 2-generalized Motzkin path, consider its last descent, i.e. the final sequence of the path free of U steps. Construct a set of new paths as follows: either replace each h step with a U step and add a D step at the end, or just add an h step at the end, or simply add an H step at the end. In the first two cases the length of the path is increased by 1, whereas in the last case it is increased by 2. It is now easy to show that this construction is encoded precisely by the mixed succession rule Ω. We also remark that another interpretation of sequence A128720 is provided in [17], namely using skew Dyck paths. It would be interesting to use the above mixed rule to describe a construction for this latter combinatorial structure as well.

To conclude this example, we also notice that, according to [1], the rule Ω is alternatively described by the A-matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 1 & 1 & \cdots \end{pmatrix},$$

which implies that the associated ECO matrix is actually a Riordan array. Its A-sequence has generating function:

$$\begin{aligned} A(t) &= \frac{1-t+t^2+\sqrt{1-2t+7t^2-10t^3+5t^4}}{2(1-t)} \\ &= 1+2t^2+t^3-t^4+6t^6+5t^7-16t^8+\cdots, \end{aligned}$$

and this shows that a direct dependence of row $n+1$ from row n is very unlikely. Using the theory of Riordan arrays [18] we can determine the formal series $d(t), h(t)$ defining our ECO matrix. More precisely:

$$h(t) = tA\big(h(t)\big) = \frac{1}{2}\left(1 - \sqrt{\frac{1-3t-t^2}{1+t-t^2}}\right).$$

Since column 0 is not privileged, we have $d(t) = \frac{h(t)}{t}$. Denoting by $S(t)$ the generating function of the row sums of the array, we get:

$$\begin{aligned} S(t) &= \frac{d(t)}{1-h(t)} = \frac{1-t-t^2-\sqrt{(1-t-t^2)(1-3t-t^2)}}{2t^2} \\ &= 1+t+3t^2+6t^3+16t^4+40t^5+109t^6+297t^8+\cdots. \end{aligned}$$

Using standard methods of asymptotic analysis [12], it is not difficult to find asymptotic values for the coefficients of $S(t)$.

3. If L is as in (2), defining Bell numbers, the sequence $(f_n^{(1)})_{n\in\mathbf{N}}$ starts $1, 1, 3, 7, 22, 75, \ldots$ and is not recorded in [17]. Thanks to our theory, it is possible to give a combinatorial interpretation to such a sequence, by performing an ECO construction described by the mixed rule:

$$\Omega: \quad \begin{cases} (1) \\ (k) \overset{+1}{\rightsquigarrow} (k)^{k-1}(k+1) \\ \quad \overset{+2}{\rightsquigarrow} (k) \end{cases}.$$

We call a *lacunary partition* of $[n] = \{1, 2, \ldots, n\}$ any partition of a subset S of $[n]$ such that $[n] \setminus S$ is a disjoint union of intervals of even cardinality. For instance, the partition $\{\{1, 8, 12\}, \{2\}, \{3, 9\}\}$ is a lacunary partition of [14]. An ECO construction for the class of lacunary partitions works as follows: given a lacunary partition π of $[n]$, construct a set of new lacunary partitions by either adding the block $\{n+1\}$, or adding $n+1$ to each of the block of π, or else leaving π unchanged, but thinking of it as a lacunary partition of $[n+2]$. Performing one of the first two operations leads to a lacunary partition of $[n+1]$, whereas the last one produces a lacunary partition of $[n+2]$. The reader can check that such a construction is encoded by Ω.

Now consider $(b)\mathbf{1}^{+1}M^{+2}$. In this case, the sequence $(l_n^{(s)})_{n\in\mathbf{N}}$ is the one determined by $\mathbf{1}$, and so $l_n^{(s)} = 1$, for all $n \in \mathbf{N}$. The polynomial $\mu_r^{(s)}(x)$ is the level polynomial of the rule associated with M with axiom (s). Applying Theorem 4.1, for the sequence $(f_n^{(b)})_{n\in\mathbf{N}}$ determined by $(b)\mathbf{1}^{+1}M^{+2}$ we get:

$$f_n^{(b)} = \sum_{k\geqslant 0}\binom{n-k}{k}\sum_i \mu_{k,i}^{(b)} = \sum_{k\geqslant 0}\binom{n-k}{k} m_k^{(b)},$$

