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Recursive Matrices and Umbral Calculus

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1. INTRODUCTION

Two major objectives can be seen to guide much recent work in enumeration: (1) to single out a limited variety of recurrences for numerical sequences which will encompass counting problems of wide-enough type; (2) to recover from empirical data an underlying set-theoretic structure which would reveal the source of the given recursion.

We are here concerned with the first of these objectives, though the eventual understanding of the second is tacitly present, if only as a goal.

We noticed the coincidence of several computations which, similar as they are in retrospect, had failed to realize their kinship. On leafing through the unique assembly of recursively solvable combinatorial problems in Comtet's and Sloane's invaluable collections, one is struck by the repeated occurrence of one and the same kind of double recursion. More strikingly, the same recursion is seen to occur in the polynomial sequences of the Umbral Calculus of Roman and Rota (see [25]).

Everywhere, the Lagrange inversion formula for power series plays a pivotal role. Much work is nowadays going into the unraveling of the ever-deeper layers of combinatorial significance of this formula, both in the ordinary case and in its as yet partially worked out noncommutative and q-analogs (Andrews, Foata, Garsia, Gessel, Joni, Raney, Reiner, Schützenberger, to name but a few). Whatever their origins, the identities abutting Lagrange inversion are expressed by integers alone. This suggests not only a hidden set-theoretic layer, but a characteristic-free generalization as well: this generalization is the central theme of our work.

We define a monoid of infinite matrices—"recursive matrices" for short. The entries of these matrices give the sought-out recursion, for example, that for coefficients of binomial and Sheffer polynomials and factor sequences, as well as that of the special sequences recently introduced by Roman in [26].

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Sequences of polynomials $p_n(x)$ satisfying the identities

$$p_n(x+y) = \sum_k p_k(x)\, p_{n-k}(y)$$

without the requirement that $p_n(x)$ be of degree n have been studied by Reiner, Morris and others. These sequences were not covered in the Umbral Calculus of Roman and Rota, nor has their classification been carried out. This problem is important, inasmuch as it is roughly equivalent to the problem of finding a complete set of conjugacy invariants for formal power series by a method which will hopefully extend to other formal groups.

With these aims in mind, we place ourselves in a characteristic-free setting. Our most pleasing result (Theorem 3) is perhaps a recursive (characteristic-free) matrix equivalent of the Lagrange Inversion Formula, here displayed as being nothing but the operation of pivoting a recursive matrix along the secondary diagonal. From this involutory interpretation of Lagrange inversion, it is an easy step to generalize the crucial Transfer Formula (see, e.g., [25, 26, 32]) to a characteristic-free setting. Thus, we submit that functional composition in arbitrary characteristic is most effectively and easily computed by recursive matrices in conjunction with the characteristic-free generalization of the Transfer Formula.

Applications and special functions are left to a subsequent work. In particular, we expect to show that new sequences of binomial and allied types in characteristic $p > 0$ exist in profusion and relate to computations in formal groups, notably Witt vectors. By way of application, we conclude with a generalization of Roman's sequences to a characteristic-free setting, dropping to boot the requirement that f_n be of degree n.

This work follows our previous study of polynomial sequences of integral type (see [1]). There, we showed that the Umbral Calculus should be cogently sharpened by techniques which keep all coefficients integral throughout. The present paper can be read independently of previous work, though some items listed in the bibliography may aid the reader in search of motivation.

2. Preliminaries and Notations

In the following, the letter $\mathbf{Z}$ will denote the ring of integers, and $\mathbf{N}$, $\mathbf{Z}^+$, $\mathbf{Z}^-$ will denote the set of nonnegative, positive and negative integers, respectively. Moreover, $\mathbf{A}$ will be a commutative integral domain, with unity, of characteristic $p \geqslant 0$. $\mathbf{U}$ will denote the group of the units of $\mathbf{A}$.

A sequence $\alpha := (a_i)$ with $i \in \mathbf{Z}$, $a_i \in \mathbf{A}$, is said to be a *Laurent sequence* whenever there exists an integer n such that, for every $k < n$, $a_k = 0$.

Clearly, the zero-sequence, which will be denoted by ζ, is a Laurent sequence. For every nonzero sequence $\alpha := (a_i)$, the *degree* of α will be the least integer n such that $a_n \neq 0$. We will denote the degree of α by $\deg(\alpha)$. Moreover, we will say that the zero sequence has degree $+\infty$.

We will denote by L^+ the set of all Laurent sequences, and, for every $n \in \mathbf{Z}$, L_n^+ shall be the set of all Laurent sequences whose degree is not less than n. Then, for every $n \in \mathbf{Z}$, we have

$$\zeta \in L_n^+.$$

L^+ is structured as a $\mathbf{Z}$-graded $\mathbf{A}$-algebra under the natural sum and scalar product and the following product: if $\alpha := (a_i)$ and $\beta := (b_i)$ are Laurent sequences, we set

$$\alpha\beta := (c_i)$$

with

$$c_i := \sum_k a_k b_{i-k}.$$

The zero element of L^+ is the zero-sequence ζ, and the identity of L^+ is the sequence

$$\upsilon := (\delta_{0i}),$$

where δ_{ij} is the Krönecker symbol.

Given two Laurent sequences α, β such that $\alpha\beta = \upsilon$, we will say that β is the *reciprocal sequence* of α, and we write

$$\alpha^{-1} := \beta.$$

Clearly, a given sequence $\alpha := (a_i) \in L^+$ of degree n has (unique) reciprocal sequence if and oly if $a_n \in \mathbf{U}$. In this case, if we denote by $\beta := (b_i)$ the reciprocal sequence of α, then $\deg(\beta) = -n$ and $b_{-n} = a_n^{-1}$.

The set of all Laurent sequences which admit reciprocal sequence will be indicated by R^+.

Moreover, L^+ becomes a complete topological $\mathbf{A}$-algebra when $\{L_n^+ ; n \in \mathbf{Z}\}$ is chosen as a basis of neighbourhoods of ζ. We say that a given sequence of Laurent sequences (α_i), with $i \in \mathbf{Z}$, *converges* to $\alpha \in L^+$ if $\lim_{i \in \mathbf{Z}^+} \alpha_i = \alpha$. Again, if $\beta := (b_i)$ is a Laurent sequence, and (α_i), $i \in \mathbf{Z}$ is a sequence of Laurent sequences, we say that the series $\sum_{i \in \mathbf{Z}} b_i \alpha_i$ *converges* to $\alpha \in L^+$, and we will write

$$\sum_{i \in \mathbf{Z}} b_i \alpha_i = \alpha$$

if the sequence (σ_n), with $n \in \mathbf{Z}$, and

$$\sigma_n = \sum_{i \leqslant n} b_i \alpha_i$$

converges to α. Note that if (α_i), $i \in \mathbf{Z}$ is a sequence of Laurent sequences convergent to ζ, and if $\beta := (b_i)$, $i \in \mathbf{Z}$ is a Laurent sequence, then the series $\sum_{i \in \mathbf{Z}} b_i \alpha_i$ always converges.

