LUCAS SEQUENCE, ITS PROPERTIES AND GENERALIZATION

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CERTIFICATE

This is to certify that the thesis entitled **"LUCAS SEQUENCE ITS, PROPERTIES AND GENERALIZATION"** is a bonafide review work carried out by **Mr. Biswajit Barik** (**Roll No: 411MA2130.**) a student of Master of Science in Mathematics at National Institute of Technology, Rourkela, during the year 2013. In partial fulfilment of the requirements for the award of the Degree of Master of Science in Mathematics under the guidance of **Dr. Gopal Krishna Panda,** National Institute of Technology, Rourkela and that project has not formed the basis for the award previously of any degree, diploma, associateship, fellowship, or any other similar title.

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Chapter 1

INTRODUCTION

The cocept of Fibonacci numbers was first discovered by Leonardo de Fibonacci de pisa. the fibonacci series was derived from the solution to a problem about rabbits. The problem is: Suppose there are two new born rabbits, one male and the other female. Find the number of rabbits produced in a year if

- Each pair takes one month to become mature:
- Each pair produces a mixed pair every month, from the second month:
- All rabbits are immortal

Suppose, that the original pair of rabbits was born on January 1. They take a month to become mature, so there is still only one pair on February 1. On March 1, they are two months old and produce a new mixed pair, so total is two pair. So continuing like this, there will be 3 pairs in April, 5 pairs in May and so on.

The numbers $1, 1, 2, 3, 5, 8, ...$ are Fibonacci numbers.They have a fascinating property: Any Fibonacci number,except the first two,is the sum of the two immediately preceding Fibonacci numbers.(At the given rate, there will be 144 pairs rabbit on December 1)

This yeild the following recursive defination of the *nth* Fibonacci number F_n

$$F_1 = 1$$

$$F_2 = 1$$

$$\vdots$$

$$F_n = F_{n-1} + F_{n-2}, n \geq 3$$

Closely related to Fibonacci numbers are the Lucas numbers $1, 3, 4, 7, 11, ...$ named after Lucas.Lucas numbers L_n are defined recursively as follows

$$L_1 = 1$$

$$L_2 = 3$$

$$\vdots$$

$$L_n = L_{n-1} + L_{n-2}, n \geq 3$$

In Chapter 4, we introduce the k-Fibonacci numbers and the generation is justified.

There is a huge interest of modern science in the application of the Golden Section and Fibonacci numbers.The Fibonacci numbers F_n are the terms of the sequence $0, 1, 2, 3, 5, ...$ wherein each term is the sum of the two previous terms,beginning with the values $F_0 = 0$,and $F_1 = 1$. On the other hand the ratio of two consecutive Fibonacci numbers converges to Golden mean, or Golden section, $\phi = \frac{1+\sqrt{5}}{2}$, which appers in modern research, particularly physics of the high energy particles or theoretical physics.

In this section, we present a generalization of the classical Fibonacci numbers by mean of a recurrence equation with a parameter k. In the sequel, we generalize some properties of the classical Fibonacci sequence.

Chapter 2

Preliminaries

In this chapter we recall some definitions and known results on elementary number theory. this chapter serves as base and background for the study of subsequent chapters. We shall keep on refereing back to it as and when required.

2.1 Division Algorithm:

Let a and b be two integers, where $b > 0$. Then there exist unique integers q and r such that $a = bq + r, 0 \leq r < b$.

Definition 2.1.1. *(Divisibility) An integer a is said to be divisible by an integer $d \neq 0$ if there exists some integer c such that $a = dc$.*

Definition 2.1.2. *If a and b are integers, not both zero, then the greatest common divisor of a and b, denoted by* gcd (a,b) *is the positive integer d satisfying*

1. *$d|a$ and $d|b$.*
2. *if $c|a$ and $c|b$ then $c|d$.*

Definition 2.1.3. *(Relatively Prime) Two integer a and b, not both of which are zero, are said to be relatively prime whenever $gcd(a, b) = 1$.*

2.2 Euclidean Algorithm

Euclidean algorithm is a method of finding the greatest common divisor of two given integers. This is a repeated application of division algorithm.

Let a and b two integers whose greatest common divisor is required. since $gcd(a, b) = gcd(|a|, |b|)$, it is enough to assume that a and b are positive integers. Without loss of generality, we assume $a > b > 0$. Now by division algorithm, $a = bq_1 + r_1$, where $0 \leq r_1 < b$. If it happens that $r_1 = 0$, then $b|a$ and $gcd(a, b) = b$. If $r_1 \neq 0$, by division algorithm $b = r_1q_2 + r_2$, where $0 \leq r_2 < r_1$. If $r_2 = 0$, the process stops. If $r_2 \neq 0$ by division algorithm $r_1 = r_2q_3 + r_3$, where $0 \leq r_3 < r_2$. The process continues until some zero reminder appears. This must happens because the reminders $r_1, r_2, r_3, ...$ forms a decreasing sequence of integers and since $r - 1 < b$, the sequence contains at most b non-negative integers. Let us assume that $r_{n+1} = 0$ and r_n is the last non-zero reminder. We have the following relation:

$$
\begin{aligned}
a &= bq_1 + r_1, 0 < r_1 < b \\
b &= r_1q_2 + r_2, 0 < r_2 < r_1 \\
r_1 &= r_2q_3 + r_3, 0 < r_3 < r_2 \\
&\vdots \\
r_{n-2} &= r_{n-1}q_n + r_n, 0 < r_n < r_{n-1} \\
r_{n-1} &= r_nq_{n+1} + 0
\end{aligned}
$$

Then

$$gcd(a, b) = r_n.$$

2.3 Fundamental Theorem of Arithmetic:

Any positive integer is either 1 or prime, or it can be expressed as a product of primes, the representation being unique except for the order of the prime factors.

