Some identities for determinants of structured matrices

Estelle L. Basor *
Department of Mathematics
California Polytechnic State University
San Luis Obispo, CA 93407

Torsten Ehrhardt †
Fakultät für Mathematik
Technische Universität Chemnitz
09107 Chemnitz, Germany

Abstract

In this paper we establish several relations between the determinants of the following structured matrices: Hankel matrices, symmetric Toeplitz + Hankel matrices and Toeplitz matrices. Using known results for the asymptotic behavior of Toeplitz determinants, these identities are used in order to obtain Fisher-Hartwig type results on the asymptotics of certain skewsymmetric Toeplitz determinants and certain Hankel determinants.

1 Introduction

In this paper we prove identities that involve the determinants of several types of structured matrices such as Hankel matrices, symmetric Toeplitz + Hankel matrices and skewsymmetric Toeplitz matrices. After having established these identities we show how they can be used in order to obtain asymptotic formulas for these determinants.

Let us first recall the underlying notation. Given a sequence $\{a_n\}_{n=-\infty}^{\infty}$ of complex numbers, we associate the formal Fourier series

$$a(t) \;=\; \sum_{n=-\infty}^{\infty} a_n t^n, \qquad t \in \mathbb{T}. \tag{1.1}$$

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The $N \times N$ Toeplitz and Hankel matrices with the (Fourier) symbol a are defined by

$$T_N(a) \;=\; (a_{j-k})_{j,k=0}^{N-1}, \qquad H_N(a) \;=\; (a_{j+k+1})_{j,k=0}^{N-1}. \tag{1.2}$$

Usually a represents an L^1-function defined on the unit circle, in which case the numbers a_n are the Fourier coefficients,

$$a_n \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} a(e^{i\theta})e^{-in\theta}\,d\theta, \qquad n \in \mathbb{Z}. \tag{1.3}$$

To a given symbol a we associate the symbol $\tilde{a}(t) := a(t^{-1})$. The symbol a is called even (odd) if $\tilde{a}(t) = \pm a(t)$, i.e., $a_{-n} = \pm a_n$.

For our purposes it is important to define another type of Hankel matrix. Given a function $b \in L^1[-1,1]$ with moments defined by

$$b_n \;=\; \frac{1}{\pi}\int_{-1}^{1} b(x)(2x)^{n-1}\,dx, \qquad n \geq 1, \tag{1.4}$$

the $N \times N$ Hankel matrices generated by the (moment) symbol b are defined by

$$H_N[b] \;=\; (b_{1+j+k})_{j,k=0}^{N-1}. \tag{1.5}$$

We indicate the difference in the definition by using the notation $H_N(\cdot)$ and $H_N[\cdot]$. The function b is called even if $b(x) = b(-x)$.

Our motivation to prove in the following sections identities for the above mentioned determinants comes from several problems. The best known problem, called the Fisher-Hartwig conjecture, concerns the asymptotics of the determinants of Toeplitz matrices for singular symbols. One would like to be able to compute the asymptotics of the determinant of $T_N(a)$ when the symbol a has jump discontinuities, zeros, or other singularities of a certain form. A history of this problem and many known results and applications can be found in [4] or [5]. In section five of this paper we prove some Fisher-Hartwig type results for certain skewsymmetric Toeplitz matrices.

Another interesting problem is to compute asymptotically the determinants of the matrices

$$T_N(a) + H_N(a)$$

where the symbol a also has singularities. The interest in these asymptotics, especially in the case where a is even, arose in random matrix theory (see [1] and the references therein). The determination of these asymptotics will be done in a forthcoming paper [2].

Finally, Hankel matrices defined by the moments of a function given on a line segment play an important role in orthogonal polynomial theory and again in random matrix theory. We refer the reader to [10] for orthogonal polynomial connections and to [8] for a general account of random matrix theory. In section five we prove two results for the asymptotics of the determinants of the Hankel moment matrices. These results allow the function b to have jump discontinuities, but require the function to be even.

The paper is organized as follows. Sections 2, 3, and 4 contain all the linear algebra type results which prove the exact identities for the various types of matrices and are self-contained. The asymptotic results are contained in section 5 and use the results of the previous sections and some already known results for Toeplitz matrices.

2 Hankel determinants versus symmetric Toeplitz + Hankel determinants

We begin with a preliminary result which will allow us to show the relationship with symmetric Toeplitz plus Hankel matrices and the Hankel moment matrices.

Proposition 2.1 *Let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers such that $a_n = a_{-n}$ and let $\{b_n\}_{n=1}^{\infty}$ be a sequence defined by*

$$b_n = \sum_{k=0}^{n-1} \binom{n-1}{k} (a_{1-n+2k} + a_{2-n+2k}). \tag{2.6}$$

Define the one-sided infinite matrices

$$A = (a_{j-k} + a_{j+k+1})_{j,k=0}^{\infty}, \qquad B = (b_{j+k+1})_{j,k=0}^{\infty}, \tag{2.7}$$

and the upper triangular one-sided infinite matrix

$$D = \begin{pmatrix} \xi(0,0) & \xi(1,1) & \xi(2,2) & \dots \\ & \xi(1,0) & \xi(2,1) & \dots \\ & & \xi(2,0) & \\ 0 & & & \ddots \end{pmatrix} \quad \textit{where} \quad \xi(n,k) = \binom{n}{[\frac{k}{2}]}. \tag{2.8}$$

Then $B = D^T AD$.