Recall that a *pseudobasis* of L^+ is a sequence (α_n), with $\alpha_n \in L^+$, $n \in \mathbf{Z}$, convergent to ζ, and such that every $\gamma \in L^+$ can be written in a unique way as $\gamma = \sum_{n \in \mathbf{Z}} b_n \alpha_n$, with $(b_n) \in L^+$.

It is easily seen that the sequence

$$(\alpha^i), \qquad i \in \mathbf{Z},$$

of all integer powers of a sequence $\alpha \in R^+$ converges to ζ if and only if α has positive degree. The set of all Laurent sequences in R^+ with positive degree will be denoted by C^+. Moreover, the sequence (α^i) is a pseudobasis if and only if α is in C^+ and its degree is exactly one. Such a series will be called a *generator* of L^+. Obviously, the Laurent sequence

$$\tau := (\delta_{1i}), \qquad i \in \mathbf{Z},$$

is a generator of L^+ and

$$\tau^n = (\delta_{ni}), \qquad n \in \mathbf{Z}.$$

It follows that, if $\alpha := (a_i) \in L^+$, then

$$\alpha = \sum_i a_i \tau^i;$$

hence, τ will be called the *canonical generator*.

Given $\lambda \in C^+$, it is possible to define an operator $C_\lambda : L^+ \to L^+$ as follows: if $\alpha \in L^+$, $\alpha = \sum_i a_i \tau^i$, we set

$$C_\lambda(\alpha) = \sum_i a_i \lambda^i.$$

It can be shown that the semigroup of all operators defined above is precisely the semigroup $\mathrm{End}(L^+)$ of all continuous endomorphisms of L^+. In the sequel, we will write—as is usually done—

$$\alpha \circ \lambda := C_\lambda(\alpha).$$

If α and β are in C^+, and

$$\alpha \circ \beta = \tau,$$

β will be said the *inverse* of α, and we will write

$$\tilde{\alpha} := \beta.$$

It is immediate that α has (unique) inverse if and only if it is a generator of L^+; we will denote by I^+ the set of all generators of L^+.

We recall that the *formal derivative* of a Laurent sequence $\alpha = \sum_{i \in \mathbf{Z}} a_i \tau^i$ is the Laurent sequence

$$D\alpha := \sum_{i \in \mathbf{Z}} i a_i \tau^{i-1}.$$

As is well known, the *chain rule* holds for formal derivative; that is, for every $\alpha \in L^+$ and $\beta \in C^+$,

$$D(\alpha \circ \beta) = ((D\alpha) \circ \beta)\, D\beta.$$

Furthermore, if $\alpha \in I^+$

$$\operatorname{Res}(\alpha^{-1} D\alpha) = 1$$

and for every $n \neq -1$

$$\operatorname{Res}(\alpha^n D\alpha) = 0,$$

where—as usual—if $\alpha = \sum_i a_i \tau^i$,

$$\operatorname{Res}(\alpha) = a_{-1}.$$

Similarly, a sequence $\alpha := (a_i)$, with $i \in \mathbf{Z}$, $a_i \in \mathbf{A}$, is said to be an *inverse Laurent sequence* whenever there exists an integer n, such that, for every $k > n$,

$$a_k = 0.$$

Clearly, the zero-sequence is also an inverse Laurent sequence. For every nonzero inverse Laurent sequence $\alpha := (a_i)$ we define the *degree* of α to be the largest $n \in \mathbf{Z}$ such that $a_n \neq 0$. We will denote the degree of α by $\deg^-(\alpha)$ in order to distinguish it from $\deg(\alpha)$ when α is both a Laurent and an inverse Laurent sequence. Moreover, we will say that the zero sequence ζ has degree $-\infty$ in L^-, i.e., $\deg^-(\zeta) = -\infty$.

We will denote by L^- the set of all inverse Laurent sequences and, for every $n \in \mathbf{Z}$, L_n^- shall be the set of all inverse Laurent sequences whose degree is not greater than n. Then $\zeta \in L^-$ for every $n \in \mathbf{Z}$.

L^- is structured as a $\mathbf{Z}$-graded $\mathbf{A}$-algebra under the natural sum, scalar product and the following product: if $\alpha := (a_i)$ and $\beta := (b_i)$ are inverse Laurent sequences, we set

$$\alpha\beta = (c_i)$$

with

$$c_i = \sum_k a_k b_{i-k}.$$

The zero element and the identity of L^- are, respectively, ζ and υ.

As for L^+, an inverse Laurent sequence $\alpha = (a_i)$ of degree n admits a (unique) reciprocal sequence if and only if $a_n \in \mathbf{U}$. The set of all inverse Laurent sequences which admit reciprocal sequence will be indicated by R^-.

L^- becomes a complete topological $\mathbf{A}$-algebra when $\{L_n^- ; n \in \mathbf{Z}\}$ is chosen as a basis of neighbourhoods of ζ. We will say that a given sequence of inverse Laurent sequences (α_i), with $i \in \mathbf{Z}$, *converges* to $\alpha \in L^-$ if $\lim_{i \in \mathbf{Z}^-} \alpha_i = \alpha$.

Again, if $\beta := (b_i)$ in an inverse Laurent sequence and (α_i), $i \in \mathbf{Z}$, is a sequence of inverse Laurent sequences, we say that the series $\sum_{i \in \mathbf{Z}} b_i \alpha_i$ converges to $\alpha \in L^-$, and we will write

$$\sum_{i \in \mathbf{Z}} b_i \alpha_i = \alpha$$

if the sequence $(\sigma_n) = (\sum_{i \leqslant n} b_i \alpha_i)$, with $n \in \mathbf{Z}$, converges to α.

Also in this case, if (α_i), $i \in \mathbf{Z}$, is a sequence of inverse Laurent sequences convergent to ζ, and if $\beta := (b_i)$ is an inverse Laurent sequence, then the series $\sum_{i \in \mathbf{Z}} b_i \alpha_i$ always converges.

Pseudobases in L^- are defined as in L^+. The sequence (α^i) of all integer powers of a given $\alpha \in R^-$ converges to ζ if and only if α has positive degree.

The set of all inverse Laurent sequences of positive degree will be denoted by C^-. Analogously the notion of *generator* is given, and it can be shown that the generators of L^- are precisely the inverse Laurent sequences of degree one which are in R^-; the set of all generators of L^- will be indicated by I^-.

In particular, the sequence τ previously defined is also a generator of L^- and, if $\alpha = (a_i)$ is an inverse Laurent sequence, we have

$$\alpha = \sum_{i \in \mathbf{Z}} a_i \tau^i.$$

Also in this case, if $\alpha = (a_i) \in L^-$ and $\lambda \in C^-$, we set

$$\alpha \circ \lambda = \sum_{i \in \mathbf{Z}} a_i \lambda^i.$$

An inverse Laurent sequence α is a generator if and only if there exists a unique $\tilde{\alpha} \in C^-$ such that

$$\alpha \circ \tilde{\alpha} = \tau.$$

For any sequence $\alpha := (a_n)$, $n \in \mathbf{Z}$, $a_n \in \mathbf{A}$, we set

$$\alpha^{\cdot} := (b_n)$$

with $b_n = a_{-n}$, for every $n \in \mathbf{Z}$.