2.4 Congruence:

Let m be fixed positive integer. Two integers a and b are said to be congruent modulo m if $a - b$ is divisible by m and symbolically this is denoted by $a \equiv b(modm)$. We also used to say a is congruent to b modulo m.

Some Properties of Congruence:

1. If $a \equiv a(modm)$.

2. If $a \equiv b(modm)$, then $b \equiv a(modm)$.

3. If $a \equiv b(modm), b \equiv c(modm)$, then $a \equiv c(modm)$.

4. If $a \equiv b(modm)$, then for any integer c

$(a + c) \equiv (b + c)(modm); ac \equiv bc(modm)$.

Definition 2.4.1. *(Fibonacci Numbers) Fibonacci Numbers are the numbers in the integer sequence defined by the recurrence relation* $F_n = F_{n-1} + F_{n-2}$ *for all* $n \geq 2$ *with* $F_0 = 0$ *and* $F_1 = 1$.

Definition 2.4.2. *(Lucas Numbers) Lucas Numbers are the numbers in the integer sequence defined by the recurrence relation* $L_n = L_{n-1} + L_{n-2}$ *for all* $n > 1$ *and* $L_0 = 2$ *and* $L_1 = 1$.

2.5 Golden Ratio and Golden Rectangle

The Golden Ratio denoted by ϕ, is an irrational mathematical constant, approximately 1.61803398874989. In mathematics two quantities are in the golden ratio of the sum of quantities to the larger quantity is equal to the ratio of the larger quantity to the smaller one. Two quantities a and b are said to be in the golden ratio if

$$\frac{a+b}{a} = \frac{a}{b} = \phi.$$

Then

$$\frac{a+b}{a} = 1 + \frac{a}{b} = 1 + \frac{1}{\phi}$$
$$1 + \frac{1}{\phi} = \phi$$
$$\phi^2 = \phi + 1$$
$$\phi^2 - \phi - 1 = 0$$
$$\phi = \frac{1+\sqrt{5}}{2}$$
$$\phi = 1.61803398874989$$
$$\phi \simeq 1.618.$$

Definition 2.5.1. *(Golden Rectangle) A golden rectangle is one whose side lengths are in golden ratio, that is, approximately 1 :* $\frac{1+\sqrt{5}}{2}$.

Construction of Golden Rectangle A Golden Rectangle can be constructed with only straightedge and compass by this technique

1. Construct a simple square.
2. Draw a line from the midpoint of one side of the square to an opposite corner.
3. Use the line as radius to draw an arc that defines the height of the rectangle.
4. Complete the golden rectangle.

2.6 Divisibility

Theorem 2.6.1. *For any integers a,b,c*

1. *If* $a|b$ *and* $c|d$*, then* $ac|bd$.
2. *If* $a|b$ *and* $b|c$*, then* $a|c$.
3. *If* $a|b$ *and* $a|c$*, then* $a|(bx+cy)$ *for arbitrary integers* x *and* y.

Proof.

1. Since $a|b$ and $c|d$ then there exits $r,s \in Z$ such that $b = ra$ and $d = cs$. Now $bd = ra.sc = rs.ac \Rightarrow ac|bd$.

2. Since $a|b$ and $b|c$ then there exits $r, s \in Z$ such that $b = ra$ and $c = sb$. Now $c = sb = sra \Rightarrow a|c$.

3. Since $a|b$ and $a|c$ then there exits $r, s \in Z$ such that $b = ar$ and $c = as$. But then $bx + cy = arx + asy = a(rx + sy)$ whatever the choice of x and y. Since $rx + sy$ is an integer then $a|(bx + cy)$.

Chapter 3

Properties of Fibonacci and Lucas Numbers

3.1 The Simplest Properties of Fibonacci Numbers

Theorem 3.1.1. *The sum of the first n fibonacci numbers is equal to* $F_{n+2} - 1$.

Proof. We have

$$\begin{aligned} F_1 &= F_3 - F_2, \\ F_2 &= F_4 - F_3, \\ &\vdots \\ F_{n-1} &= F_{n+1} - F_n, \\ F_n &= F_{n+2} - F_{n+1}. \end{aligned}$$

Adding up these equations term by term, we get $F_1+F_2+F_3+....+F_n = F_{n+2}-F_2 = F_{n+2}-1$.

Theorem 3.1.2. *The sum of first n fibonacci with odd suffixes is equal to* F_{2n}.