Proof. The assertion is equivalent to the statement that for all $n, m \geq 0$ the following identity holds:

$$b_{n+m+1} = \sum_{j=0}^{n} \sum_{k=0}^{m} (a_{n-j-m+k} + a_{n-j+m-k+1}) \xi(n,j) \xi(m,k), \tag{2.9}$$

where b_{n+m+1} is given by

$$b_{n+m+1} \;=\; \sum_{r=0}^{n+m} \binom{n+m}{r} (a_{2r-n-m} + a_{2r-n-m+1}). \tag{2.10}$$

In order to prove this identity it is sufficient to prove that for each $s \geq 0$ the terms $a_s = a_{-s}$ occur as many times in (2.9) as in (2.10). In fact, a_s and a_{-s} occurs in (2.9) exactly $N_1 + N_2 + N_3$ times if $s \geq 1$ and $N_1 = N_2$ times if $s = 0$, where

$$\begin{aligned}
N_1 &= \sum_{\substack{0 \leq j \leq n \\ 0 \leq k \leq m \\ j-k=n-m-s}} \binom{n}{[\frac{j}{2}]}\binom{m}{[\frac{k}{2}]} = \sum_{\substack{0 \leq j \leq n \\ m+1 \leq k \leq 2m+1 \\ j+k=n+m-s+1}} \binom{n}{[\frac{j}{2}]}\binom{m}{[\frac{k}{2}]}, \\
N_2 &= \sum_{\substack{0 \leq j \leq n \\ 0 \leq k \leq m \\ j-k=n-m+s}} \binom{n}{[\frac{j}{2}]}\binom{m}{[\frac{k}{2}]} = \sum_{\substack{n+1 \leq j \leq 2n+1 \\ 0 \leq k \leq m \\ j+k=n+m-s+1}} \binom{n}{[\frac{j}{2}]}\binom{m}{[\frac{k}{2}]}, \\
N_3 &= \sum_{\substack{0 \leq j \leq n \\ 0 \leq k \leq m \\ j+k=n+m+1-s}} \binom{n}{[\frac{j}{2}]}\binom{m}{[\frac{k}{2}]}.
\end{aligned}$$

In the expression for N_1 we have made a change of variables $k \mapsto 2m+1-k$ and in N_2 a change of variables $j \mapsto 2n+1-j$. Hence it follows that

$$N_1 + N_2 + N_3 \;=\; \sum_{\substack{j,k \geq 0 \\ j+k=n+m+1-s}} \binom{n}{[\frac{j}{2}]}\binom{m}{[\frac{k}{2}]}.$$

Moreover, $N_1 = N_2 = \frac{N_1+N_2+N_3}{2}$ for $s = 0$ since then $N_3 = 0$.

On the other hand, a_s and a_{-s} occurs in (2.10) exactly $M_1 + M_2$ times if $s \geq 1$ and $M_1 = M_2$ times if $s = 0$, where

$$M_1 \;=\; \binom{n+m}{[\frac{n+m+s}{2}]}, \qquad M_2 \;=\; \binom{n+m}{[\frac{n+m-s}{2}]}.$$

Thus we are done as soon as we have shown that $M_1 + M_2 = N_1 + N_2 + N_3$ for each $s \geq 0$.

We distinguish two cases. If $n + m + 1 - s$ is even, then we substitute $j \mapsto 2j$, $k \mapsto 2k$, and $j \mapsto 2j + 1$, $k \mapsto 2k + 1$ in the above expression for $N_1 + N_2 + N_3$ and arrive at

$$\begin{aligned}
N_1 + N_2 + N_3 &= \sum_{\substack{j,k \geq 0 \\ 2j+2k=n+m+1-s}} \binom{n}{j}\binom{m}{k} + \sum_{\substack{j,k \geq 0 \\ 2j+2k=n+m-1-s}} \binom{n}{j}\binom{m}{k}. \\
&= \binom{n+m}{\frac{n+m+1-s}{2}} + \binom{n+m}{\frac{n+m-1-s}{2}} \;=\; M_1 + M_2.
\end{aligned}$$

If $n+m+1-s$ is odd, then we substitute $j \mapsto 2j$, $k \mapsto 2k+1$, and $j \mapsto 2j+1$, $k \mapsto 2k$ in the expression for $N_1+N_2+N_3$ and obtain

$$\begin{aligned} N_1+N_2+N_3 &= 2 \sum_{\substack{j,k\geq 0\\ 2j+2k=n+m-s}} \binom{n}{j}\binom{m}{k} \\ &= 2\binom{n+m}{\frac{n+m-s}{2}} \;=\; M_1+M_2, \end{aligned}$$

which also completes the proof. □

Theorem 2.2 *Let* $\{a_n\}_{n=-\infty}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *fulfill the assumptions of the previous proposition. For* $N \geq 1$ *define the matrices*

$$A_N = (a_{j-k}+a_{j+k+1})_{j,k=0}^{N-1}, \qquad B_N = (b_{j+k+1})_{j,k=0}^{N-1}. \tag{2.11}$$

Then $\det A_N = \det B_N$.