The correspondence now defined is obviously an involutory bijection. It is easily seen that it maps L^+ onto L^- and vice versa, and induces a continuous isomorphism between these topological $\mathbf{A}$-algebras.

In the sequel, accordingly with the current terminology, the elements of L^+ and the elements of L^- will be called *Laurent series* and *inverse Laurent series*, respectively.

Given a matrix $\mathbf{M} := (m_{ij})$ with $i,j \in \mathbf{Z}$ and $m_{ij} \in \mathbf{A}$, its *kth row-generating function* shall be the sequence

$$\mathbf{M}(k) := (m_{kj}), \qquad \text{with} \quad j \in \mathbf{Z}.$$

Similarly, the *hth column generating function* of M shall be the sequence

$$\mathbf{M}[h] := (m_{ih}), \qquad \text{with} \quad i \in \mathbf{Z}.$$

A matrix M will be called a *Laurent matrix* (*inverse Laurent matrix*) whenever all its row (column) generating functions are Laurent sequences (inverse Laurent sequences). Moreover, a matrix $\mathbf{M} := (m_{ij})$, with $i,j \in \mathbf{Z}$ and $m_{ij} \in \mathbf{A}$ will be called *diagonally finite* whenever, for every $(h,k) \in \mathbf{Z}^2$, there exists only a finite set of ordered pairs $(i,j) \in \mathbf{Z}^2$ such that

$$m_{ij} \neq 0, \qquad i \geqslant h, \quad j \leqslant k.$$

Obviously, a diagonally finite matrix is both a Laurent and an inverse Laurent matrix, while the converse is false. In the $\mathbf{A}$-module D of all diagonally finite matrices the usual matrix product is defined; under this product, D becomes a (noncommutative) $\mathbf{A}$-algebra.

We define now two linear operators F and G over the A-module of all matrices $\mathbf{M}: \mathbf{Z} \times \mathbf{Z} \to \mathbf{A}$ as follows: if $\mathbf{M} := (m_{ij})$, we set

$$F\mathbf{M} := (p_{ij})$$

with

$$p_{ij} = m_{i-1,j}, \qquad i,j \in \mathbf{Z},$$

and

$$G\mathbf{M} := (q_{ij})$$

with

$$q_{ij} = m_{i,j-1}, \qquad i, j \in \mathbf{Z}.$$

We remark that, if $\mathbf{M}$ is a Laurent matrix, then

$$F\mathbf{M}(i) = \mathbf{M}(i-1)$$

and

$$G\mathbf{M}(i) = \tau\mathbf{M}(i);$$

similarly, if $\mathbf{M}$ is an inverse Laurent matrix, then

$$F\mathbf{M}[j] = \tau\mathbf{M}[j]$$

and

$$G\mathbf{M}[j] = \mathbf{M}[j-1].$$

Given a Laurent series

$$\alpha = \sum_{i \in \mathbf{Z}} a_i \tau^i \neq \zeta$$

the linear operator

$$\alpha(G) := \sum_{i \in \mathbf{Z}} a_i G^i$$

is well defined over the **A**-module of all Laurent matrices.

Analogously, given an inverse Laurent series

$$\gamma = \sum_{i \in \mathbf{Z}} c_i \tau^i \neq \zeta$$

the linear operator

$$\gamma(F) := \sum_{i \in \mathbf{Z}} c_i F^i$$

is well defined over the **A**-module of all inverse Laurent matrices.

3. Representation of the Umbral Semigroup

We define a semigroup over the set of all ordered pairs (α, β) where $\alpha \in C^+$ and $\beta \in L^+$, $\beta \neq \zeta$, by means of the operation

$$(\alpha, \beta)(\gamma, \delta) := (\alpha \circ \gamma, (\beta \circ \gamma)\, \delta).$$

This semigroup will be called the *umbral semigroup over* **A** and will be denoted by $US(\mathbf{A})$. This structure turns out to be a monoid, (τ, υ) being its unity.

We single out three subgroups of $US(\mathbf{A})$, namely, the *umbral group* $UG(\mathbf{A})$, the *homogeneous group* $HG(\mathbf{A})$ and the *Appell group* $AG(\mathbf{A})$, defined as

$$G(\mathbf{A}) := \{(a, \beta); \alpha \in I^+, \beta \in R^+\},$$

$$HG(\mathbf{A}) := \{(\alpha, \upsilon); \alpha \in I^+\},$$

$$AG(\mathbf{A}) := \{(\tau, \beta); \beta \in R^+\}.$$

By the way we note that $US(\mathbf{A})$ contains the *homogeneous semigroup*

$$HS(\mathbf{A}) := \{(\alpha, \upsilon); \alpha \in C^+\},$$

which is canonically isomorphic to the compositional semigroup of Laurent series over **A**, and the *Appell semigroup*

$$AS(\mathbf{A}) := \{(\tau, \beta); \beta \in L^+, \beta \neq \zeta\},$$

which is canonically isomorphic to the multiplicative semigroup of L^+.

Our next goal will be to exhibit a faithful representation of the semigroup $US(\mathbf{A})$ by means of a class of diagonally finite matrices.

First of all, we are interested in Laurent matrices M satisfying a linear problem of the kind

$$\begin{aligned} (F\alpha(G) - I)\, \mathbf{M} &= \mathbf{0}, \\ \mathbf{M}(i) &= \beta. \end{aligned} \qquad (*)$$

with $\alpha \in R^+$ and $\beta \in L^+$.

It is immediate, by construction, that for any given $\alpha \in R^+$ and $\beta \in L^+$ the linear problem $(*)$ has a unique solution in the **A**-module of all Laurent matrices.

We immediately get the following result:

3.1. Proposition. *Let $\alpha \in R^+$, $\beta \in L^+$ and $i \in \mathbf{Z}$; then the Laurent matrix* $\mathbf{M}$ *is the unique solution of problem* (*) *if and only if*

$$\mathbf{M}(k) = \alpha^{k-i}\beta$$

for every $k \in \mathbf{Z}$.

Proof. First of all we remark that

$$(\alpha(G)\,\mathbf{M})(k) = \alpha\mathbf{M}(k)$$

for every $k \in \mathbf{Z}$. Let now $\mathbf{M}$ be the solution problem (*); by assumption we have

$$\mathbf{M}(i) = \beta;$$

by induction, suppose that

$$\mathbf{M}(k) = \alpha^{k-i}\beta$$

for some $k \in \mathbf{Z}$; then

$$\mathbf{M}(k+1) = (F^{-1}\mathbf{M})(k) = (\alpha(G)\,\mathbf{M})(k) = \alpha\mathbf{M}(k) = \alpha^{k+1-i}\beta$$

and analogously

$$\mathbf{M}(k-1) = \alpha^{k-1-i}\beta.$$

The converse is immediate. ∎

By the preceding result, we can consider—without loss of generality—only problems of the kind

$$\begin{aligned} (F\alpha(G) - I)\,\mathbf{M} &= \mathbf{0}, \\ \mathbf{M}(0) &= \beta, \end{aligned} \tag{**}$$

with $\alpha \in R^+$, $\beta \in L^+$. The unique solution $\mathbf{M}$ of problem (**) will be said the *(α, β)-recursive matrix*. The series α and β will be called the *recurrence rule* and the *boundary value*, respectively. Hence, we are led to define an application R from the set $R^+ \times L^+$ to the set of all Laurent matrices which maps the pair (α, β) into the (α, β)-recursive matrix.

