JOURNAL OF FUNCTIONAL ANALYSIS **50**, 387–413 (1983)

Toeplitz and Wiener–Hopf Determinants with Piecewise Continuous Symbols

ESTELLE BASOR

California State Polytechnic University, San Luis Obispo, California 93407

AND

HAROLD WIDOM

University of California, Santa Cruz, California 95064

Communicated by the Editors

Received June 26, 1982

The asymptotics for determinants of Toeplitz and Wiener–Hopf operators with piecewise continuous symbols are obtained in this paper. If $W_\alpha(\sigma)$ is the Wiener–Hopf operator defined on $L_2(0, \alpha)$ with piecewise continuous symbol σ having a finite number of discontinuities at ξ_r, then under appropriate conditions it is shown that

$$\det W_\alpha(\sigma) \sim G(\sigma)^\alpha \, \alpha^{\Sigma \lambda_r^2} K(\sigma),$$

where $G(\sigma) = \exp(\log \sigma)\hat{}(0)$, $\lambda_r = (1/2\pi) \log[\sigma(\xi_r+)/\sigma(\xi_r-)]$ and $K(\sigma)$ is a completely determined constant. An analogous result is obtained for Toeplitz operators. The main point of the paper is to obtain a result in the Wiener–Hopf case since the Toeplitz case had been treated earlier. In the Toeplitz case it was discovered that one could obtain asymptotics fairly easily for symbols with several singularities if, for each singularity one could find a single example of a symbol with a singularity of that kind whose associated asymptotics were known. Fortunately in the Toeplitz case such asymptotics were known. The difficulty in the Wiener–Hopf case is that there was not a single singular case where the determinant was explicitly known. This problem was overcome by using the fact that Wiener–Hopf determinants when discretized become Toeplitz determinants whose entries depend on the size of the matrix. No theorem on Toeplitz matrices can be applied directly but these theorems are modified to obtain the desired results.

1. INTRODUCTION

The asymptotic formulas for the determinants of finite Toeplitz matrices $T_N(\phi)$ and finite Wiener–Hopf operators $W_\alpha(\sigma)$ associated with smooth symbols ϕ and σ are by now classical. To remind the reader, and to fix

0022-1236/83 $3.00

notation, ϕ is a function defined on the unit circle with Fourier coefficients ϕ_k and $T_N(\phi)$ is the $N \times N$ matrix

$$(\phi_{i-j}) \qquad i, j = 0,..., N-1.$$

The asymptotic formula

$$\det T_N(\phi) \sim G(\phi)^N E(\phi) \qquad (N \to \infty), \tag{1.1}$$

where

$$G(\phi) = \exp(\log \phi)_0, \qquad E(\phi) = \exp \sum_{k=1}^{\infty} k(\log \phi)_k (\log \phi)_{-k}.$$

This is correct if, for example, ϕ is continuous and

$$\sum |k| \, |\phi_k|^2 < \infty, \qquad \phi(e^{i\theta}) \neq 0, \qquad \Delta \arg \phi(e^{i\theta}) = 0. \tag{1.2}$$

The operator $W_\alpha(\sigma)$ acts on $L_2(0, \alpha)$ and has distribution kernel $\delta(x-y) - k(x-y)$, where

$$(\delta - k)(z) = \hat{\sigma}(z) = \frac{1}{2\pi} \int e^{-i\xi z} \sigma(\xi) \, d\xi.$$

If $1 - \sigma \in L_1$, then $I - W_\alpha(\sigma)$ is of trace class so det $W_\alpha(\sigma)$ is well defined. If in addtion σ is continuous and

$$\int |z| \, |k(z)|^2 \, dz < \infty, \qquad \sigma(\xi) \neq 0, \qquad \Delta \arg \sigma(\xi) = 0,$$

then

$$\det W_\alpha(\sigma) \sim G(\sigma)^\alpha E(\sigma) \qquad (\alpha \to \infty),$$

where

$$G(\sigma) = \exp(\log \sigma)\hat{}(0), \qquad E(\sigma) = \exp \int_0^\infty z(\log \sigma)\hat{}(z)(\log \sigma)\hat{}(-z) \, dz.$$

The asymptotics of Toeplitz determinants for singular symbols (those significantly violating conditions (1.2)) were first considered in [5, 8], where certain special cases (for which the determinants could be evaluated explicitly) were worked out and conjectures made as to the nature of the asymptotic formula for certain classes of singular symbols. The conjectures were proved in [9] for one class of singular symbols and then extended to a wider class in [1]. These proofs were difficult. Recently ([2, 3]) it was discovered that one could obtain fairly easily the asymptotics for certain symbols with several singularities if, for each singularity, one could find a single example of a symbol with a singularity of that kind whose associated asymptotics were known. Thus the special cases of [5, 8] do actually yield some general results.