Proof. We know

$$F_1 = F_2,$$
$$F_3 = F_4 - F_2,$$
$$F_5 = F_6 - F_4,$$
$$\vdots$$
$$F_{2n-1} = F_{2n} - F_{2n-2}.$$

Adding up these equations term by term, we obtain

$$F_1 + F_3 + F_5 + ... + F_{2n-1} = F_{2n}.$$

Theorem 3.1.3. $F_1^2 + F_2^2 + ... + F_n^2 = F_n F_{n+1}$.

Proof. We know that

$$F_k F_{k+1} - F_{k-1} F_k = F_k(F_{k+1} - F_{k-1}) = F_k^2$$
$$F_1^2 = F_1 F_2$$
$$F_2^2 = F_2 F_3 - F_1 F_2$$
$$\vdots$$
$$F_n^2 = F_n F_{n+1} - F_{n-1} F_n.$$

Adding up these equations term by term, we get

$$F_1^2 + F_2^2 + ... + F_n^2 = F_n F_{n+1}.$$

Theorem 3.1.4. $F_{n+m} = F_{n-1} F_m + F_n F_{m+1}$.

Proof. We shall prove the theorem by the method of induction on m. for $m = 1$, we get $F_{n+1} = F_{n-1} F_1 + F_n F_{1+1} = F_{n-1} + F_n$ Which is true. Suppose that it is true for $m = k$ and

$m = k + 1$, we shall prove it is also true that $m = k + 2$.

Let

$$F_{n+k} = F_{n-1}F_k + F_nF_{k+1}$$

and

$$F_{n+(k+1)} = F_{n-1}F_{k+1} + F_nF_{k+2}.$$

Adding these two equations, we get

$$F_{n+(k+2)} = F_{n-1}F_{k+2} + F_nF_{k+3}.$$

Hence

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}.$$

Theorem 3.1.5. $F_{n+1}^2 = F_nF_{n+2} + (-1)^n$

Proof. We shall prove the theorem by induction on n. We have since, $F_2^2 = F_1F_3 - 1 = 1$, the assertion is true for n = 1. let us assume that the theorem is true for n = $1, 2, ..., k$. Then adding $F_{n+1}F_{n+2}$ to both sides, we get

$$F_{n+1}^2 + F_{n+1}F_{n+2} = F_{n+1}F_{n+2} + F_nF_{n+2} + (-1)^n.$$

Which implies that $F_{n+1}(F_{n+1} + F_{n+2}) = F_{n+2}(F_n + F_{n+1}) + (-1)^n$. This simplifies to $F_{n+1}F_{n+3} = F_{n+2}^2 + (-1)^n$. Finally we have, $F_{n+2}^2 = F_{n+1}F_{n+2} + (-1)^{n+1}$.

3.2 Number-Theoretic Properties of Fibonacci Numbers

Theorem 3.2.1. *For the Fibonacci sequence, gcd*$(F_n, F_{n+1}) = 1$ *for every* $n \geq 1$.

Proof. let $gcd(F_n, F_{n+1}) = d > 1$. Then $d|F_n$ and $d|F_{n+1}$. Then $F_{n+1} - F_n = F_{n-1}$ will also be divisible by d. Again, We know that $F_n - F_{n-1} = F_{n-2}$. This implies $d|F_{n-2}$. Working backwards, the same argument shows that $d|F_{n-3}, d|F_{n-4}$,... and finally that $d|F_1 = 1$. This is impossible. Hence $gcd(F_n, F_{n+1}) = 1$ for every $n \geq 1$

Theorem 3.2.2. *For $m \geq 1$, $n \geq 1$, F_{nm} is divisible by F_m*

Proof. We shall prove the theorem by induction on n. For n = 1 the theorem is true. Let us assume that $f_m|F_{nm}$, for n=1,2,3,.....k. Now $F_{m(k+1)} = F_{mk} + F_m = F_{mk-1}F_m = F_{mk}F_{m+1} + F_m$. The right hand site of the equation is divisible by F_m. Hence $d|F_{m(k+1)}$.

Lemma 3.2.1. *if $m = nq + r$, then $gcd(F_m, F_n) = gcd(F_r, F_n)$.*

Proof. Observe that

$gcd(F_m, F_n) = gcd(Fnq + r, F_n) = gcd(F_{nq-1}F_r + F_{qn}F_{r+1}, F_n) = gcd(F_{nq-1}F_r, F_n)$. Now we claim that $gcd(F_{nq-1,F_n}) = 1$. Let $d = gcd(F_{nq-1}, F_n)$. Then $d|F_{nq-1}$ and $d|F_n$. Also that $F_n|F_{nq}$. Therefore $d|F_{nq}$. This d is the positive common divisor of F_{nq} and F_{nq-1}. but $gcd(F_{nq-1}, F_{nq}) = 1$. This is an absurd. Hence $d = 1$.

Theorem 3.2.3. *The greatest common divisor of two Fibonacci number is again a Fibonacci number.*

Proof. Let F_m and F_n be two Fibonacci Number.