where, of course, $(m_n^{(s)})_{n\in\mathbf{N}}$ is the sequence associated with the rule operator M with axiom (s). As before, we can easily recover the generating function $f^{(b)}(x)$ of $f_n^{(b)}$ starting from that of $m_n^{(b)}$, thus obtaining:

$$f^{(b)}(x) = \frac{1}{1-x}\cdot m^{(b)}\left(\frac{x^2}{1-x}\right).$$

Examples. We leave as an open problem that of finding an ECO construction described by $(1)\mathbf{1}^{+1}M^{+2}$ for each of the structures mentioned in the next examples. In the first of them, we also provide a rigorous proof (via Riordan arrays) of the fact that a certain sequence on [17] comes out, whereas the details of the remaining examples are left to the reader.

1. If M is the rule operator of Catalan numbers described in (3), the mixed rule $(1)\mathbf{1}^{+1}M^{+2}$ determines sequence A090344 in [17]. Such a sequence counts Motzkin paths without horizontal steps at odd height.
   According to [1], the double mixed succession rule associated with $(1)\mathbf{1}^{+1}M^{+2}$ corresponds to the A-matrix
   $$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \end{pmatrix},$$
   which gives rise to a *vertically stretched Riordan array* (see [7]):

$n \setminus k$	1	2	3	4
0	1			
1	1			
2	1	1		
3	1	2		
4	1	4	1	
5	1	7	3	
6	1	13	8	1
7	1	24	18	4

   A proper Riordan array can be obtained by simply shifting column $k+1$ up by k positions. It has the following (infinite) A-matrix:
   $$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \iddots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \end{pmatrix}.$$
   To get the generating function of the A-sequence we use the equality $A(t) = \sum_{j=0}^{\infty} t^j A(t)^{-j} P^{[j]}(t)$ (see [1]), where $P^{[j]}(t)$ is the generating function of row j in the A-matrix. Here $P^{[0]}(t) = 1+t$ and $P^{[j]}(t) = t^j$, for $j > 0$, and the solution of the above equation is $A(t) = \frac{1+t+t^2+\sqrt{1+2t+3t^2-2t^3+t^4}}{2}$.
   To compute $h(t)$, using formula $h(t) = tA(h(t))$, we get an equation of degree two whose solution is:
   $$h(t) = \frac{1}{2t}\left(1 - \sqrt{\frac{1-t-4t^2}{1-t}}\right).$$
   Since $d(t) = \frac{1}{1-t}$, the original (stretched) Riordan array is:
   $$\left(\frac{1}{1-t}, \frac{1}{2}\left(1 - \sqrt{\frac{1-t-4t^2}{1-t}}\right)\right),$$
   and the generating function of the sequence of its row sums is:
   $$S(t) = \sum_{n=0}^{\infty} S_n t^n = \frac{d(t)}{1-h(t)} = \frac{1}{2t^2}\left(1 - \sqrt{\frac{1-t-4t^2}{1-t}}\right).$$
   Also in this case it is possible to get asymptotic values for the S_n's by using standard asymptotic analysis.
2. Taking for M the rule operator of Motzkin numbers in (4) (and choosing (1) as axiom), we get sequence A026418 of [17]. It counts ordered trees having no branches of length 1, according to the number of edges.
3. If M is as in (2), defining Bell numbers, the resulting sequence $(f_n^{(1)})_{n \in \mathbf{N}}$ starts $1, 1, 2, 3, 6, 11, 23, 47, 103, \ldots$ which is not in [17].

5.2. A factorial-like rule operator

Consider the rule operator $L = \mathbf{x}^2 D$ associated with the production rule

$$\omega: \quad (k) \rightsquigarrow (k+1)^k.$$

We start by determining the family of sequences related to L.

Lemma 5.1. *If $(l_n^{(b)})_{n\in\mathbf{N}}$ is the sequence determined by L with axiom (b), then we have, for all $n \in \mathbf{N}$:*

$$l_n^{(b)} = (n+b-1)_{b-1} = \frac{(n+b-1)!}{(b-1)!} = n!\binom{n+b-1}{b-1},$$

where $(x)_y = x(x-1)\cdot\ldots\cdot(x-y+1)$ denotes the usual falling factorial.

Proof (sketch). Use a simple induction argument. For $b=1$ it is well known [11] that $l_n^{(1)} = n!$. Now observe that the recursion defined by the production rule implies that $l_{n+1}^{(b)} = b l_n^{(b+1)}$, whence the thesis. □

According to our program, we start by computing the general form of a rule operator commuting with L.