Note that $R(\alpha, \beta) = R(\gamma, \delta)$ if and only if

$$\alpha = \gamma, \qquad \beta = \delta,$$

whenever $\beta \neq \zeta$, while

$$R(\alpha, \zeta) = R(\gamma, \zeta) = \mathbf{0}$$

for every $\alpha, \gamma \in R^+$.

If $\mathrm{char}(\mathbf{A}) = p > 0$, the entries of any recursive matrix can be computed directly by means of coefficients of the boundary value and of the first $p-1$ powers of the recurrence rule and of its reciprocal series. In fact, we have:

3.2. PROPOSITION. *Suppose* $\mathrm{char}(\mathbf{A}) = p > 0$. *Let* $\alpha \in R^+$, $\beta \in L^+$, *with* $\beta = \sum_i b_i \tau^i$, *and set* $\alpha^i = \sum_j a_{ij} \tau^j$ *for every* $i \in \mathbf{Z}$. *Let* $(m_{ij}) = R(\alpha, \beta)$. *For every* $i, j \in \mathbf{Z}$, *with* $i = \sum_{h=0}^{n} i_h p^h$ $(i_n \neq 0)$, *where* $0 \leqslant i_h \leqslant p-1$ *if* $i \geqslant 0$ *and* $1 - p \leqslant i_h \leqslant 0$ *if* $p < 0$, *we have*

$$m_{ij} = \sum_{(k_s) \mathbf{Z}^n} \left(\prod_{s=0}^{n} a_{i_s, k_s}^{p^s} \right) b_{j-k},$$

where $k = \sum_{h=0}^{n} k_h p^h$.

Proof. It is straightforward consequence of Proposition 3.1. ∎

In particular, we get:

3.3. COROLLARY. *Let* $\mathrm{char}(\mathbf{A}) = p > 0$, *and let* $M = (m_{ij})$ *be a recursive matrix with boundary value* v. *For every* $i, j \in \mathbf{Z}$, *with* $i = \sum_{h=0}^{n} i_h p^h$, *where* $0 \leqslant i_h \leqslant p-1$ *if* $i \geqslant 0$, *and* $1 - p \leqslant i_h \leqslant 0$ *if* $i < 0$, *we have*

$$m_{ij} = \sum_{(k_s) \mathbf{Z}^n} \prod_{s=0}^{n} m_{i_s, k_s}^{p^s},$$

where the sum above is extended to all n-tuples (k_s) *such that* $\sum_{s=0}^{n} k_s p^s = j$.

The preceding result yields a generalization of the so-called Lucas factorization formula for p-binomial coefficients.

3.4. PROPOSITION. *The Laurent matrix* $R(\alpha, \beta)$ *is a diagonally finite matrix if and only if* $\alpha \in C^+$.

Proof. Immediate by Proposition 3.1. ∎

We are now interested in studying the column-generating functions of a diagonally finite recursive matrix. We have:

3.5. PROPOSITION. *The diagonally finite matrix* $\mathbf{M}$ *is the* (α, β)*-recursive matrix if and only if*

$$\mathbf{M}(0) = \beta$$

and

$$\mathbf{M}[j] = \tau \sum_{i \in \mathbf{Z}} a_i \mathbf{M}[j-i]$$

for every $j \in \mathbf{Z}$.

Proof. Let $\mathbf{M} = R(\alpha, \beta)$, with $\alpha \in C^+$ and $\beta \in L^+$: then we have $\mathbf{M}(0) = \beta$; furthermore

$$\mathbf{M}[j] = \tau(F^{-1}\mathbf{M})[j] = \tau(\alpha(G)\,\mathbf{M})[j] = \sum_{i \in \mathbf{Z}} \tau a_i \mathbf{M}[j-i].$$

Analogously we have the converse. ∎

A recursive matrix with boundary value υ will be called a *homogeneous matrix*, and a recursive matrix with recurrence rule τ will be called an *Appell matrix*. The identity matrix is the unique Appell homogeneous matrix. The row-generating functions law (Proposition 3.1) leads us to note that the sum of two recursive matrices with the same recurrence rule α is the recursive matrix with recurrence rule α and whose boundary value is the sum of the boundary values. Furthermore, we have:

3.6. PROPOSITION. *Let $\alpha \in R^+$, $\beta \in L^+$, $\beta := (b_i)$; then*

$$R(\alpha, \beta) = \sum_{i \in \mathbf{Z}} b_i R(\alpha, \tau^i).$$

By Proposition 3.2, it follows that the product of two recursive matrices $R(\alpha, \beta)$ and $R(\gamma, \delta)$ is well defined whenever the series γ belongs to C^+; furthermore, we have

THEOREM 1. *The map R induces a semigroup monomorphism from the umbral semigroup $US(\mathbf{A})$ to the semigroup D of diagonally finite matrices. Precisely, we have*

$$R(\alpha, \beta) \times R(\gamma, \delta) = R(\alpha \circ \gamma, (\beta \circ \gamma)\, \delta)$$

for every $\alpha, \gamma \in C^+$, β, $\delta \in L^+$, $\beta \neq \zeta \neq \delta$.

Proof. Set

$$R(\alpha, \beta) = \mathbf{P} = (p_{ni}),$$

$$R(\gamma, \delta) = \mathbf{Q} = (q_{ni}),$$

and

$$\mathbf{P} \times \mathbf{Q} = \mathbf{M} = (m_{ni}).$$

We have

$$\begin{aligned}\mathbf{M}(n) &= \sum_i \sum_j p_{nj} q_{ji} \tau^i = \sum_j p_{nj} \mathbf{Q}(j) \\ &= \sum_j p_{nj} \gamma^j\, \delta = (\mathbf{P}(n) \circ \gamma)\delta \\ &= \delta((\alpha^n \beta) \circ \gamma) = (\alpha \circ \gamma)^n (\beta \circ \gamma)\, \delta.\end{aligned}$$

By Proposition 3.1, we get

$$\mathbf{M} = R(\alpha \circ \gamma, (\beta \circ \gamma)\,\delta). \quad \blacksquare$$

As an immediate consequence, we succeed in splitting in a unique way any given recursive nonzero matrix into the product of a homogeneous matrix and an Appell matrix.

3.7. Corollary. *Let $\alpha \in R^+$, $\beta \in L^+$, $\beta \neq \zeta$; then*

$$R(\alpha, \beta) = R(\alpha, \upsilon) \times R(\tau, \beta).$$

The diagonally finite recursive matrices which admit two-sided inverse are those whose recurrence rule and boundary value are in I^+ and R^+, respectively. Hence:

3.8. Proposition. *The largest subgroup of the semigroup of diagonally finite recursive matrices is the image under R of the umbral group $UG(\mathbf{A})$.*

Note that the terminology settled for special substructures of $US(\mathbf{A})$ and special subsets of recursive matrices is consistent, that is, R maps the homogeneous semigroup $HS(\mathbf{A})$ into the semigroup of all homogeneous recursive matrices and the Appell semigroup $AS(\mathbf{A})$ into the semigroup of all nonzero Appell matrices.