An attempt to treat the Wiener–Hopf case in a similar way runs into the problem that (at least as far as the authors know) there is not a single singular case where the determinant is explicitly known. Of course one could try the methods of [1, 9] which probably have continuous analogs but, because of their difficulty, this did not seem to us a happy idea. Instead we began with the fact, observed and exploited by Dyson [4], that Wiener–Hopf determinants when discretized become Toeplitz determinants and the asymptotics of the latter could be used to obtain, quite formally, the asymptotics of the former. The problem here is that after discretizing not only does the size of the matrix $T_N(\phi)$ depend on N (which in turn depends on α) but so also does the function ϕ. Thus no theorem on Toeplitz matrices can be applied directly. But the methods of proof can be used and we do use them here to obtain asymptotic results in the Wiener–Hopf case.

We shall consider symbols whose only singularities are a finite number of jump discontinuities, for which the method of [2] is most successful. Since our symbol ϕ associated with a fixed σ will vary with N we shall have to redo the main theorem of [2]. This will not be an unfortunate redundancy (we hope the reader will agree) since we have also modified it to be able to give a simpler form to the answer, even for a fixed ϕ, than one had hitherto realized existed. Thus although the main point of this paper is to obtain a result in the Wiener–Hopf case, which is new, we also make a minor contribution in the Toeplitz case.

To state the formula for $T_N(\phi)$ we suppose ϕ has a continuously defined argument (although this is probably not absolutely necessary) and set

$$\lambda_r = \frac{1}{2\pi} \log[\phi(e^{i\theta_r+})/\phi(e^{i\theta_r-})].$$

These are real valued. The assertion is

$$\det T_N(\phi) \sim G(\phi)^N N^{\Sigma \lambda_r^2} E(\phi) \prod_r g(\lambda_r). \tag{1.3}$$

Here $G(\phi)$ is as before except that one is careful always to use a $\log \phi$ with continuous imaginary part (which exists by assumption). The definition of $E(\phi)$ is modified, now and hereafter, to

$$E(\phi) = \exp \sum_{k=1}^{\infty} \left\{ k(\log \phi)_k (\log \phi)_{-k} - k^{-1} \sum \lambda_r^2 \right\}.$$

The factors $g(\lambda_r)$ in (1.3) are given by

$$g(\lambda) = e^{(1+\gamma)\lambda^2} \prod_{k=1}^{\infty} \left(1 + \frac{\lambda^2}{k^2}\right)^k e^{-\lambda^2/k},$$

where γ is Euler's constant.

The formula for Wiener–Hopf operators if σ has finitely many jump discontinuities at the points ξ_r is just a little more complicated. We set

$$\lambda_r = \frac{1}{2\pi} \log[\sigma(\xi_{r+})/\sigma(\xi_{r-})],$$

let $G(\sigma)$ and $g(\lambda)$ be as defined before, but now define

$$E(\sigma) = \exp \int_0^{\infty} \left\{ z(\log \sigma)\hat{}(z)(\log \sigma)\hat{}(-z) - \frac{1-e^{-z}}{z} \sum \lambda_r^2 \right\} dz.$$

The assertion is

$$\det W_\alpha(\sigma) \sim G(\sigma)^\alpha \, \alpha^{\sum \lambda_r^2} E(\sigma) \prod_r g(\lambda_r). \tag{1.4}$$

We state the precise results now. The C^2 assumptions could be relaxed but this would result in more complicated statements.

THEOREM I. *If ϕ is bounded away from zero, is piecewise C^2 with a finite number of jump discontinuities, and has a continuously defined argument then* (1.3) *holds.*

THEOREM II. *If σ is bounded away from zero, is piecewise C^2 with a finite number of jump discontinuities, has a continuously defined argument which vanishes at $\pm\infty$, and if*

$$1 - \sigma(\xi) \in L_1, \qquad (1 + \xi^2)\, \sigma''(\xi) \in L_2,$$

then (1.4) *holds.*

As the reader can easily verify the series defining $E(\phi)$ and the integral defining $E(\sigma)$ are (conditionally) convergent under the assumptions of the theorems.

