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Generalized Fibonacci-Like Sequence and its Properties

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Abstract

The Fibonacci Sequence can be generalized in several ways. Similar is the case with Lucas Sequence. In this paper, we study Generalized Fibonacci-Like sequence $\{M_n\}$ defined by the recurrence relation

$$M_n = M_{n-1} + M_{n-2}, \text{ for all } n \geq 2$$

with $M_0 = 2m$ and $M_1 = 1+m$, m being a fixed positive integer. The associated initial conditions are the sum of initial conditions of Fibonacci sequence and m times the initial conditions of Lucas sequence respectively. We shall define Binet's formula and generating function of Generalized Fibonacci-Like sequence. Mainly, Induction method and Binet's formula will be used to establish properties of Generalized Fibonacci-Like sequence.

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1. Introduction

Even though Fibonacci numbers were introduced in 1202 in Fibonacci's book Liber abaci, they remain fascinating and mysterious to people today. The Fibonacci sequence is a source of many nice and interesting identities as appears in the work of Vajda[10], Harris[11], and Carlitz[7].

The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, $F_0 = 0$, $F_1 = 1$. (1.1)

The sequence of Lucas numbers $\{L_n\}$ is defined by

$L_n = L_{n-1} + L_{n-2}$, $n \geq 2$, $L_0=2$, $L_1=1$. (1.2)

The Binet's formula for Fibonacci sequence is given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \qquad (1.3)$$

where $\alpha = \dfrac{1+\sqrt{5}}{2} = \text{Golden ratio} \approx 1.618$

and $\beta = \dfrac{1-\sqrt{5}}{2} \approx -0.618$.

Similarly, the Binet's formula for Lucas sequence is given by

$$L_n = \alpha^n + \beta^n = \left\{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

In this paper, we present various properties of the Generalized Fibonacci-Like sequence $\{M_n\}$ defined by

$$M_n = M_{n-1} + M_{n-2}, \text{ for all } n \geq 2 \qquad (1.4)$$

with $M_0 = 2m$ and $M_1 = 1+m$, m being a fixed positive integer.

Here the initial conditions M_0 and M_1 are the sum of initial conditions of Fibonacci sequence and m times the initial conditions of Lucas sequence respectively.

i.e. $M_0 = F_0 + mL_0$, $M_1 = F_1 + mL_1$.

The few terms of the sequence $\{M_n\}$ are

2m, 1+m, 3m+1, 4m+2, 7m + 3, and so on.

2. Preliminary results of Generalized Fibonacci-Like sequence

We need to introduce some basic results of Generalized Fibonacci-Like sequence and Fibonacci Sequence.

The relation between Fibonacci Sequence and Generalized Fibonacci-Like sequence can be written as $M_n = F_n + mLn$, $n \geq o$. (2.1)
The recurrence relation (1.1) has the characteristic equation
$x^2 - x - 1 = 0$ which has two roots

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}$$

Now notice a few things about α and β:
$\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$ and $\alpha\beta = -1$.
using these two roots, we obtain Binet's formula of recurrence relation (1.4)

$$M_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} + m\left(\alpha^n + \beta^n\right)$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} + m\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

The generating function of $\{M_n\}$ is defined as

$$\sum_{n=0}^{\infty} M_n x^n = \frac{2m + (1-m)x}{1 - x - x^2}$$

Using partial fractions , we obtain

$$\sum_{n=0}^{\infty} M_n x^n = \frac{1}{2\sqrt{5}} \sum_{n=0}^{\infty} \left[\frac{(-1)^n a_1}{a^{n+1}} + \frac{b_1}{b^{n+1}}\right] x^n$$

where $a = \frac{1+\sqrt{5}}{2}$, $a_1 = (5m-1) - \sqrt{5}\,(1-m)$

$b = \frac{-1+\sqrt{5}}{2}$, $b_1 = (5m-1) + \sqrt{5}\,(1-m)$

3. Properties of Generalized Fibonacci-Like Sequence

Despite its simple appearance the Generalized Fibonacci-Like sequence $\{M_n\}$ contains a wealth of subtle and fascinating properties[3,5,6,8]

Sums of Generalized Fibonacci-Like terms:

Theorem 3.1. Sum of first n terms of the Generalized Fibonacci-Like sequence$\{M_n\}$ is

$$M_1 + M_2 + M_3 + ... + M_n = \sum_{k=1}^{n} M_k = M_{n+2} - (3m+1) \quad (3.1)$$

This identity becomes

$$M_1 + M_2 + ... + M_{2n} = \sum_{k=1}^{2n} M_k = M_{2n+2} - (3m+1) \quad (3.2)$$

Theorem 3.2. Sum of the first n terms with odd indices is

$$M_1 + M_3 + M_5 + ... + M_{2n-1} = \sum_{k=1}^{n} M_{2k-1} = M_{2n} - 2m \quad (3.3)$$

Theorem 3.3. Sum of the first n terms with even indices is

$$M_2 + M_4 + M_6 + ... + M_{2n} = \sum_{k=1}^{n} M_{2k} = M_{2n+1} - (1+m) \quad (3.4)$$

The identities from 3.1 to 3.3 can be derived by induction method.

If we subtract equation (3.4) termwise from equation (3.3), we get alternating sum of first n numbers

$$M_1 - M_2 + M_3 - M_4 + ... + M_{2n-1} - M_{2n}$$
$$= M_{2n} - 2m - M_{2n+1} + 1 + m$$
$$= - M_{2n+1} - m + 1 \quad (3.5)$$

Adding M_{2n+1} to both sides of equation (3.5), we get

$$M_1 - M_2 + M_3 - M_4 + ... + M_{2n-1} - M_{2n} + M_{2n+1}$$
$$= - M_{2n-1} - m + 1 + M_{2n+1}$$
$$= M_{2n} - m + 1 \quad (3.6)$$

Combining (3.5) and (3.6), we obtain

$$M_1 - M_2 + M_3 - M_4 + ... + (-1)^{n+1} M_n = (-1)^{n+1} M_{n-1} - m + 1 \quad (3.7)$$

Theorem 3.4 Sum of the squares of first n terms of the Generalized Fibonacci-Like Sequence is

$$M_1^2 + M_2^2 + M_3^2 + ... + M_n^2 = \sum_{k=1}^{n} M_k^2 = M_n \; M_{n+1} - 2m\,(1+m) \quad (3.8)$$

Now we state and prove some nice identities similar to those obtained for Fibonacci and Lucas sequences [1,2,4,9]

Theorem 3.5. For every integer $n \geq 0$,

$$mM_{n+2} - m\,M_{n+1} = m\,M_n \quad (3.9)$$

Theorem 3.6. For every positive integer n,

$M_n^2 = M_n M_{n+1} - M_{n-1} M_n, \quad n \geq 1$ (3.10)

Theorem 3.7. For every positive integer n,

$M_{n+1} M_{n-1} - M_n^2 = (-1)^{n+1} (5m^2-1)$ (3.11)

Proof. we shall use mathematical induction over n.

It is easy to see that for n = 1,

$$M_2 M_0 - M_1^2 = (-1)^2 (5m^2-1)$$

$$5m^2 - 1 = 5m^2-1, \text{ which is true.}$$

Assume that the result is true for n = k. Then

$$M_{k+1} M_{k-1} - M_k^2 = (-1)^{k+1} (5m^2 - 1) \quad (3.12)$$

Adding $M_k M_{k+1}$ to each side of equation (3.12), we get

$$M_{k+1} M_{k-1} - M_k^2 + M_k M_{k+1} = (-1)^{k+1} (5m^2 - 1) + M_k M_{k+1}$$

$$M_{k+1} (M_{k-1} + M_k) - M_k^2 = (-1)^{k+1} (5m^2 - 1) + M_k M_{k+1}$$

$$M_{k+1}^2 - M_k (M_k + M_{k+1}) = (-1)^{k+1} (5m^2 - 1)$$

$$M_{k+1}^2 - M_k M_{k+2} = (-1)^{k+1} (5m^2 - 1)$$

$$-(M_k M_{k+2} - M_{k+1}^2) = (-1)^{k+1} (5m^2 - 1)$$

$$M_k M_{k+2} - M_{k+1}^2 = (-1)^{k+2} (5m^2 - 1)$$

Which is precisely our identity when n = k + 1.

Therefore, the result is true for n = k+1 also.