Finally, we remark that the set of all Appell matrices is a commutative subalgebra of the (noncommutative) algebra of all diagonally finite matrices. This subalgebra $A(\mathbf{A})$ is canonically isomorphic to L^+, since

$$R(\tau, \alpha) + R(\tau, \beta) = R(\tau, \alpha + \beta),$$
$$R(\tau, \alpha) \times R(\tau, \beta) = R(\tau, \alpha\beta)$$

for every $\alpha, \beta \in L^+$.

Moreover:

3.9. Proposition. *Every $\mathbf{A}$-algebra of recursive matrices with unity element is a subalgebra of the algebra $A(\mathbf{A})$ of Appell matrices.*

Proof. Let M be an $\mathbf{A}$-algebra of recursive (diagonally finite) matrices with unity element, and let, for $\mathbf{M} \in M$,

$$\mathbf{M} = R(\alpha, \beta)$$

with $\alpha \in C^+$; then the matrix $\mathbf{M} + \mathbf{I}$ must also belong to M, that is,

$$\mathbf{M} + \mathbf{I} = R(\gamma, \delta)$$

for some $\gamma \in C^+$ and $\delta \in L^+$. By Proposition 3.1, we get

$$\alpha^i \beta + \tau^i = \gamma^i \delta$$

for every $i \in \mathbf{Z}$; in particular, for $i = 0$, we obtain

$$\beta + \upsilon = \delta$$

and, for $i = 1, -1$,

$$(\alpha - \gamma)\beta = \gamma - \tau,$$
$$(\gamma - \alpha)\alpha^{-1}\gamma^{-1}\beta = (\tau - \gamma)\tau^{-1}\gamma^{-1}.$$

That is,

$$(\gamma - \tau)\alpha^{-1}\gamma^{-1} = (\gamma - \tau)\tau^{-1}\gamma^{-1},$$

which implies

$$\alpha = \gamma = \tau;$$

hence $\mathbf{M}$ is an Appell matrix. ∎

The canonical isomorphism between L^+ and $A(\mathbf{A})$ allows us to regard the latter as a (complete) topological algebra; the recursive matrix

$$\mathbf{D} = R(\tau, \tau)$$

turns out to be a generator of $A(\mathbf{A})$. Now, it is possible to give a characterization of Appell matrices which recalls some results of the classical umbral calculus:

3.10. PROPOSITION. *A Laurent matrix* $\mathbf{M}$ *is an Appell matrix if and only if*

$$\mathbf{M} \times \mathbf{D} - \mathbf{D} \times \mathbf{M} = \mathbf{0}.$$

Proof. It is easily seen that

$$\mathbf{M} \times \mathbf{D} = G\mathbf{M}$$

and

$$\mathbf{D} \times \mathbf{M} = F^{-1}\mathbf{M};$$

hence, the statement follows. ∎

4. General Lagrange Inversion

In this section we will consider a further class of infinite matrices which arise from linear problems of the kind

$$\begin{aligned} (G\alpha(F) - I)\, \mathbf{M} &= \mathbf{0}, \\ \mathbf{M}[j] &= \beta \end{aligned} \tag{*}$$

with $\alpha \in R^-$ and $\beta \in L^-$.

It is immediate, by construction, that, for any given $\alpha \in R^-$ and $\beta \in L^-$, the linear problem (*) has a unique solution in the $\mathbf{A}$-module of all inverse Laurent matrices.

4.1. Proposition. *Let $\alpha \in R^-$, $\beta \in L^-$ and $j \in \mathbf{Z}$; then the inverse Laurent matrix $\mathbf{M}$ is the unique solution of the problem (*) if and only if*

$$\mathbf{M}[k] = \alpha^{k-j}\beta$$

for every $k \in \mathbf{Z}$.

Proof. It is in close analogy with that given for Proposition 3.1. ∎

This result allows us to consider—without loss of generality—only problems of the kind

$$\begin{aligned} (G\alpha(F) - I)\, \mathbf{M} &= \mathbf{0}, \\ \mathbf{M}[0] &= \beta \end{aligned} \tag{**}$$

with $\alpha \in R^-$, $\beta \in L^-$. The unique solution $\mathbf{M}$ of problem (**) will be called the (inverse) $[\alpha, \beta]$-recursive matrix, and will be indicated as $\mathbf{M} = C(\alpha, \beta)$; the series α and β will be called the *recurrence rule* ad the *boundary value*, respectively.

Note that $C(\alpha, \beta) = C(\gamma, \delta)$ if and only if

$$\alpha = \gamma, \qquad \beta = \delta,$$

whenever $\beta \neq \zeta \neq \delta$, while

$$C(\alpha, \zeta) = C(\gamma, \zeta) = \mathbf{0}$$

for every $\alpha, \gamma \in R^-$.

4.2. Proposition. *The inverse Laurent matrix $C(\alpha, \beta)$ is a diagonally finite matrix if and only if α has positive degree.*

Proof. Follows immediately by Proposition 4.1. ∎

In the sequel, an inverse recursive matrix will be called an *inverse homogeneous matrix* if its boundary series is υ.

Note that an inverse recursive matrix with recurrence rule τ is also a recursive matrix; furthermore

$$C(\tau, \beta) = R(\tau, \beta^{\cdot})$$

for every $\beta \in L^{-}$.

We have:

4.3. PROPOSITION. *Let $\alpha \in R^{-}$, $\beta \in L^{-}$, $\beta := (b_i)$; then*

$$C(\alpha, \beta) = \sum_{i \in \mathbf{Z}} b_i C(\alpha, \tau^i).$$

We now define a linear operator T on the $\mathbf{A}$-module of all matrices

$$\mathbf{M}: \mathbf{Z} \times \mathbf{Z} \to \mathbf{A};$$

if $\mathbf{M} = (m_{ij})$, then

$$T\mathbf{M} := (m_{-j,-i}).$$

T is an involution, maps Laurent matrices into inverse Laurent matrices and leaves the submodule of all diagonally finite matrices fixed. Moreover, if the product $\mathbf{M} \times \mathbf{N}$ is defined, then

$$T(\mathbf{M} \times \mathbf{N}) = T\mathbf{N} \times T\mathbf{M}.$$

A first link between recursivity and inverse recursivity is given by the following result, whose proof is straightforward:

4.4. PROPOSITION. *Let $\alpha \in R^{-}$ and $\beta \in L^{-}$; then, we have*

$$TC(\alpha, \beta) = R((\alpha^{-1})^{\cdot}, \beta^{\cdot}).$$

The proof of the following proposition becomes very easy by means of the preceding result:

4.5. PROPOSITION. *Let $\alpha \in C^{-}$, $\beta, \delta \in L^{-}$ and $\gamma \in R^{-}$; then*

$$C(\alpha, \beta) \times C(\gamma, \delta) = C(\gamma \circ \alpha, (\delta \circ \alpha)\beta).$$