Let us assume that $m \geq n$. Then by applying Euclidian Algorithm to m and n, We get the following system of equations

$$m = q_1 n + r_1, 0 \leq r_1 < n$$

$$n = q_2 r_1 + r_2, 0 \leq r_2 < r_1$$

$$r_1 = q_3 r_2 + r_2, 0 \leq r_3 < r_2, ...$$

$$r_{n-2} = q_n r_{n-1} + r_n, 0 \leq r_n < r_{n-1}$$

$$r_{n-1} = q_{n+1} r_n + 0$$

Then form the previous lemma

$$\begin{aligned} gcd(F_m, F_n) &= gcd(F_{r_1}, F_n) \\ &= gcd(F_{r_1}, F_{r_1}) \\ &\vdots \\ &= gcd(F_{r_{n-2}}, F_{r_n}). \end{aligned}$$

Since $r_n|r_{n-1}$, then $F_{r_n}|F_{r_{n-1}}$. Therefore $gcd(F_{r_{n-1}}, F_{r_n}) = F_{r_n}$. But r_n, being the last non-zero reminder Euclidian Algorithm for m and n, is equal to $gcd(m, n)$. Thus $gcd(F_m, F_n) = F_d$, Where $d = gcd(m, n)$.

Theorem 3.2.4. *In the Fibonacci sequence, $F_m|F_n$ if and only if $m|n$.*

Proof. If $F_m|F_n$, then $gcd(F_m, F_n) = F_m$. But we know that $gcd(F_m, F_n) = F_{gcd(m,n)}$. This implies that $gcd(m, n) = m$. Hence $m|n$.

Theorem 3.2.5. *The sequence of ratio of successive Fibonacci Numbers $F_{n+1}|F_n$ converges to Golden Ratio i.e., $lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi$.*

Proof. We consider the sequence $r_n = \frac{F_{n+1}}{F_n}$, for n = 1,2,3,....Then by definition of Fibonacci Numbers, we have $r_n = \frac{F_{n+1}}{F_n} = \frac{F_n+F_{n-1}}{F_n} = 1 + \frac{1}{r_{n-1}}$.
When $n \to \infty$, then we can write the above equation in limits:

$$\begin{aligned} x &= 1 + \frac{1}{x} \\ x^2 &= 1 + x = x^2 - x - 1 = 0 \\ x &= \frac{1+\sqrt{5}}{2} = \phi \end{aligned}$$

Hence,

$$lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi$$

3.3 Binet's Formulae for Fibonacci and Lucas Numbers

Lemma 3.3.1. *Let* $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$*, so that* α *and* β *are both roots of the equation* $x^2 = x + 1$*. Then* $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$*, for all* $n \geq 1$*.*

Proof. When $n = 1$, $F_1 = 1$ Which is true. let us assume that it is true for $n = 1, 2, ..., n$. Then $F_{k-1} = \frac{\alpha^{k-1} - \beta^{k-1}}{\sqrt{5}}$ and $F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}}$. Adding these two equations, we get $F_k + F_{k-1} = \frac{\alpha^k}{\sqrt{5}}\,(1 + \alpha^{-1}) + \frac{\beta^k}{\sqrt{5}}\,(1 + \beta^{-1})$. Then $F_{k+1} = \frac{\alpha(k+1) + \beta(k+1)}{\sqrt{5}}$.

Lemma 3.3.2. *Let* $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$*, so that* α *and* β *are both roots of the equation* $x^2 = x + 1$*. Then* $L_n = \alpha^n + \beta^n$*, for all n ≥1.*

Proof. For n = 1, $L_1 = 1$. Then the theorem is true for n = 1. Let us assume that it is true for n = 1,2,...,k. We have to prove that it is true for n = k+1. Now

$$L_k + L_{k-1} = \alpha^k + \alpha^{k-1} + \beta^k + \beta^{k-1}$$
$$L_{k+1} = \alpha^k(1 + \alpha^{-1}) + \beta^k(1 + \beta^{-1})$$
$$L_{k+1} = \alpha^k(1 + \alpha - 1) + \beta^k(1 + \beta - 1)$$
$$L_{k+1} = \alpha^{k+1} + \beta^{k+1}$$

3.4 Relation Between Fibonacci and Lucas Numbers

Theorem 3.4.1. $L_n = F_{n-1} + F_{n+1}$*, for n>1.*

Proof. We know that

$$L_{k+1} = L_k + L_{k-1}$$
$$L_{k+1} = (F_{k-1} + F_{k+1}) + (F_{k-2} + F_k)$$
$$L_{k+1} = (F_{k-1} + F_{k-2}) + (F_k + F_{k+1})L_{k+1} \qquad = F_k + F_{k+2}$$

Theorem 3.4.2. *For all $n \geq 1$, $F_{2n} = L_n F_n$.*

Proof. Now

$$\begin{aligned} L_n F_n &= \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)(\alpha^n + \beta^n) \\ L_n F_n &= \frac{1}{\sqrt{5}}(\alpha^{2n} - \beta^{2n}) \\ L_n F_n &= F_{2n}. \end{aligned}$$

Lemma 3.4.1. $L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n$ *for* $n \geq 1$.

Proof. Induction:

$$\begin{aligned} L_{n+1}^2 - L_n L_{n+2} &= L_{n-1}L_{n+1} - L_n^2 \\ &= -5(-1)^n. \end{aligned}$$

3.5 Further relation between the Lucas and Fibonacci Numbers

Lemma 3.5.1. $2F_{m+n} = F_m L_n + F_n L_m$.