Hence, $M_{n+1} M_{n-1} - M_n^2 = (-1)^{n+1} (5m^2 - 1) \; \forall \; n \geq 1.$

Theorem 3.8. Let n be a positive integer. Then

$$M_{2n} = \sum_{k=0}^{n} \binom{n}{k} M_{n-k} \quad (3.13)$$

Theorem 3.9. For every positive integer n,

$$M_3 + M_6 + M_9 + \ldots\ldots + M_{3n} = 1/2 \left[M_{3n+2} - (3m+1) \right] \quad (3.14)$$

Proof. By using Binet's formula, we have

$$M_3 + M_6 + M_9 + \ldots\ldots + M_{3n}$$

$$= \frac{\alpha^3 - \beta^3}{\sqrt{5}} + m(\alpha^3 + \beta^3) + \frac{\alpha^6 - \beta^6}{\sqrt{5}} + m(\alpha^6 + \beta^6) + \ldots.. + \frac{\alpha^{3n} - \beta^{3n}}{\sqrt{5}} + m(\alpha^{3n} + \beta^{3n})$$

$$= \frac{1}{\sqrt{5}}\left[\left(\alpha^3 + \alpha^6 + \alpha^9 + ...\alpha^{3n}\right) - \left(\beta^3 + \beta^6 + ... + \beta^{3n}\right)\right]$$
$$+ m\left[\left(\alpha^3 + \alpha^6 + \alpha^9 + ...\alpha^{3n}\right) + \left(\beta^3 + \beta^6 + ... + \beta^{3n}\right)\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{\alpha^{3n+3} - \alpha^3}{\alpha^3 - 1}\right) - \left(\frac{\beta^{3n+3} - \beta^3}{\beta^3 - 1}\right)\right] + m\left[\frac{\alpha^{3n+3} - \alpha^3}{\alpha^3 - 1} + \frac{\beta^{3n+3} - \beta^3}{\beta^3 - 1}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\frac{\alpha^{3n+2} - \alpha^2}{2} - \left(\frac{\beta^{3n+2} - \beta^2}{2}\right)\right] + m\left[\frac{\alpha^{3n+2} - \alpha^2}{2} + \frac{\beta^{3n+2} - \beta^2}{2}\right]$$
$$= \frac{1}{2}\left[\frac{\alpha^{3n+2} - \beta^{3n+2}}{\sqrt{5}} + m\left(\alpha^{3n+2} + \beta^{3n+2}\right)\right]$$
$$- \frac{1}{2}\left[\frac{\alpha^2 - \beta^2}{\sqrt{5}} + m\left(\alpha^2 + \beta^2\right)\right]$$
$$= \frac{1}{2}\left(M_{3n+2} - M_2\right)$$
$$= \frac{1}{2}\left[\mathrm{M}_{3n+2} - (3\mathrm{m} + 1)\right].$$

Theorem 3.10. For every positive integer n,

$$M_5 + M_8 + M_{11} + ... + M_{3n+2} = \frac{M_{3n+4} - (7m + 3)}{2} \tag{3.15}$$

This can be derived same as theorem 3.9

4. Connection Formulae

Theorem 4.1. Let n be a positive integer. Then

$$M_{n+1} + M_{n-1} = (1+m)L_n + 2mL_{n-1},\ n \geq 1 \tag{4.1}$$

Proof. We shall prove this identity by induction over n.

For n = 1,we have

$$M_2 + M_0 = (1+m)\ L_1 + 2mL_0$$
$$5m + 1 = 5m + 1, \text{which is true}$$

Suppose that the identity holds for n = k-2 and n = k-1. Then,

$$M_{k-1} + M_{k-3} = (1+m)\ L_{k-2} + 2m\ L_{k-3} \tag{4.2}$$
$$M_k + M_{k-2} = (1+m)\ L_{k-1} + 2m\ L_{k-2} \tag{4.3}$$

Adding equation (4.2) and equation (4.3), we get

$$\left(M_{k-1} + M_k\right) + \left(M_{k-3} + M_{k-2}\right) = (1+m)\left(L_{k-2} + L_{k-1}\right) + 2m\left(L_{k-3} + L_{k-2}\right)$$

i.e. $$M_{k+1} + M_{k-1} = (1+m)\, L_k + 2m\, L_{k-1}$$
which is precisely our identity when n = k.
Hence, $M_{n+1} + M_{n-1} = (1+m)L_n + 2mL_{n-1}\ \forall\, n \geq 1.$