Proof. We have

$$\begin{aligned}
T(C(\alpha,\beta)\times C(\gamma,\delta)) &= TC(\gamma,\delta)\times TC(\alpha,\beta)\\
&= R((\gamma^{-1})^{\cdot},\delta^{\cdot})\times R((\alpha^{-1})^{\cdot},\beta^{\cdot})\\
&= R((\gamma^{-1})^{\cdot}\circ(\alpha^{-1})^{\cdot},(\delta^{\cdot}\circ(\alpha^{-1})^{\cdot})\beta^{\cdot})\\
&= R((\gamma^{\cdot}\circ(\alpha^{-1})^{\cdot})^{-1},(\delta\circ\alpha)^{\cdot}\beta^{\cdot})\\
&= R(((\gamma\circ\alpha)^{-1})^{\cdot},((\delta\circ\alpha)\beta)^{\cdot})\\
&= TC(\gamma\circ\alpha,(\delta\circ\alpha)\beta). \qquad \blacksquare
\end{aligned}$$

4.6. COROLLARY. *Let $\alpha\in R^{-}$, $\beta\in L^{-}$; then, we have*

$$C(\alpha,\beta)=C(\tau,\beta)\times C(\alpha,\upsilon).$$

It is now clear that the study of inverse recursive matrices can be reduced to that of recursive matrices by means of the linear operator T. The goal is now to find whenever an inverse recursive matrix is a recursive matrix. First of all, we have to investigate conditions under which the equations

$$(F\alpha(G)-I)\,\mathbf{M}=\mathbf{0},\qquad \alpha\in R^{+},$$

and

$$(G\gamma(F)-I)\,\mathbf{M}=\mathbf{0},\qquad \gamma\in R^{-},$$

have common solutions.

We have:

THEOREM 2. *Let $\alpha\in R^{+}$, $\gamma\in R^{-}$; then the equations*

$$(F\alpha(G)-I)\,\mathbf{M}=\mathbf{0},$$
$$(G\gamma(F)-I)\,\mathbf{M}=\mathbf{0}$$

have a nontrivial common solution in the $\mathbf{A}$*-algebra of all diagonally finite matrices if and only if $\alpha\in I^{+}$, $\gamma\in I^{-}$ and $\gamma=(\tilde{\alpha}^{-1})^{\cdot}$. If that is the case, the* $\mathbf{A}$*-modules of solutions coincide.*

Proof. Let $\alpha\in I^{+}$ and $\gamma=(\tilde{\alpha}^{-1})^{\cdot}$; the following equations are equivalent:

$$\begin{aligned}
(F\alpha(G)-I)\,\mathbf{M}&=\mathbf{0},\\
\alpha(G)\,\mathbf{M}&=F^{-1}\mathbf{M},\\
G\mathbf{M}&=\tilde{\alpha}(F^{-1})\,\mathbf{M},\\
G\mathbf{M}&=\tilde{\alpha}^{\cdot}(F)\,\mathbf{M},
\end{aligned}$$

$$(\tilde{\alpha}^{\cdot})^{-1}(F)\,G\mathbf{M} = \mathbf{M},$$
$$(G\gamma(F) - I)\,\mathbf{M} = \mathbf{0}.$$

Hence the **A**-modules of all solutions of the given equations coincide. ∎

4.7. PROPOSITION. *Let $\alpha \in I^+$; then*

$$R(\alpha, D\alpha) = C((\tilde{\alpha}^{-1})^{\cdot}, (\tau\tilde{\alpha}^{-1})^{\cdot}).$$

Proof. Let $\mathbf{M} := R(\alpha, D\alpha)$; by Theorem 2, $\mathbf{M}$ is an inverse recursive matrix with recurrence rule $(\tilde{\alpha}^{-1})^{\cdot}$ and, by previous remarks,

$$\mathbf{M}[-1] = \tau^{-1}. \quad ∎$$

Finally, we come to our main result:

THEOREM 3. *Let $\alpha \in I^+$ and $\beta \in L^+$; then*

$$R(\alpha, \beta) = C((\tilde{\alpha}^{-1})^{\cdot}, (\tau\tilde{\alpha}^{-1}(\beta \circ \tilde{\alpha})\,D\tilde{\alpha})^{\cdot}).$$

Proof. We have

$$\begin{aligned}
R(\alpha, \beta) &= R(\tau, (\beta D\alpha^{-1}) \circ \tilde{\alpha}) \times R(\alpha, D\alpha)\\
&= C(\tau, ((\beta D\alpha^{-1}) \circ \tilde{\alpha})^{\cdot}) \times C((\tilde{\alpha}^{-1})^{\cdot}, (\tau\tilde{\alpha}^{-1})^{\cdot})\\
&= C((\tilde{\alpha}^{-1})^{\cdot}, (\tau\tilde{\alpha}^{-1})^{\cdot}((\beta D\alpha^{-1}) \circ \tilde{\alpha})^{\cdot})\\
&= C((\tilde{\alpha}^{-1})^{\cdot}, (\tau\tilde{\alpha}^{-1}(\beta \circ \tilde{\alpha})((D\alpha^{-1}) \circ \tilde{\alpha}))^{\cdot})\\
&= C((\tilde{\alpha}^{-1})^{\cdot}, (\tau\tilde{\alpha}^{-1}(\beta \circ \tilde{\alpha})((D\alpha) \circ \tilde{\alpha})^{-1})^{\cdot})\\
&= C((\tilde{\alpha}^{-1})^{\cdot}, (\tau\tilde{\alpha}^{-1}(\beta \circ \tilde{\alpha})\,D\tilde{\alpha})^{\cdot}). \quad ∎
\end{aligned}$$

As an immediate consequence of Theorem 3 we get a generalization of the classical Lagrange–Bürmann inversion formula:

4.8. PROPOSITION. *For every $\alpha \in I^+$ and $\beta \in L^+$, we have*

$$\sum_{n \in \mathbf{Z}} \operatorname{Res}(\alpha^n\beta)\,\tau^n = (\tau(\beta \circ \tilde{\alpha})\,D\tilde{\alpha})^{\cdot}.$$

5. AN EXTENSION OF ROMAN'S THEORY OF SPECIAL SEQUENCES

As stated in the Introduction, the results we have developed thus far are meant to incorporate and generalize much of the work of the last decade relating to special sequences of polynomials. Our point of view has been that of taking the matrices defined by the coefficients of these polynomials and treating them algebraically. Some advantages of this method are, besides achieving a characteristic-free approach, the recasting of a number of phenomena, such as the Lagrange inversion formula, in an algebraic light, thus best lending them to further extension.

It is, however, of relevance to show that the analytic methods used by previous authors can be recovered in full, indeed greatly extended by simple applications of our results.