Proof. By induction:

$$\begin{aligned} F_{n+m+1} &= F_{n+m} + F_{n+m-1} \\ &= \frac{1}{2}(F_n L_m + F_m L_n) + \frac{1}{2}(F_n L_{m-1} + F_{m-1} L_n) \\ &= \frac{1}{2}(F_n(L_m + Ln - 1)) + L_n(F_m + F_{m-1}) \\ &= \frac{1}{2}(F_n L_{m+1} + L_n F_{m+1}) \end{aligned}$$

Theorem 3.5.1. *Two further relations:*

(a) $L_n^2 - 5F_n^2 = 4(-1)^n$

(b) $L_{n+1}L_n - 5F_{n+1}F_n = 2(-1)^n$

Proof.

$$\begin{aligned}(a) L_n^2 - 4((-1)^n + F_n^2) &= (F_{n+1} + F_{n-1})^2 - 4(F_{n-1}F_{n+1}) \\ &= (F_{n+1} - F_{n-1})^2 \\ &= F_n^2\end{aligned}$$

$$\begin{aligned}(b) L_{n+2}L_{n+1} - 5F_{n+2}F_{n+1} &= (L_{n+1} + L_n)L_{n+1} - 5(F_{n+1}F_n)F_{n+1} \\ &= L_{n+1}^2 + L_nL_{n+1} - 5F_{n+1}^2 - 5F_nF_{n+1} \\ &= L_{n+1}^2 - 5F_{n+1}^2 + 2(-1)^n \\ &= 4(-1)^{n+1} + 2(-1)^n \\ &= 2(-1)^n\end{aligned}$$

Chapter 4

K-Lucas Number

Definition 3.1 For any integer number $k \geq 1$,the k-th Fibonacci sequence, say $F_{k,n}$ is defined recurrently by:

$$F_{k,0} = 0$$

$$F_{k,1} = 1$$

$$\vdots$$

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} for n \geq 1$$

As particular case

• if $k = 1$, we obtain the classical Fibonacci sequence 0,1,2,3,5,8,...

• if $k = 2$, the Pell sequence appears 0,1,2,5,12,29,70,...

• if $k = 3$, we obtain the sequence $F_{3,n} = 0, 1, 3, 10, 33, 109, ...$

From definition of the k-Fibonacci numbers, the first of them are presented in Table 1, and from these expression, one may deduce the value of any k-Fibonacci number by simple substitution. For example, the seventh element of the 4-Fibonacci sequence,

$F_{4,n}$ is $F_{4,7} = 4^6 + 5.4^4 + 6.4^2 + 1 = 5437$.

$$F_{k,1} = 1$$

$$F_{k,2} = k$$

$$F_{k,3} = k^2 + 1$$

$$F_{k,4} = k^3 + 2k$$

$$F_{k,5} = k^4 + 3k^2 + 2$$

$$F_{k,6} = k^5 + 4k^3 + 3k$$

$$F_{k,7} = k^6 + 5k^4 + 6k^2 + 1$$

$$F_{k,8} = k^7 + 6k^5 + 10k^3 + 4k$$

There are a large number of k-Fibonacci sequence indexed in *The Online Encyclopedia of Integer Sequence*, from now on **OEIS**, being the first

$$\bullet F_{1,n} = \{0, 1, 1, 2, 3, 5, 8, ...\} : A000045$$

$$\bullet F_{2,n} = \{0, 1, 2, 5, 12, 29, ...\} : A000129$$

$$\bullet F_{3,n} = \{0, 1, 3, 10, 33, 109, ...\} : A006190$$

4.1 Generation of the *k*-Lucas numbers from the *k*-Fibonacci numbers

we introduce some sequences obtained from the k-Fibonacci sequences and then some properties of the k-Lucas number will be proved.

Theorem 4.1.1. *For any integer n, number $(k^2+4)F_{k,n}^2 + 4(-1)^n$ is a perfect square.*

Proof. From the Binet formula for the k-Fibonacci numbers.

Binet formula: $(f_{k,n} = \sigma_k^n - (\sigma_k)^{-n}$, Where $\sigma_k = \frac{k+\sqrt{k^2+4}}{2}$ is the positive root of the characteristic equation $r^2 - k.r - 1 = 0$ associated to the recurrence relation defining k-

Fibonacci numbers.)