Proof. A_N and B_N are the $N \times N$ sections of the infinite matrices A and B of the previous proposition. Let D_N be the $N\times N$ sections of the infinite matrix D. Because of the triangular structure of D, it follows that $B_N = D_N^T A_N D_N$. Noting that the entries on the diagonal of D are equal to $\xi(n,0)=1$, we obtain the desired assertion. □

The previous theorem shows the connection between the determinants of a symmetric Toeplitz + Hankel matrix on the one hand and a Hankel determinant on the other hand. We now express this relationship by using the standard notation for these matrices.

Theorem 2.3 *Let* $a \in L^1(\mathbb{T})$ *be an even function, and define* $b \in L^1[-1,1]$ *by*

$$b(\cos\theta) = a(e^{i\theta})\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}. \tag{2.12}$$

Then $\det(T_N(a)+H_N(a)) = \det H_N[b]$.

Proof. The moments of b are given by

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-1}^{1} b(x)(2x)^{n-1}\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} a(e^{i\theta})(1+\cos\theta)(2\cos\theta)^{n-1}\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} a(e^{i\theta})(1+e^{-i\theta})(e^{i\theta}+e^{-i\theta})^{n-1}\,d\theta \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} a(e^{i\theta})\left(\sum_{k=0}^{n-1}(e^{i(n-1-2k)\theta}+e^{i(n-2-2k)\theta})\binom{n-1}{k}\right)d\theta$$
$$= \sum_{k=0}^{n-1}\binom{n-1}{k}(a_{-n+1+2k}+a_{-n+2+2k}).$$

Here we have made a change of variables $x=\cos\theta$ and written $(e^{i\theta}+e^{-i\theta})^{n-1}$ using the binomial formula. With regard to (2.6) and Theorem 2.2 this completes the proof. □

In regard to relation (2.12) we remark that $b \in L^1[-1,1]$ in and only if $a(e^{i\theta})(1+\cos\theta) \in L^1(\mathbb{T})$.

Thus at this point we have shown that if a and b satisfy the relation (2.12), then

$$\det H_N[b] = \det(T_N(a)+H_N(a)).$$

But actually more can be done in the case that the symbol a satisfies a quarter wave symmetry property. Then, in fact, certain Hankel moment determinants can be written as Toeplitz determinants. The symbol $b(x) \in L^1[-1,1]$ of these Hankel determinants is of the form

$$b(x) = b_0(x)\sqrt{\frac{1+x}{1-x}} \tag{2.13}$$

where $b_0(-x)=b_0(x)$ for all $x\in[-1,1]$.

We first begin with the following auxiliary result. In what follows, let W_N stand for the matrix acting on $\mathbb{C}^N$ by

$$W_N : (x_0,x_1,\dots,x_{N-1}) \mapsto (x_{N-1},\dots,x_1,x_0),$$

and let I_N denote the $N\times N$ identity matrix.

Proposition 2.4 *Let $a \in L^1(\mathbb{T})$ and assume that $a(-t)=a(t^{-1})=a(t)$. Define*

$$d(e^{i\theta}) = a(e^{i\theta/2}). \tag{2.14}$$

Then $\det(T_N(a)+H_N(a)) = \det T_N(d)$.

Proof. Note first that $d(t)$ is well defined since $a(t)=a(-t)$. Moreover, $a_{2n+1}=0$ and $a_{2n}=d_n$. By rearranging rows and columns of $T_N(a)+H_N(a)$ in an obvious way, it is easily seen that this matrix is similar to

$$\begin{pmatrix} (a_{2j-2k})_{j,k=0}^{N_1-1} & 0 \\ 0 & (a_{2j-2k})_{j,k=0}^{N_2-1}\end{pmatrix} + \begin{pmatrix} 0 & (a_{2j+2k+2})_{j=0,\ k=0}^{N_1-1,N_2-1} \\ (a_{2j+2k+2})_{j=0,\ k=0}^{N_2-1,N_1-1} & 0\end{pmatrix}$$

where $N_1 = \left[\frac{N+1}{2}\right]$ and $N_2 = \left[\frac{N-1}{2}\right]$. This matrix equals

$$\begin{pmatrix} T_{N_1}(d) & H_{N_1,N_2}(d) \\ H_{N_2,N_1}(d) & T_{N_2}(d) \end{pmatrix},$$

where $H_{N_1,N_2}(d)$ and $H_{N_2,N_1}(d)$ are Hankel matrices of size $N_1 \times N_2$ and $N_2 \times N_1$, respectively. Multiplying the last matrix from the left and the right with the diagonal matrix $\operatorname{diag}(W_{N_1}, I_{N_2})$ we obtain the matrix $T_N(d)$. Notice in this connection that $d_n = d_{-n}$ since $a(t^{-1}) = a(t)$. □

Corollary 2.5 *Let $b \in L^1[-1,1]$ and suppose (2.13) holds with $b_0(-x) = b_0(x)$ for all $x \in [-1,1]$. Define the function*

$$d(e^{i\theta}) \;=\; b_0(\cos\frac{\theta}{2}). \tag{2.15}$$

Then $\det H_N[b] = \det T_N(d)$.