Theorem 5.1. *Let M be a rule operator such that $M(1) = x^a$, for some $a \in \mathbf{N}$. Then, M commutes with L if and only if*

$$M = L_{[a]} = \frac{\mathbf{x}^{a+1}}{(a-1)!} D^a \mathbf{x}^{a-1}.$$

Proof. Suppose that M commutes with L. On the polynomial 1 it is

$$M(x) = M\bigl(\mathbf{x}^2 D(1)\bigr) = \mathbf{x}^2 D\bigl(M(1)\bigr) = \mathbf{x}^2 D\bigl(x^a\bigr) = a x^{a+1}.$$

Now suppose by induction that $M(x^n) = n\binom{a+n-1}{a-1}x^{a+n}$. Then:

$$\begin{aligned} M\bigl(x^{n+1}\bigr) &= \frac{1}{n} M\bigl(\mathbf{x}^2 D\bigl(x^n\bigr)\bigr) = \frac{1}{n}\mathbf{x}^2 D\bigl(M\bigl(x^n\bigr)\bigr) \\ &= \frac{1}{n}\mathbf{x}^2 D\left(n\binom{a+n-1}{a-1}x^{a+n}\right) = (n+1)\binom{a+n}{a-1}x^{a+n+1}. \end{aligned}$$

We have thus showed that $M(x^n) = n\binom{a+n-1}{a-1}x^{a+n}$, that is $M = L_{[a]} = \frac{\mathbf{x}^{a+1}}{(a-1)!}D^a\mathbf{x}^{a-1}$, as desired.

For the converse, we leave to the reader the proof of the fact that the operator $L_{[a]} = \frac{\mathbf{x}^{a+1}}{(a-1)!}D^a\mathbf{x}^{a-1}$ commutes with L. □

Consider now the case $a=2$, so to obtain the operator $M = L_{[2]} = \mathbf{x}^3 D \mathbf{x}^2$. The mixed succession rule $\Omega_b = (b)L^{+1}M^{+2}$ is

$$\Omega_b: \quad \begin{cases} (b) \\ (k) \overset{+1}{\rightsquigarrow} (k+1)^k \\ \quad\;\; \overset{+2}{\rightsquigarrow} (k+2)^{k(k+1)} \end{cases}. \tag{8}$$

The first levels of its generating tree, when $b=1$, appear in Fig. 3.

In order to apply Theorem 4.1, we need to know the sequence $l_n^{(s)}$ determined by the production rule associated with L with axiom (s) and the level polynomials $\mu_r^{(s)}(x)$ of the generating tree related

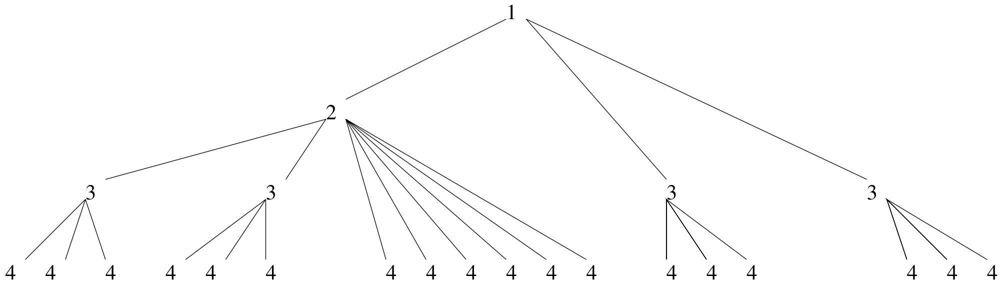


Fig. 3. The first levels of the generating tree of $(1)L^{+1}M^{+2}$.

to M with axiom (s). The first information is provided by Lemma 5.1, that is $l_n^{(s)} = n!\binom{n+s-1}{s-1}$. As far as the polynomials $\mu_r^{(s)}(x)$ are concerned, we observe that, in the generating tree associated with M, only one label appears at any given level; more precisely, the only label at level r is $(s+2r)$. Therefore $\mu_r^{(s)}(x)$ consists of only one monomial, whose coefficient is found in the following, simple lemma.