We obtain

$$F_{k,n}^2 = \frac{\sigma_k^{2n} - 2(-1)^n + \sigma_k^{-2n}}{k^2+4}$$

If n is even $(k^2+4)F_{k,n}^2 + 4 = \sigma_k^{2n} - 2 + \sigma_k^{-2n} + 4 = (\sigma_k^n + \sigma_k^{-n})^2$

If n is odd $(k^2+4)F_{k,n}^2 - 4 = \sigma_k^{2n} + 2 + \sigma_k^{-2n} - 4 = (\sigma_k^n - \sigma_k^{-n})^2$

Now we show that first sequences obtained after finding the square root of the numbers of the preceding form and we show also the reference code of these in **OEIS**:

- $L_1 = L_{1,n} = \{2, 1, 3, 4, 7, 11, 18, 29, ...\}$: A000032
- $L_2 = L_{2,n} = \{2, 2, 6, 14, 34, 82, 198, 478, ...\}$: A002203
- $L_3 = L_{3,n} = \{2, 3, 11, 36, 119, 393, 1298, 4287, ...\}$: A006497

First one is the well known Lucas sequence and second one is the Pell-Lucas sequence. For this reason, we have decide to call them *The Lucas Sequence.*

Elements of these sequences, say $L_k = L_{k,n}$, verify the following recurrence law:

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1} \text{ for n} \geq 1$$

With initial condition $L_{k,0} = 2$ and $L_{k,1} = k$

Classical Lucas number $L_{1,n}$ are related with the Artin's Constant. Sequence $F_{k,n}$ and $L_{k,n}$ are called conjugate sequence in a k-Fibonacci Lucas sequence.

Expression of first k-Lucas numbers are presented in Table 2,and from these expression ,anyone may deduce the value of any k-Lucas number by simple substitution on the corresponding $L_{k,n}$ as we have done for $F_{k,n}$. First k-Lucas number are showed in Table.

$$L_{k,0} = 2$$

$$L_{k,1} = k$$

$$L_{k,2} = k^2 + 2$$

$$L_{k,3} = k^3 + 3k$$

$$L_{k,4} = k^4 + 4k^2 + 2$$

$$L_{k,5} = k^5 + 5k^3 + 5k$$

$$L_{k,6} = k^6 + 6k^4 + 9k^2 + 2$$

$$L_{k,7} = k^7 + 7k^5 + 14k^3 + 7k$$

Particular cases:

- For $k = 1$, the classical Lucas sequence appears: {2,1,3,4,7,11,18,...}
- For $k = 2$, we obtain the Pell-Lucas sequence: {2,2,6,14,34,82,198,...}

Theorem 4.1.2. ***(Binet formula)*** k-*Lucas numbers are given by the formula* $L_{k,n} = \sigma_k^n - (\sigma_k)^{-n}$ *with* $\sigma_k = \frac{k+\sqrt{k^2+4}}{2}$.

proof. Characteristic equation of the recurrence is $r^2 - kr - 1 = 0$,which solution are $\sigma_k = \frac{k+\sqrt{k^2+4}}{2}$ and $\acute{\sigma_k} = \frac{k-\sqrt{k^2+4}}{2}$.So, solution of equation is $L_{k,n} = C_1\sigma_k^n + C_2\sigma_k^m$. By doing $n = 0 \to L_{k,o} = 2$ and $n = 1 \to L_{k,1} = k$, we obtain the values $C_1 = C_2 = 1$

Finally,talking into account $\sigma_k.\acute{\sigma_k} = -1 \to \acute{\sigma_k} = -\frac{1}{\sigma_k}$ and then $L_{k,n}$.

- If $k = 1$ we obtain the classical Lucas numbers, and then $\sigma_1 = \frac{1+\sqrt{5}}{2}$ is well known as the Golden ratio, Φ , while $\acute{\sigma_1}$ is usually written as ϕ. in this notation the general term of the classical Lucas sequence is given by

$$L_n = \phi^n + \varphi^n = (\tfrac{1+\sqrt{5}}{2})^n + (\tfrac{1-\sqrt{5}}{2})^n$$

• If k = 2,then $\sigma_2 = 1 + \sqrt{2}$ and it is known as *the silver ratio* and correspondence sequence is the Pell-Lucas sequence in which $LP_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$

• Finally, if $k = 3$, then $\sigma_3 = \frac{3+\sqrt{13}}{2}$ is known as the *the bronze ratio*

Theorem 4.1.3. ***(Catalan Identity)*** *For* $r > n$, $L_{k,n-r}L_{k,n+r} - L^2_{k,n} = (-1)^{n+r}L_{k,2r} + 2(-1)^{n+1}$.

Proof. By replacing Binet Identity in expression $L_{k,n-r}L_{k,n+r} - L^2_{k,n}$, and taking into account σ_k . $\acute{\sigma_k}$ = -1 $\rightarrow \acute{\sigma_k} = -\sigma_k$, we find out

$$\begin{aligned} L_{k,n-r}L_{k,n+r} - L^2_{k,n} &= (\sigma_k^{n-r} + (-\sigma_k)^{-n+r})(\sigma_k^{n+r} + (-\sigma_k)^{-n-r}) - (\sigma_k^n + (-\sigma_k)^{-n})^2 \\ &= (-1)^{n+r}(\sigma_k^{-2r} + \sigma_k^{2r}) - 2(-1)^n \\ &= (-1)^{n+r}L_{k,2r} + 2(-1)^{n+1} \end{aligned}$$

In the particular case when r = 1, the Simson (or Cassini) formula for the k-Lucas number is obtained: $L_{k,n-1}L_{k,n+1} - L^2_{k,n} = (-1)^{n+1}(k^2+4)$ only taking into account that $L_{k,2} = k^2+2$. If r = 2, then it is $L^2_{k,n} - L_{k,n-2}L_{k,n+2} = (-1)^{n-1}L_{k,4} + 2(-1)^n = (-1)^{n-1}k^2(k^2+4)$