It hardly needs be recalled that a great variety of polynomial sequences arising in classical analysis, combinatorics, probability and sundry other sources obey the recursion we have studied. While a detailed examination of classical special cases will have to wait for a later publication, we only mention by way of typical examples the *Stirling polynomials*:

$$S_n(x) = \sum_{k=0}^{n} S_{n,k} x^k.$$

The coefficients of these polynomials, the *Stirling numbers of the second kind*, satisfy the well-known recursion

$$\binom{h+k}{n} S_{n,h+k} = \sum_{j=0}^{n} \binom{n}{j} S_{j,h} S_{n-j,k}$$

and thus the matrix $\mathbf{S} := (s_{ij})$, whose entries are

$$s_{ij} := \frac{i!}{j!} S_{j,i},$$

is one of our recursive matrices. Similarly, the coefficients of the *Laguerre polynomials*,

$$L_n(x) = \sum_{k=0}^{n} L_{n,k} x^k$$

with

$$L_{n,k} = (-1)^k \frac{n!}{k!} \binom{n-1}{k-1},$$

satisfy the same recursion:

$$\binom{h+k}{n} L_{n,h+k} = \sum_{j=0}^{n} \binom{n}{j} L_{j,h} L_{n-j,k}.$$

The Hermite polynomials

$$H_n(x) = \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k \frac{n!}{k!(n-2k)!} x^{n-2k}$$

satisfy the recursion

$$H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} H_k(x) y^{n-k},$$

which again falls under our theory.

These examples taken at random hint to the further development we have in mind.

Not long ago S. Roman discovered the remarkable fact that these sequences of polynomials could be obtined by Wiener–Hopf truncation of inverse Laurent series. For example, he could associate to the Hermite polynomials $H_n(x)$ a family of inverse Laurent series

$$H_n(x) = \sum_{k=0}^{\infty} \frac{c_n}{c_k c_{n-k}} \left(-\frac{1}{2}\right)^k x^{n-2k},$$

where coefficients c_i are those introduced in [27] and the index n ranges over positive and negative integers alike. For n positive, the ordinary Hermite polynomials are obtained by removing terms of negative degree. Amazingly, all identities relating to Hermite polynomials (and the same happens for all Sheffer sequences of polynomials) are still valid. This phenomenon indicates that formal properties of special polynomials can perhaps be understood in the light of this double extension.

In closing, we briefly show that Roman's results can be obtained as an application of our methods, with a further generalization, not considered by Roman, to Laurent sequences not restricted to change by one degree at each step.

In this section, for sake of readability and accordingly with Roman's notation, we will denote Laurent and inverse Laurent series in functional form, that is,

$$\alpha(x) := \sum_{i \in \mathbf{Z}} a_i x^i.$$

A sequence of inverse Laurent series (ϕ_n), with $n \in \mathbf{Z}$ and

$$\phi_n(x) = \sum_k c_{k,n} x^k,$$

is said to be a *homogeneous sequence* whenever

(i) $\deg^-(\phi_m) < \deg^-(\phi_{m+1})$

for some integer m, and

(ii) $c_{h+k,n} = \sum_j c_{h,j} c_{k,n-j}$

for every integer h, k and n.

Condition (i) ensures finiteness of the sum in condition (ii).

We explicitely note that no more conditions are given on degrees of the inverse Laurent series ϕ_n.

If (ϕ_n) is a sequence of inverse Laurent series, with $n \in \mathbf{Z}$ and

$$\phi_n(x) = \sum_k c_{k,n} x^k,$$

the matrix

$$\mathbf{C} := (c_{kn})$$

will be said the *canonical matrix* of the sequence (ϕ_n).

The next statement shows the natural link between homogeneous sequences and homogeneous recursive matrices.

5.1. PROPOSITION. *A sequence (ϕ_n) of inverse Laurent series, with $n \in \mathbf{Z}$ and*

$$\phi_n(x) = \sum_k c_{k,n} x^k,$$

is a homogeneous sequence if and only if its canonical matrix is (α, υ)-recursive, with $\alpha \in C^+$. Moreover,

$$\alpha(x) = \sum_j c_{1,j} x^j.$$

Proof. Follows immediately by Proposition 3.1. ∎

The degrees of a homogeneous sequence are nondecreasing. Precisely, we have

5.2. COROLLARY. *If (ϕ_n) is a homogeneous sequence and $\mathbf{C} := R(\alpha, \upsilon)$ is its canonical matrix, then, for every integer n,*

$$\deg^-(\phi_{nh+i}) = n, \qquad i = 0, 1, 2, \ldots, h-1,$$

with $h = \deg(\alpha)$.

For every Laurent series

$$\alpha(x) := \sum_{i \in \mathbf{Z}} a_i x^i$$

and for every inverse Laurent series

$$\phi(x) := \sum_{i \in \mathbf{Z}} c_i x^i,$$

set, as usual,

$$\langle \alpha \mid \phi \rangle := \sum_{i \in \mathbf{Z}} a_i c_i.$$

Homogeneous sequences yield Roman's special sequences as a special case. In fact, we have:

5.3. PROPOSITION. *A sequence (ϕ_n) of inverse Laurent series, with $n \in \mathbf{Z}$ and*

$$\deg^-(\phi_m) < \deg^-(\phi_{m+1})$$

for some integer m, is a homogeneous sequence if and only if

$$\langle \alpha\beta \mid \phi_n \rangle = \sum_j \langle \alpha \mid \phi_j \rangle \langle \beta \mid \phi_{n-j} \rangle$$

for every $\alpha, \beta \in L^+$.

Proof. Let $\phi_n(x) = \sum_j c_{k,n} x^k$; the assertion follows from the identities:

$$\langle x^{i+1} \mid \phi_n \rangle = c_{i+1,n} = \sum_j c_{i,j} c_{1,n-j} = \sum_j \langle x^i \mid \phi_n \rangle \langle x \mid \phi_{n-j} \rangle. \quad \blacksquare$$

A sequence of inverse Laurent series (ϕ_n), with

$$\phi_n(x) = \sum_k c_{k,n} x^k, \qquad n \in \mathbf{Z},$$

is said to be a *Sheffer sequence* whenever there exists a (unique) homogeneous sequence (ψ_n),

$$\psi_n(x) = \sum_k d_{k,n} x^k$$

such that

(iii) $c_{h+k,n} = \sum_j c_{h,j} d_{k,n-j}$

for all integers h, k and n.

The homogeneous sequence (ψ_n) will be said the *allied sequence* of (ϕ_n).

The following generalizations of Proposition 5.1 and 5.2 are straightforward:

5.4. PROPOSITION. *A sequence (ϕ_n) of inverse Laurent series, with $n \in \mathbf{Z}$ and $\phi_n(x) = \sum_k c_{k,n} x^k$, is a Sheffer sequence if and only if its canonical matrix is (α, β)-recursive, with $\alpha \in C^+$; moreover,*

$$\beta(x) = \sum_j c_{0,j} x^j$$

and

$$\alpha(x) = \beta^{-1}(x) \sum_j c_{1,j} x^j.$$

5.5. PROPOSITION. *A sequence (ϕ_n) in L^-, with $n \in \mathbf{Z}$, is a Sheffer sequence if and only if there exists a homogeneous sequence (ψ_n) such that the identities*

$$\langle \alpha\beta \mid \phi_n \rangle = \sum_k \langle \alpha \mid \phi_k \rangle \langle \beta \mid \psi_{n-k} \rangle$$

hold for every $\alpha, \beta \in L^+$ and for every integer n.