Here is an outline of the proofs of the theorems. The main lemma (our modification of the main theorem of [2]) says that often (1.3) holds for a product of two symbols if it holds for each and if they have no common point of discontinuity. The functions may in general depend on N; but if they do not and if they both satisfy the hypothesis of Theorem I, then nothing else is required. From this it will quickly follow that Theorem I holds in general if it holds in the special case

$$\phi(e^{i\theta}) = e^{\lambda(\pi-\theta)}, \qquad 0 < \theta \leqslant 2\pi$$

and if it holds for all ϕ without jumps. That it holds in this special case (where, observe, $E(\phi) = 1$) was shown in [5]; the determinant is a Cauchy determinant and the asymptotics were not difficult. If ϕ has no jumps, then (13) reduces to (1.1) which holds because conditions (1.2) are satisfied. But (and this will be useful) it also follows from the main lemma applied to the Wiener–Hopf factors $\phi_\pm$ of ϕ. For each of them (1.1) is trivially true since

$$\det T_N(\phi_\pm) = G(\phi_\pm)^N$$

exactly and $E(\phi_\pm) = 1$.

For Theorem II it suffices to do the case of any convenient function σ with a single jump. Replacing the kernel of $W_\alpha(1-\sigma)$ by one which is constant on x and y intervals of length α/N will have the effect, under a simple assumption, of replacing $W_\alpha(\sigma)$ by an operator with the same determinant as $T_N(\phi)$, where

$$\phi(e^{i\theta}) = \sigma(N\theta/\alpha). \tag{1.5}$$

It suffices for this that $1-\sigma$ have compact support and that N/α be large enough. So we choose N depending on α so that $N/\alpha \to \infty$ and (1.4) will follow if we can show that

$$\det T_N(\phi) \sim \det W_\alpha(\sigma) \tag{1.6}$$

and that (1.3) holds for our variable ϕ. The former will be shown to hold if

$$\alpha^2/N \to 0. \tag{1.7}$$

As for the latter we cannot apply Theorem I since ϕ is not fixed. Instead we use the main lemma twice as in the proof of that theorem. For the main lemma to be applicable we must assume that

$$N = O(\alpha^{3-\delta})$$

for some $\delta > 0$ which is fortunately not inconsistent with (1.7).

Much of the work in this paper will consist of estimation of trace norms of various operators. In the next section we present first some lemmas on s-numbers of operators (ranging from the well known to the perhaps new). These are applied to prove (1.6) as well as the fact, needed later, that if ϕ or σ has no discontinuities, then the corresponding Hankel operator is trace class. (For ϕ this is very easy, for σ less so.) The following three sections contain the proofs of the main lemma and Theorem I and II, respectively. For general facts concerning s numbers, trace class operators, and determinants we refer the reader to [7].

2. Preliminaries

We use the following notation: The Hilbert–Schmidt norm of an operator, as well as the L_2 norm of a kernel, is denoted by $\| \ \|_2$; the trace norm of an operator is $\| \ \|_1$. In Lemma 2.1, M denotes any measure space, J an interval in $(-\infty, \infty)$ and Z_N the set of integers in $[0, N)$. Moreover, ∂_y, ∂_y^2 denote first and second partial derivatives with respect to y and

$$\Delta_j K(x, j) = K(x, j+1) - K(x, j).$$

LEMMA 2.1. (i) *For any operator A, $s_n(A) \leqslant n^{-1/2} \|A\|_2$.*

(ii) *If A is the operator from $L_2(J)$ to $L_2(M)$ with kernel $K(x, y)$ we have $s_{2n}(A) \leqslant n^{-3/2} |J| \, \|\partial_y K(x, y)\|_2$, $s_{3n}(A) \leqslant n^{-5/2} |J|^2 \, \|\partial_y^2 K(x, y)\|_2$.*

(iii) *If A is the operator from $L_2(Z_N)$ to $L_2(M)$ with kernel $K(x, j)$ we have $s_{2n-1}(A) \leqslant n^{-3/2} N \|\Delta_j K(x, j)\|_2$.*