Finally, if $k = 1$, then $L^2_n - L_{n-1}L_{n+1} = (-1)^n 5$

Corollary 4.1.1. ***(Gelin-Cesaro's Identity)***

$$L_{k,n-2}L_{k,n-1}L_{k,n+1}L_{k,n+2} - L^4_{k,n} + k^2(k^2+4)^2 = (-1)^n(k^4+3k^2-4)L^2_{k,n}$$

If $k = 1$, for the classical Lucas number, we obtain

$$L_{n-2}L_{n-1}L_{n+1}L_{n+2} + 25 = L^4_n$$

Theorem 4.1.4. ***(Convolution theorem)*** *For* $m \geq 1$, $L_{k,n+1}L_{k,m} + L_{k,m}L_{k,m-1} = (k^2 + 4)F_{k,n+m}$ *is verified.*

Proof. By induction. For m = 1:

$$\begin{aligned} L_{k,n+1}L_{k,1} + L_{k,n}L_{k,0} &= kL_{k,n+1} + 2L_{k,n} \\ &= L_{k,n+2} + L_{k,n} \\ &= F_{k,n+3} + 2F_{k,n+1} + F_{k,n-1} \\ &= kF_{k,n+2} + 3F_{k,n+1} + F_{k,n+1} - kF_{k,n} \\ &= k^2F_{k,n+1} + kF_{k,n} + 4F_{k,n+1} - kF_{k,n} \\ &= (k^2+4)F_{k,n+1} \end{aligned}$$

Let us suppose formula is true until m - 1:

$L_{k,n+1}L_{k,m-1} + L_{k,n} + L_{k,m-2} = (k^2+4)F_{k,n+m+1}$

Then

$$\begin{aligned} (k^2+4)F_{k,n+m} &= (k^2+4)(kF_{k,n+m-1} + F_{k,n+m-2}) \\ &= k(L_{k,n+1}L_{k,m-1} + L_{k,n}L_{k,m-2}) + (L_{k,n+1}L_{k,m-2} + L_{k,n}L_{k,m-3}) \\ &= L_{k,n+1}L_{k,m} + L_{k,n}L_{k,m-1} \end{aligned}$$

Particular case:

• If $k = 1$, for both the classical Lucas and classical Fibonacci sequences, formula $L_{n+1}L_m + L_nL_{m-1} = 5F_{n+m}$ is obtained.

• If $m = n+1$, we obtain again formula $L^2_{k,n+1} + L^2_{k,n} = (k^2+4)F^2_{k,2n+1}$

• In this last case, if k = 1, for the classical sequence it is obtained $L^2_{n+1} + L^2_n = F_{2n+1}$

• If $m = 1$, then it is $L_{k,n+1}L_{k,1} + L_{k,n}L_{k,0} = (k^2+4)F_{k,n+1}$

$\rightarrow kL_{k,n+1} + 2L_{k,n} = (k^2+4)F_{k,n+1}$

$\rightarrow kL_{k,n+2} + L_{k,n} = (k^2+4)F_{k,n+1}$

And consequently, changing n by n - 1 we obtain again formula

$$L_{k,n+1} + L_{k,n-1} = (k^2+4)F_{k,n}$$

Theorem 4.1.5. ***(D'Ocagne identity)*** *If* $m \geq n$*:* $L_{k,m}L_{k,n+1} - L_{k,m+1}L_{k,n} = (-1)^{n+1}(k^2 + 4)F_{k,m-n}$

Proof. By induction. for n = 0 and applying again formula

$$\begin{aligned} L_{k,m}L_{k,1} - L_{k,m+1}L_{k,0} &= kL_{k,m} - 2L_{k,m+1} \\ &= -(L_{k,m-1} + L_{k,m+1}) \\ &= -(k^2+4)F_{k,m} \end{aligned}$$

Let us suppose formula is true until n - 1:

$L_{k,m}L_{k,1} - L_{k,m+1}L_{k,n-2} = (-1)^{n-1}(k^2+4)F_{k,m-(n-2)}$

and

$L_{k,m}L_{k,n} - L_{k,m+1}L_{k,n-1} = (-1)^{n}(k^2+4)F_{k,m-(n-1)}$

Then

$L_{k,m}L_{k,n+1} - L_{k,m+1}L_{k,n} + L_{k,m}(kL_{k,n} + L_{k,n-1}) - L_{k,m+1}(kL_{k,n-1} + L_{k,n-2})$

$= k(L_{k,m}L_{k,n} - L_{k,m+1}L_{k,n-1}) + (L_{k,m}L_{k,n-1} - L_{k,m+1}L_{k,n-2})$

$= (-1)^n(k^2+4)[kF_{k,m-(n-1)} - F_{k,m-(n-2)}] = (-1)^{n+1}(k^2+4)F_{k,n-1}$

As a particular case, if $n = m - 1$, as $F_{k,1} = 1$, it is

$L^2_{k,m} - L_{k,m+1}L_{k,m-1} = (-1)^m(k^2+4)$ and the cassini identity is obtained

4.2 A new relation between the *k*-Lucas numbers

For n, r $\geq$ 0, it is $L_{k,n}L_{k,n+r} = L_{k,2n+r} + (-1)^n L_{k,r}$.