Proof. Since $b_0(x) = b_0(-x)$ it follows from definition (2.12) that $a(-t) = a(t^{-1}) = a(t)$. Now we can apply Theorem 2.3 and Proposition 2.4 in order to obtain the identity $\det H_N[b] = \det(T_N(a) + H_N(a)) = \det T_N(d)$. □

Concerning the previous corollary, we wish to emphasize that the function d is even, and hence the matrices $T_N(d)$ are symmetric.

3 Symmetric Toeplitz + Hankel determinants versus skewsymmetric Toeplitz determinants

The main result of this section has been established in [7, Lemma 18] and proved in [6, Lemma 1] and [9, Proof of Thm. 7.1(a)]. We give a slightly simplified and self-contained proof here.

Theorem 3.1 *Let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers such that $a_{-n} = a_n$. Let c_n be defined by*

$$c_n \;=\; \sum_{k=-n+1}^{n} a_k \quad \text{for } n > 0, \tag{3.16}$$

and put $c_0 = 0$ and $c_{-n} = -c_n$. Then $\det T_{2N}(c) = (\det(T_N(a) + H_N(a)))^2$.

Proof. First of all we multiply the matrix $T_{2N}(c)$ from the left and right with $\operatorname{diag}(W_N, I_N)$. We obtain the matrix

$$\begin{pmatrix} T_N(\tilde{c}) & H_N(\tilde{c}) \\ H_N(c) & T_N(c) \end{pmatrix} = \begin{pmatrix} -T_N(c) & -H_N(c) \\ H_N(c) & T_N(c) \end{pmatrix}$$

by observing that $\tilde{c} = -c$. Next we claim that

$$\begin{pmatrix} T_N(1-t) & 0 \\ T_N(t) & I_N \end{pmatrix} \begin{pmatrix} -T_N(c) & -H_N(c) \\ H_N(c) & T_N(c) \end{pmatrix} \begin{pmatrix} T_N(1-t^{-1}) & T_N(t^{-1}) \\ 0 & I_N \end{pmatrix}$$
$$= \begin{pmatrix} I_N & 0 \\ 0 & T_N(1+t) \end{pmatrix} \begin{pmatrix} X_N & -T_N(a) - H_N(a) \\ T_N(a) + H_N(a) & 0 \end{pmatrix} \begin{pmatrix} I_N & 0 \\ 0 & T_N(1+t^{-1}) \end{pmatrix}$$

with a certain matrix X_N. If we take the determinant of this equation, we obtain the desired determinant identity.

In order to proof the above matrix identity it suffices to show that the following three equations hold:

$$T_N(c) - T_N(t)T_N(c)T_N(t^{-1}) + H_N(c)T_N(t^{-1}) - T_N(t)H_N(c) = 0, \qquad (3.17)$$

$$-T_N(t)T_N(c)T_N(1-t^{-1}) + H_N(c)T_N(1-t^{-1}) = T_N(1+t)\left(T_N(a) + H_N(a)\right), \qquad (3.18)$$

$$T_N(1-t)T_N(c)T_N(t^{-1}) + T_N(1-t)H_N(c) = \left(T_N(a) + H_N(a)\right)T_N(1+t^{-1}). \qquad (3.19)$$

Notice that (3.19) can be obtained from (3.18) by passing to the transpose. Moreover, by employing (3.17) equation (3.18) reduces to

$$T_N(1-t)\left(T_N(c) + H_N(c)\right) = T_N(1+t)\left(T_N(a) + H_N(a)\right). \qquad (3.20)$$

Let us first prove (3.17). We introduce the $N \times 1$ column vectors $e_0 = (1, 0, 0, \ldots, 0)^T$ and $\gamma_N = (0, c_1, c_2, \ldots, c_{N-1})^T$. Then

$$T_N(c) - T_N(t)T_N(c)T_N(t^{-1}) = \gamma_N e_0^T - e_0\gamma_N^T = T_N(t)H_N(c) - H_N(c)T_N(t^{-1}),$$

whence indeed (3.17) follows.