Theorem 4.2. Let n be a positive integer. Then

$$M_{n+1} - M_{n-1} = (1+m)\, F_n + 2m\, F_{n-1}, n \geq 1 \qquad (4.4)$$

Theorem 4.3. For every integer n ≥ 0,

$$M_{n+1} = F_{n+1} + m(L_{n+1}),\ n \geq 0 \qquad (4.5)$$

Theorem 4.4. For every integer n ≥ 0,

$M_{2n} = F_{2n} + mL_{2n}$, $n \geq 0$ (4.6)

Theorem 4.5. Let n be a positive integer. Then

$$\begin{vmatrix} M_n & F_n & 1 \\ M_{n+1} & F_{n+1} & 1 \\ M_{n+2} & F_{n+2} & 1 \end{vmatrix} = [F_n\, M_{n+1} - M_n\, F_{n+1}]$$

Proof. Let $\Delta = \begin{vmatrix} M_n & F_n & 1 \\ M_{n+1} & F_{n+1} & 1 \\ M_{n+2} & F_{n+2} & 1 \end{vmatrix}$ (4.7)

Suppose $M_n = a, M_{n+1} = b,\quad M_{n+2} = a+b$
$F_n = p,\ F_{n+1} = q,\quad F_{n+2} = p+q$ (4.8)

Substituting the value of equation (4.8) in equation (4.7), we get

$$\Delta = \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 - R_2$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_3$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ b-(a+b) & q-(p+q) & 0 \\ a+b & p+q & 1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ -a & -p & 0 \\ a+b & p+q & 1 \end{vmatrix}$$

$$\Delta = [pb - aq] \tag{4.9}$$

Substituting the values of the equation (4.8) in equation (4.9), we get

$$\Delta = [F_n M_{n+1} - M_n F_{n+1}]$$

Hence, $\begin{vmatrix} M_n & F_n & 1 \\ M_{n+1} & F_{n+1} & 1 \\ M_{n+2} & F_{n+2} & 1 \end{vmatrix} = [F_n M_{n+1} - M_n F_{n+1}]$

Theorem 4.6. For every integer n ≥ 2,

$$\begin{vmatrix} M_n & M_{n+1} & M_{n+2} \\ M_{n+2} & M_n & M_{n+1} \\ M_{n+1} & M_{n+2} & M_n \end{vmatrix} = 2\left(M_n^3 + M_{n+1}^3\right) \tag{4.10}$$

Theorem 4.7. For every positive integer n,

$$\begin{vmatrix} M_n & L_n & 1 \\ M_{n+1} & L_{n+1} & 1 \\ M_{n+2} & L_{n+2} & 1 \end{vmatrix} = \left(L_n M_{n+1} - M_n L_{n+1}\right) \tag{4.11}$$

Theorem 4.8. For every positive integer n,

$$\begin{vmatrix} 1+M_n & M_{n+1} & M_{n+2} \\ M_n & 1+M_{n+1} & M_{n+2} \\ M_n & M_{n+1} & 1+M_{n+2} \end{vmatrix} = 1 + M_n + M_{n+1} + M_{n+2} \tag{4.12}$$

Theorem 4.9. For every positive integer n,

$$\begin{vmatrix} M_n + M_{n+1} & M_{n+1} + M_{n+2} & M_{n+2} + M_n \\ M_{n+2} & M_n & M_{n+1} \\ 1 & 1 & 1 \end{vmatrix} = 0 \qquad (4.13)$$

The identities from (4.6) to (4.9) can be proved similarly as (4.5)

5. Conclusion

There are many known identities established for Fibonacci and Lucas sequences. This paper describes comparable identities of Generalized Fibonacci-Like sequence . We have also developed connection formulas for Generalized Fibonacci-Like sequence, Fibonacci sequence and Lucas sequence respectively.It is easy to discover new identities simply by varying the pattern of known identities and using inductive reasoning to guess new results. Of course, the ideas can be extended to more general recurrent sequences in obvious way.

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