From the preceding results it follows that a Sheffer sequence is a spanning set for L^- if and only if the boundary value of its canonical matrix is in R^+. Furthermore, a Sheffer sequence which is a spanning set is a basis for L^- whenever the recurrence rule of its canonical matrix is in I^+. Such sequences will be called *recursive bases* for L^-.

As usual, given any series $\alpha \in C^+$, its *conjugate sequence* (ϕ_n) is the (unique) sequence in L^- defined by

$$\phi_n(x) = \sum_k \langle \alpha^k \mid x^n \rangle x^k.$$

As an immediate consequence of Proposition 3.1, we have

5.6. PROPOSITION. *A sequence (ϕ_n) of inverse Laurent sequences is a homogeneous sequence if and only if it is the conjugate sequence of a Laurent series $\alpha \in C^+$. Furthermore, α is the recurrence rule of the canonical matrix of (ϕ_n).*

The preceding result can be generalized to the Sheffer case as follows:

given an ordered pair (α, β), with $\alpha \in C^+$ and $\beta \neq \zeta$, its *conjugate sequence* (ϕ_n) is the (unique) sequence in L^- defined by

$$\phi_n(x) = \sum_k \langle \alpha^k \beta \mid x^n \rangle \, x_k.$$

We have:

5.7. Proposition. *A sequence (ϕ_n) in L^- is a Sheffer sequence if and only if it is the conjugate sequence of an ordered pair (α, β), with $\alpha \in C^+$ and $\beta \in L^+$, $\beta \neq \zeta$. Furtherore, α and β are the recurrence rule and the boundary value of its canonical matrix, respectively.*

Another classical notion is that of *associated sequence* of a series: a Laurent series $\alpha \in C^+$ and a sequence (ϕ_n) in L^- will be called *associated* whenever the identities

$$\langle \alpha^k \mid \phi_n \rangle = \delta_{n,k} \qquad (*)$$

hold for every integer n and k.

5.8. Proposition. *A sequence (ϕ_n) in L^- is a homogeneous recursive basis if and only if it is the associated sequence for a series α in I^+. Moreover,*

$$\tilde{\alpha}(x) = \sum_k \langle x \mid \phi_k \rangle \, x^k.$$

Proof. We remark that, if $\mathbf{C}$ is the canonical matrix of the sequence (ϕ_n) and N is the (α, υ)-recursive matrix, then identities $(*)$ can be re-written as

$$\mathbf{N} \times \mathbf{C} = \mathbf{I};$$

by Theorem 1 we have tha assertion. ∎

As a consequence of Proposition 5.8 we have the following expansion formulas:

5.9. Proposition. *Let (ϕ_n) be the associated sequence for $\alpha \in I^+$; then, for every $\sigma \in L^-$, we have*

$$\sigma(x) = \sum_k \langle \alpha^k \mid \sigma \rangle \, \phi_k(x).$$

5.10 Proposition. *Let (ϕ_n) be the associated sequence for $\alpha \in I^+$; then, for every $\gamma \in L^+$, we have*

$$\gamma(x) = \sum_k \langle \gamma \mid \phi_k \rangle \, \alpha^k(x).$$

Given any Laurent series α, we define a continuous linear operator

$$\boldsymbol{\alpha}: L^- \to L^-$$

by setting

$$\boldsymbol{\alpha}\phi(x) := \boldsymbol{\alpha}^{\bullet}(x)\,\phi(x)$$

for every $\phi \in L^-$.

In particular, being $\tau(x) = x$, by this definition $\boldsymbol{\tau}$ is the linear operator such that $\boldsymbol{\tau}x^n = x^{n-1}$ for every integer n, and hence it is nothing but Roman's derivative D introduced in [26].

As usual, we are interested in characterizing the matrix of a given operator $\boldsymbol{\alpha}$, with $\alpha \in L^+$, with respect to the canonical pseudobasis (x^i) of L^-.

5.11. Proposition. *For every $\alpha \in L^+$, the matrix of the linear operator $\boldsymbol{\alpha}$ is the Appell matrix $R(\tau, \alpha)$. Conversely, given any continuous linear operator*

$$T: L^- \to L^-$$

whose matrix is (τ, α)-recursive, we have

$$T = \boldsymbol{\alpha}.$$

Proof. Let $\mathbf{M} := (m_{ij})$ be the canonical matrix of the operator $\boldsymbol{\alpha}$. By assumption, we have, for every $j \in \mathbf{Z}$,

$$\alpha^{\bullet}(x)\,x^j = \sum_{i \in \mathbf{Z}} m_{ij} x^i,$$

that is, $\mathbf{M} = C(\tau, \alpha^{\bullet}) = R(\tau, \alpha)$. ∎

Hence, the algebra Σ of operators $\boldsymbol{\alpha}$, with $\alpha \in L^+$, is topologically isomorphic to the algebra $A(\mathbf{A})$ of Appell matrices. This allows us to expand any $\boldsymbol{\alpha} \in \Sigma$ as

$$\boldsymbol{\alpha} = \alpha(\boldsymbol{\tau}).$$

A well-known result on shift-invariant operators of the classical umbral calculus can be read in the present theory as follows:

5.12. Proposition. *A continuous linear operator*

$$T: L^- \to L^-$$

is in Σ if and only if it commutes with $\boldsymbol{\tau}$.

Given any Laurent series α of positive degree a sequence (ϕ_n) in L^- will be called *related* to α whenever

$$\mathbf{a}\phi_n = \phi_{n-1} \qquad (**)$$

for every integer n.

The following result is nothing but the classical Transfer Formula, without boundary conditions.

5.13. Proposition. *The sequence (ϕ_n) is related to $\alpha \in C^+$ if and only if the canonical matrix of (ϕ_n) is an inverse recursive matrix with recurrence rule $(\alpha^{-1})^{\bullet}$.*

Proof. Conditions $(**)$ can be read as:

$$\phi_{n-1} = \phi_n \alpha^{\bullet}$$

and this gives the assertion. ∎

5.14. Proposition. *Let $\alpha \in C^+$, $\deg(\alpha) = h > 0$, and let (ϕ_n) be a sequence related to α; then*

$$\deg^-(\phi_n) = nh + \deg^-(\phi_0)$$

for every integer n. Hence, (ϕ_n) is a basis for L^- if and only if $\alpha \in I^+$ and $\phi_0 \in R^-$.

Theorem 2 leads us to prove the following results:

5.15. Proposition. *A sequence (ϕ_n) related to $\alpha \in C^+$ is a Sheffer sequence if and only if $\alpha \in I^+$.*

5.16. Proposition. *If a sequence (ϕ_n) is associated to $\alpha \in I^+$, then (ϕ_n) is related to α.*

5.17. Proposition (Transfer Formula). *A sequence (ϕ_n) related to $\alpha \in I^+$ is the sequence associated to α if and only if*

$$\phi^{\bullet}_{-1}(x) = xD\tilde{\alpha}(x).$$

Proof. Follows immediately by Theorem 3. ∎

Needless to add, the group of classical Sheffer and umbral operators is faithfully represented by the group of all invertible recursive matrices.

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