Next we remark that from the definition of the sequences $\{a_n\}_{n=-\infty}^{\infty}$ and $\{c_n\}_{n=-\infty}^{\infty}$ it follows that $c_n - c_{n-1} = a_n + a_{n-1}$ for all $n \in \mathbb{Z}$. Introducing the column vectors $\hat{\gamma}_N = (c_1, \ldots, c_N)^T$, $\alpha_N = (a_0, \ldots, a_{N-1})^T$ and $\hat{\alpha}_N = (a_1, \ldots, a_N)^T$, it can be readily verified that

$$\begin{aligned} T_N(1-t)T_N(c) &= (c_{j-k} - c_{j-k-1})_{j,k=0}^{N-1} - e_0\hat{\gamma}_N^T, \\ T_N(1+t)T_N(a) &= (a_{j-k} + a_{j-k-1})_{j,k=0}^{N-1} - e_0\hat{\alpha}_N^T, \\ T_N(1-t)H_N(c) &= (c_{j+k+1} - c_{j+k})_{j,k=0}^{N-1} + e_0\gamma_N^T, \\ T_N(1+t)H_N(a) &= (a_{j+k+1} + a_{j+k})_{j,k=0}^{N-1} - e_0\alpha_N^T. \end{aligned}$$

Using the above relation $c_n - c_{n-1} = a_n + a_{n-1}$, it follows that

$$\begin{aligned} T_N(1-t)T_N(c) - T_N(1+t)T_N(a) &= -e_0\hat{\gamma}_N^T + e_0\hat{\alpha}_N^T \\ T_N(1+t)H_N(a) - T_N(1-t)H_N(c) &= -e_0\alpha_N^T - e_0\gamma_N^T. \end{aligned}$$

Since $\hat{\gamma}_N - \gamma_N = \hat{\alpha}_N + \alpha_N$ by the same relation, this implies equation (3.20). □

The results of this theorem are not easy to rephrase by using the classical notation for Toeplitz and Hankel matrices. Consider, for instance, the simplest case where $a(t) \equiv 1$. Then $c_n = \mathrm{sign}(n)$ which are not the Fourier coefficients of an L^1-function. For more information on how one can nevertheless express the relationship between the symbols a and c, and how the asymptotics for certain of the above determinants can be determined we refer to [2].

4 Hankel determinants versus skewsymmetric Toeplitz determinants

The results of the previous two sections allow us to establish an identity between Hankel determinants and determinants of skewsymmetric Toeplitz matrices. The next theorem is an additional needed ingredient for the identity.

Theorem 4.1 *Let $\{c_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers such that $c_{-n} = -c_n$ for all $n \in \mathbb{Z}$. Define numbers $\{b_n\}_{n=1}^{\infty}$ by*

$$b_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \left\{ \binom{n-1}{k} - \binom{n-1}{k-1} \right\} c_{n-2k}. \tag{4.21}$$

Moreover, define the matrices

$$B_N = (b_{j+k+1})_{j,k=0}^{N-1}, \qquad C_{2N} = (c_{j-k})_{j,k=0}^{2N-1}.$$

Then $\det C_{2N} = (\det B_N)^2$.

Proof. In formula (3.16) the numbers c_n are defined in terms of the numbers $a_{-n+1}, \dots, a_n$. By a simple inspection of this formula, it is easy to see that for any given sequence $\{c_n\}_{n=-\infty}^{\infty}$ there exists a sequence $\{a_n\}_{n=-\infty}^{\infty}$ such that (3.16) and $a_n = a_{-n}$ holds for all positive n.

Now let us define the numbers b_n not by (4.21) but by (2.6). Then with B_N and C_{2N} defined as above it follows from Theorem 2.2 and Theorem 3.1 that $\det C_{2N} = (\det B_N)^2$. It remains to show that (4.21) holds.

Indeed, we have that

$$\begin{aligned}
\sum_{k=0}^{[\frac{n}{2}]}\left\{\binom{n-1}{k}-\binom{n-1}{k-1}\right\}c_{n-2k} &= \sum_{k=0}^{[\frac{n}{2}]}\left\{\binom{n-1}{k}-\binom{n-1}{k-1}\right\}\sum_{j=-n+2k+1}^{n-2k}a_j \\
&= \sum_{\substack{-n+2k+1\le j\le n-2k\\ 0\le 2k\le n}}\left\{\binom{n-1}{k}-\binom{n-1}{k-1}\right\}a_j \\
&= \sum_{j=-n+1}^{n}\ \sum_{k=0}^{\min\{[\frac{n-j}{2}],[\frac{n+j-1}{2}]\}}\left\{\binom{n-1}{k}-\binom{n-1}{k-1}\right\}a_j \\
&= \sum_{j=-n+1}^{n}\binom{n-1}{\min\left\{\left[\frac{n-j}{2}\right],\left[\frac{n+j-1}{2}\right]\right\}}a_j \\
&= \sum_{j=-n+1}^{n}\binom{n-1}{\left[\frac{n-j}{2}\right]}a_j \;=\; \sum_{k=0}^{n-1}\binom{n-1}{k}(a_{2k+1-n}+a_{2k+2-n}).
\end{aligned}$$

By formula (2.6) this is equal to b_n. □

We again express the above relationship in terms of the standard notation.