Lemma 5.2. *The generating tree associated with M having axiom (s) has $(s)^{2r}$ nodes at level r (each of which is labelled $(s+2r)$), where $(x)^y = x(x+1)\cdot\ldots\cdot(x+y-1)$ denotes the raising factorial.*

Proof. At level 0 and 1 there are 1 and $s(s+1)$ nodes, respectively. By induction, suppose to have $(s)^{2r}$ nodes at level r; each of them is labelled $(s+2r)$ and produces $(s+2r)(s+2r+1)$ sons, whence the thesis. $\square$

As a consequence, we have that $\mu_r^{(s)}(x) = (s)^{2r}x^{s+2r}$, which means that $\mu_{r,s+2r}^{(s)} = (s)^{2r}$, whereas $\mu_{r,j}^{(s)} = 0$, for $j \neq s+2i$. We are now ready to apply Theorem 4.1, thus getting for the sequence $(f_n^{(b)})_{n\in\mathbf{N}}$ determined by $(b)L^{+1}M^{+2}$ the following formula:

$$\begin{aligned} f_n^{(b)} &= \sum_{k\geqslant 0}\binom{n-k}{k}(b)^{2k}(n-2k)!\binom{n+b-1}{b+2k-1} \\ &= \frac{(n+b-1)!}{(b-1)!}\sum_{k\geqslant 0}\binom{n-k}{k} = (b)^n F_n, \end{aligned}$$

where $(F_n)_{n\in\mathbf{N}}$ is the sequence of Fibonacci numbers. To give a combinatorial interpretation for $(f_n^{(b)})_{n\in\mathbf{N}}$, consider the set $\overline{S}_n$ of *coloured permutations* of $[n]$, i.e. permutations whose elements can possibly be coloured (a coloured element will simply be overlined). We introduce the notion of a *paired coloured permutation*, i.e. a coloured permutation whose set of coloured elements is a disjoint union of intervals of even cardinality. Thus, the permutation $\overline{93}1\bar{4}6\bar{5}7\overline{28}$ is a paired coloured permutation of $\overline{S}_9$.

Proposition 5.1. *Given $b\in\mathbf{N}$, fix $\tau\in S_{b-1}$. Then $f_n^{(b)}$ is the number of paired coloured permutations $\pi\in\overline{S}_{b+n-1}$ in which the elements $1,2,\ldots,b-1$ are not coloured and appear in π as a pattern isomorphic to τ.*

Proof. Let $\pi\in\overline{S}_{b+k-1}$ be a paired coloured permutation satisfying the above hypotheses. Starting from π we perform the following construction:

1. add the noncoloured element $b+k$ in any of the $b+k$ possible positions, so to obtain $b+k$ new permutations belonging to $\overline{S}_{b+k}$;
2. add the two coloured elements $\overline{b+k}$ and $\overline{b+k+1}$ in any possible positions: this can be done in $(b+k)(b+k+1)$ different ways, and produces permutations belonging to $\overline{S}_{b+k+1}$.

The additional hypothesis that the subpermutation of π constituted by the elements $1, \dots, b-1$ must be isomorphic to τ implies that τ is the minimal permutation of our class, and any other permutation obtained using the above described construction must avoid the pattern τ in its elements $1, 2, \dots, b-1$. Thus, if a permutation $\pi \in \overline{S}_{b+k-1}$ is given label $(b+k)$, the above construction can be described by the mixed rule Ω given in (8). $\square$

Corollary 5.1. *If $b = 1$, then $f_n = f_n^{(1)} = n!F_n$, and the sequence $(f_n)_{n\in\mathbf{N}}$ enumerates the class of paired coloured permutations.*

This last sequence also appears in [17] (it is essentially sequence A005442), and can be obtained as the row sums of a particular convolution matrix (see [14]). The combinatorial interpretation of $(f_n)_{n\in\mathbf{N}}$ reported in [17] seems to be essentially different from the one given here: it would be interesting to have a bijective argument explaining how to relate them.

6. Final remarks

In the present paper we have studied doubled mixed succession rules, and, in the commutative case, we have been able to give an expression for the sequence associated with one of such rules in terms of the sequences associated with the constituent simple succession rules. The next step should be to have analogous results for more general kinds of doubled rules. For instance, one could consider two succession rules whose associated rule operators obey some weaker form of commutativity, such as $LM = qML$, for a given q (or, more generally, $LM = f(q)ML$, for some polynomial f).

Another presumably fertile line of research concerns exhaustive generation. Similarly to what has been done for classical rules [4], one can try to develop general exhaustive generation algorithms based on mixed succession rules, maybe finding a new way of defining general Gray codes depending only on the form of the mixed succession rule under consideration.

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