Proof. By induction. For $r = 0$ it is $L^2_{k,n} = (\sigma_k^n + (-\sigma_k)^n)^2 = \sigma_k^{2n} + (-\sigma_k)^{-2n} + 2(-1)^n = L_{k,2n} + (-1)^n L_{k,0}$

Let us suppose formula is true until $r-1$:

$$L_{k,n}L_{k,n+r-1} = L_{k,n}(kL_{k,n+r-1} + L_{k,n+r-2}) \tag{4.2.1}$$

$$= k(L_{k,2n+r-1} + (-1)^n L_{k,r-1}) + L_{k,2n+r-2} + (-1)^n L_{k,r-2} \tag{4.2.2}$$

$$= k(L_{k,2n+r-1} + L_{k,2n+r-2}) + (-1)^n (kL_{k,r-1} + L_{k,r-2}) \tag{4.2.3}$$

$$= L_{k,2n+r} + (-1)^n L_{k,r} \tag{4.2.4}$$

Particular case:

- If $r = 0$, then $L_{k,2n} = L_{k,n}^2 + 2(-1)^{n+1}$
- If $r = 1$, then $L_{k,n}L_{k,n+1} = L_{k,2n+r} + k(-1)^n$, and consecuently, $L_{k,2n+1} = L_{k,n}L_{k,n+1} + (-1)^{n+1}k$. In this case, if $k = 1$, for the classical Lucas numbers, relation $L_{2n+1} = L_nL_{n+1} + (-1)^{n+1}$ is verified.
- If $r = n$, then $L_{k,3n} = L_{k,n}(L_{k,n}^2 + 3(-1)^{n+1})$

if n is odd $L_{k,m.n} = \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} (m+1-i) L_{k,n}^{m-2i}$

if n is even $L_{k,m.n} = \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} (m+1-i)(-1)^i L_{k,n}^{m-2i}$

Theorem 4.2.1. ***(Sum of the first k-Lucas number)*** *Sum of the n first* k-*Lucas number is* $\sum_{i=1}^{n} L_{k,i} = 1 + \frac{1}{k}(L_{k,n} + L_{k,n+1} - 2)$

Proof. As $L_{k,n} = F_{k,n+1} + F_{k,n-1}$ and $\sum_{i=0}^{n} F_{k,i} = \frac{1}{k}(F_{k,n+1} + F_{k,n-1})$, it is

$\sum_{i=0}^{n} L_{k,i} = 2 + \sum_{i=1}^{n} L_{k,i} = 2 + \sum_{i=1}^{n} (F_{k,i-1} + F_{k,i+1})$

$= 2 + \frac{1}{k}(F_{k,n} + F_{k,n-1} - 1) + \frac{1}{k}(F_{k,n+2} + F_{k,n+1} - 1) - F_{k,i}$

$= 1 + \frac{1}{k}(F_{k,n-1} + F_{k,n+1} + F_{k,n} + F_{k,n+2} - 2)$

$= 1 + \frac{1}{k}(L_{k,n} + L_{k,n+1} - 2)$

In particular, if $k = 1$, then $\sum_{i=0}^{n} L_i = L_{n+2} - 1$

The only sequence of partial sums of k-Lucas number listed in OEIS are:

• For $k = 1$: $\{2, 3, 6, 10, 17, 28, 46, 75, ...\}$: A001610

• For $k = 2$: $\{2, 4, 10, 245, 58, 140, 338, ...\}$: A052542

Second sequence is simply twice the Pell numbers.

4.3 Generating function for the *k*-Lucas numbers

In this paragraph, the generating function for the k-Lucas sequence is given.As a result,k-Lucas sequence are seen as the coefficient of the corresponding generating function.

Let us suppose k-Fibonacci numbers are the coefficient of a potential series centered at the origin, and consider the corresponding analytic function $l_k(x)$. Function defined in such a way is callrd the generating function of the k-Lucas numbers. so,

$$l_k(x) = L_{k,0} + L_{k,1x} + L_{k,2x^2} + ... + L_{k,n}x^n + ...$$

and then,

$$\begin{aligned} kxl_k(x) &= kL_{k,0}x + kL_{k,1}x^2 + ... + kL_{k,n}x^{n+1} + ... \\ x^2l_k(x) &= L_{k,0}x^2 + L_{k,1}x^3 + ... + L_{k,nx^{n+2}+....} \\ &= (1 - kx - x^2)l_k(x) = 2 - kx \\ &= l_k(x) = \frac{2 - kx}{1 - kx - x^2} \end{aligned}$$

Bibliography

[1] Thomas Koshy; *Elementary Number Theory with Applications*; Elsevier.

[2] D.M. Burton; *Elementary Number Theory*; Tata Mc Graw Hill.

[3] Hardy G.H., Wright; *An Introduction To the Theory of Numbers*; Oxford Science Publications, Fifth Edition (1979).

[4] Gareth A. Jones, J. Marcy Jones ; *Elementary Number Theory*(2007).

[5] Sergio Falcon ; *On The K-Lucas Number*(2010), 1039-1050.