Theorem 4.2 *Let $b\in L^1[-1,1]$ and define $c\in L^1(\mathbb{T})$ by*

$$c(e^{i\theta}) = i\,\mathrm{sign}(\theta)\,b(\cos\theta), \qquad -\pi<\theta<\pi. \tag{4.22}$$

Then $\det T_{2N}(c)=(\det H_N[b])^2$.

Proof. Obviously, $c(e^{-i\theta})=-c(e^{i\theta})$. Hence $c_{-n}=-c_n$. It is sufficient to verify formula (4.21) for the Fourier coefficients and moments. First of all,

$$c_n = \frac{1}{\pi}\int_0^{\pi} b(\cos\theta)\sin(n\theta)\,d\theta.$$

Hence

$$b_n = \frac{1}{\pi}\int_0^{\pi} b(\cos\theta)\left(\sum_{k=0}^{[\frac{n}{2}]}\left\{\binom{n-1}{k}-\binom{n-1}{k-1}\right\}\sin((n-2k)\theta)\right)d\theta.$$

The expression in the big braces equals (by a change of variables $k\mapsto n-k$ in the second part of the sum)

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n-1}{k}\sin((n-2k)\theta)-\sum_{k=n-[\frac{n}{2}]}^{n}\binom{n-1}{n-k-1}\sin((2k-n)\theta)$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} \sin((n-2k)\theta) \quad = \quad (2\cos\theta)^{n-1} \sin\theta.$$

Hence

$$b_n \quad = \quad \frac{1}{\pi} \int_0^\pi b(\cos\theta)(2\cos\theta)^{n-1} \sin\theta \, d\theta.$$

Now it is easy to see that b_n are the moments of the function b. □

Regarding relation (4.22) we remark that $c \in L^1(\mathbb{T})$ if and only if $b(x)/\sqrt{1-x^2} \in L^1[-1,1]$.

At this point we have three main identities for Hankel moment determinants, one which follows from Theorem 2.3, one which follows from Corollary 2.5 and finally one which follows from the previous theorem. If we desire to find the asymptotics of the determinants of the Hankel moment matrices it is clear that the corresponding asymptotics for Toeplitz matrices need to be derived. In particular, in light of Theorem 4.2 and formula (4.22), it is desirable to compute the asymptotics of the Toeplitz determinant $\det T_{2N}(c)$, where c satisfies $c(e^{-i\theta}) = -c(e^{i\theta})$ and accordingly implies that the Toeplitz matrices are skewsymmetric. Note from this it follows that $\det T_{2N+1}(c) = 0$ for all N. However, this implies that a single asymptotic formula for the determinants, such as the one given in the classical Szegö limit theorem, or the more general Fisher-Hartwig formulas would not make sense here. In the following section we nevertheless compute the asymptotics of such Toeplitz determinants in some cases and raise a conjecture about more general cases.

5 Asymptotics of certain skewsymmetric Toeplitz determinants and Hankel determinants

Our goal of this section is to consider Toeplitz determinants with generating function $c(e^{i\theta}) = \chi(e^{i\theta})a(e^{i\theta})$ where a is an even functions and

$$\chi(e^{i\theta}) \quad = \quad i\,\mathrm{sign}(\theta), \qquad -\pi < \theta < \pi. \tag{5.23}$$

Let $t_\beta(e^{i\theta})$ stand for the function

$$t_\beta(e^{i\theta}) \quad = \quad e^{i\beta(\theta-\pi)}, \qquad 0 < \theta < 2\pi. \tag{5.24}$$

This function has a single jump at $t = 1$ whose size is determined by the parameter β.

In the following proposition we assume that a is not necessarily an even function but satisfies instead a rotation symmetry condition.

Proposition 5.1 *Assume that $a \in L^1(\mathbb{T})$ satisfies the relation $a(-t) = a(t)$ for $t \in \mathbb{T}$. Define the functions*

$$d(e^{i\theta}) = a(e^{i\theta/2}), \qquad d_1(e^{i\theta}) = t_{-1/2}(e^{i\theta})d(e^{i\theta}), \qquad d_2(e^{i\theta}) = t_{1/2}(e^{i\theta})d(e^{i\theta}).$$

Then $\det T_{2N}(a) = (\det T_N(d))^2$ *and* $\det T_{2N}(\chi a) = \det T_N(d_1) \det T_N(d_2)$.

Proof. From the assumptions $a(t) = a(-t)$ it follows that the Fourier coefficients a_{2n+1} are zero. Hence $T_{2N}(a)$ has a checkered pattern, and rearranging rows and columns it is easily seen that $T_{2N}(a)$ is similar to the matrix $\operatorname{diag}(T_N(d), T_N(d))$.

The Fourier coefficients c_{2n} of $c(t) = \chi(t)a(t)$ are equal to zero. By rearranging the rows and columns of $T_{2N}(\chi a)$ in the same way as above it becomes apparent that $T_{2N}(\chi a)$ is similar to a matrix

$$\begin{pmatrix} 0 & D_2 \\ D_1 & 0 \end{pmatrix} \qquad \text{where } D_1 = \left(c_{2(j-k)+1}\right)_{j,k=0}^{N-1} \text{ and } D_2 = \left(c_{2(j-k)-1}\right)_{j,k=0}^{N-1}.$$

From the identity

$$\chi(e^{i\theta}) \;=\; t_{-1/2}(e^{i\theta})t_{1/2}(e^{i(\theta-\pi)}) \;=\; -t_{1/2}(e^{i\theta})t_{-1/2}(e^{i(\theta-\pi)}) \tag{5.25}$$

it follows that $d_1(e^{i\theta}) = e^{-i\theta/2}c(e^{i\theta/2})$ and $d_2(e^{i\theta}) = -e^{i\theta/2}c(e^{i\theta/2})$. Hence $D_1 = T_N(d_1)$ and $D_2 = -T_N(d_2)$. Since $\det T_{2N}(c) = (-1)^N \det D_1 \det D_2$, this completes the proof. □

Hence we have reduced the computation of $\det T_{2N}(\chi a)$ to the Toeplitz determinants $T_N(d_1)$ and $T_N(d_2)$, for which in the case of piecewise continuous functions it is possible to apply the Fisher-Hartwig conjecture under certain assumptions.

The following result, which is taken from [5], makes this explicit. Therein $G(\cdot)$ is the Barnes G-function [11], $d_{0,\pm}$ are the Wiener-Hopf factors of the function d_0,

$$d_{0,\pm}(e^{i\theta}) \;=\; \exp\left(\sum_{k=1}^{\infty}[\log d_0]_{\pm k}e^{\pm ik\theta}\right), \tag{5.26}$$

and

$$d_{\pm}(e^{i\theta}) \;=\; d_{0,\pm}(e^{i\theta})\prod_{r=1}^{R}\left(1-e^{\pm i(\theta-\theta_r)}\right)^{\pm\beta_r} \tag{5.27}$$

are the generalized Wiener-Hopf factors of d.

Proposition 5.2 *Let*

$$d(e^{i\theta}) \;=\; d_0(e^{i\theta})\prod_{r=1}^{R}t_{\beta_r}(e^{i(\theta-\theta_r)}), \tag{5.28}$$

where d_0 is an infinitely differentiable nonvanishing function with winding number zero, $\theta_1, \ldots, \theta_R \in (0, 2\pi)$ are distinct numbers, and $\beta_1, \ldots, \beta_R$ are complex parameters satisfying $|\mathrm{Re}\,\beta_r| < 1/2$ for all $r = 1, \ldots, R$. Then

$$\begin{aligned} \frac{\det T_N(t_{-1/2}d)}{\det T_N(d)} &\sim N^{-1/4}G(1/2)G(3/2)d_+(1)^{-1/2}d_-(1)^{1/2}, \qquad N \to \infty, \\ \frac{\det T_N(t_{1/2}d)}{\det T_N(d)} &\sim N^{-1/4}G(1/2)G(3/2)d_+(1)^{1/2}d_-(1)^{-1/2}, \qquad N \to \infty. \end{aligned}$$

Moreover,

$$\det T_N(d) \sim F^N N^{\Omega} E, \qquad N \to \infty, \tag{5.29}$$

where $F = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log d_0(e^{i\theta})\, d\theta\right)$, $\Omega = -\sum_{r=1}^{R} \beta_r^2$, and E is another constant.

(The constant E is quite complicated, so in the interest of brevity, we omit the exact formula from this paper and refer to [3, 4, 5] for an explicit representation.)

The previous propositions yield the following results. We keep the same notation.

Corollary 5.3 *Let d be a function of the form (5.28) and assume that the same conditions as above are fulfilled. Let $a(e^{i\theta}) = d(e^{2i\theta})$. Then*

$$\frac{\det T_{2N}(\chi a)}{\det T_{2N}(a)} \sim N^{-1/2}G^2(1/2)G^2(3/2), \qquad N \to \infty, \tag{5.30}$$

and

$$\det T_{2N}(a) \sim F^{2N}N^{2\Omega}E^2, \qquad N \to \infty. \tag{5.31}$$

The following corollary gives an asymptotic formula for the determinants of Hankel moment matrices in the special case where the symbol is even.

Corollary 5.4 *Let $b \in L^1[-1, 1]$ such that $b(-x) = b(x)$. Define $d(e^{i\theta}) = b(\cos(\theta/2))$ and suppose that d is of the form (5.28). Then*

$$\det H_N[b] \sim F^N N^{\Omega-1/4}G(1/2)G(3/2)E, \qquad N \to \infty. \tag{5.32}$$

Proof. Define $a(e^{i\theta}) = b(\cos\theta)$. Then Theorem 4.2 implies that $(\det H_N[b])^2 = \det T_N(\chi a)$. Since $b(x) = b(-x)$ the function is well defined and $a(e^{i\theta}) = d(e^{2i\theta})$. Now the formula follows from Corollary 5.3 and by taking square roots. □

The interesting point in Corollary 5.3 is that the asymptotic limit of (5.30) does not depend on the underlying function a. We remark that we have proved this limit relation for certain piecewise continuous functions a subject to the condition $a(-t) = a(t)$. Our primary goal was however to determine the limit for certain functions a satisfying the relation $a(t^{-1}) = a(t)$. Our conjecture is that the asymptotic limit is given by the above expression in general also for those functions.

In order to support this hypothesis we resort to the generalization of the Fisher-Hartwig conjecture, which has not yet been proved, but is strongly suggested by examples. Since $\det T_{2N+1}(\chi a) = 0$ for all N (under the assumption $a(t^{-1}) = a(t)$), the asymptotics of $T_N(\chi a)$ can only be described by the generalized but not the original conjecture. The crucial observation is that one has several possibilities for representing χa in a form like (5.28). Indeed, from (5.25) it follows that

$$\chi(e^{i\theta})a(e^{i\theta}) = t_{-1/2}(e^{i\theta})t_{1/2}(e^{i(\theta-\pi)})a(e^{i\theta}) = -t_{1/2}(e^{i\theta})t_{-1/2}(e^{i(\theta-\pi)})a(e^{i\theta}),$$

tacitly assuming that a admits also representation of the form (5.28) with appropriate properties.

Then the generalized conjecture predicts [3, 5] that

$$\begin{aligned}\det T_N(\chi a) \sim\ & \det T_N(t_{-1/2}(e^{i\theta}))\det T_N(t_{1/2}(e^{i(\theta-\pi)}))\det T_N(a)E_1 \\ & +(-1)^N \det T_N(t_{1/2}(e^{i\theta}))\det T_N(t_{-1/2}(e^{i(\theta-\pi)}))\det T_N(a)E_2,\end{aligned}$$

where E_1 and E_2 are the "correlation" constants

$$\begin{aligned}E_1 &= E(t_{-1/2}(e^{i\theta}), t_{1/2}(e^{i(\theta-\pi)}))E(t_{-1/2}(e^{i\theta}), a)E(t_{1/2}(e^{i(\theta-\pi)}), a) \\ E_2 &= E(t_{1/2}(e^{i\theta}), t_{-1/2}(e^{i(\theta-\pi)}))E(t_{1/2}(e^{i\theta}), a)E(t_{-1/2}(e^{i(\theta-\pi)}), a)\end{aligned}$$

with $E(\cdot,\cdot)$ defined by

$$E(b,c) = \exp\left(\lim_{r\to 1-0}\sum_{k=1}^{\infty}\Big(k[\log h_r b_+]_k[\log h_r c_-]_{-k} + k[\log h_r b_-]_{-k}[\log h_r c_+]_k\Big)\right),$$

$h_r b_\pm$ and $h_r c_\pm$ denoting the harmonic extensions of the Wiener-Hopf factors of $b_\pm$ and $c_\pm$.

From all this it follows that

$$\frac{\det T_{2N}(\chi a)}{\det T_{2N}(a)} \sim (2N)^{-1/2}G^2(1/2)G^2(3/2)(E_1+E_2),$$

where a straightforward computation of the constants gives

$$\begin{aligned}E_1 &= 2^{-1/2}\left(\frac{a_+(-1)a_-(1)}{a_-(-1)a_+(1)}\right)^{1/2}, \\ E_2 &= 2^{-1/2}\left(\frac{a_+(-1)a_-(1)}{a_-(-1)a_+(1)}\right)^{-1/2}.\end{aligned}$$

The assumption that $a(t^{-1}) = a(t)$ implies that $a_-(t) = \gamma a_+(t^{-1})$ with a certain constant $\gamma \neq 0$. Hence

$$E_1 = E_2 = 2^{-1/2},$$

which leads to the conjecture that

$$\frac{\det T_{2N}(\chi a)}{\det T_{2N}(a)} \sim N^{-1/2} G^2(1/2) G^2(3/2), \qquad N \to \infty. \tag{5.33}$$

Using Theorem 4.2 we arrive at a conjecture for the Hankel moment matrices:

$$\frac{\det H_N[b]}{\sqrt{\det T_{2N}(a)}} \sim N^{-1/4} G(1/2) G(3/2), \qquad N \to \infty, \tag{5.34}$$

where $a(e^{i\theta}) = b(\cos\theta)$. We remark that this formula is in accordance with Corollary 5.4.

We end this section by noting one other result that follows from our identities and Corollary 2.5. This result applies to Hankel moment matrices with a special case of Jacobi weights and computes the asymptotics for $\det H_N[b]$ where b is of the form $b_0(x)\sqrt{\frac{1+x}{1-x}}$ with an even function b_0.

Corollary 5.5 *Suppose $b \in L^1[-1,1]$ is of the above form with an even function b_0. Let $d(e^{i\theta}) = b_0(\cos(\theta/2))$ and suppose the d is of the form (5.28). Then*

$$\det H_N[b] \sim F^N N^{\Omega} E, \qquad N \to \infty.$$

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