Proof. (i) This is immediate from

$$\|A\|_2^2 \geqslant \sum_{m=1}^{n} s_m(A)^2 \geqslant n s_n(A)^2.$$

(ii) Divide J into n equal parts and for each $y \in J$ denote by $\{y\}$ the left endpoint of the interval containing y. Let

$$K'(x, y) = K(x, \{y\})$$

and denote by A' the corresponding operator. Then

$$s_{2n}(A) \leqslant s_n(A - A') + s_{n+1}(A') = s_n(A - A') \leqslant n^{-1/2} \|K - K'\|_2,$$

since A' has rank n and where we have used (i). In estimating $\|K - K'\|_2^2$ there arise integrals of the form

$$\int_0^\lambda |f(s) - f(0)|^2 \, ds,$$

where $\lambda := n^{-1} |J|$ and $f(s) = K(x, \{y\} + s)$. Since

$$|f(s) - f(0)|^2 \leqslant s \int_0^\lambda |f'(t)|^2 \, dt \tag{2.1}$$

the above integral is at most

$$\tfrac{1}{2}\lambda^2 \int_0^{\lambda} |f'(t)|^2 \, dt.$$

Applying this to each of the functions f and adding, we obtain

$$\|K - K'\|_2^2 \leqslant \tfrac{1}{2} n^{-2} |J|^2 \|\partial_y K\|_2^2$$

and the first assertion follows.

For the second we use instead of $K'(x, y)$ the function $K''(x, y)$ which is linear in each y subinterval and for which K'' and $\partial_y K''$ are equal, respectively, to K and $\partial_y K$ at each left endpoint. Then the corresponding A'' has rank $2n$, inequality (2.1) is replaced by

$$|f(s) - f(0)|^2 \leqslant s^3 \int_0^{\lambda} |f''(t)|^2 \, dt$$

and the second assertion follows easily.

(iii) The proof is similar to that of the first part of (ii) except that we divide the interval $[0, N)$ into n subintervals of length $[N/n)$ plus perhaps one shorter interval.

In Lemma 2.2, A is the operator given by the infinite matrix

$$K(i, j), \qquad i, j = 0, 1, \ldots.$$

We define the quantities p_n and q_n by

$$p_n^2 = \sum_{i=0}^{\infty} \sum_{j=n}^{\infty} |K(i, j)|^2, \qquad q_n^2 = \sum_{i=0}^{\infty} \sum_{j=n}^{\infty} |\Delta_j K(i, j)|^2.$$

LEMMA 2.2. *For all integers $m \geqslant n > 1$ we have*

$$s_{4n}(A) \leqslant n^{-1/2} p_m + m n^{-3/2} q_n.$$

Proof. The matrix $K(i, j)$ may be written as the sum of three matrices K_1, K_2, K_3 in which K is multiplied, respectively, by

$$\chi_{[0,n)}(j), \qquad \chi_{[n,m)}(j), \qquad \chi_{[m,\infty)}(j).$$

(The χ denotes characteristic function.) Accordingly A is the sum of three operators A_1, A_2, A_3. Since A_1 has rank n, $s_{n+1}(A_1) = 0$. By Lemma 2.1(i)

$$s_n(A_3) \leqslant n^{-1/2} \|K_3\|_2 = n^{-1/2} p_m.$$

By Lemma 2.1(iii)

$$s_{2n+1}(A_2) \leqslant n^{-3/2}(m-n)\,\|\Delta_j K_2(i,j)\|_2 \leqslant n^{-3/2} m q_n.$$

Since $s_{4n}(A) \leqslant s_{n+1}(A_1) + s_{2n+1}(A_2) + s_n(A_3)$ the assertion follows.

Lemma 2.2 will not be used until the last section. Lemma 2.1 will be used now to prove (1.6).

LEMMA 2.3. *Assume $k \in C^2(R)$ and that $k'(z)$ and $k''(z)$ are $O(|z|^{-1})$ as $|z| \to \infty$. Divide the interval $[0, \alpha]$ into N equal parts and let $\{x\}$ denote the left endpoint of the interval containing x. Then the trace norm of the operator on $L_2(0, \alpha)$ with kernel*

$$k(x-y) - k(\{x\} - \{y\})$$

has limit 0 if $\alpha \to \infty$, $\alpha^2/N \to 0$.

Proof. Write the kernel as $K_1 + K_2$, where

$$K_1(x, y) = k(x-y) - k(x - \{y\}), \qquad K_2(x, y) = k(x - \{y\}) - k(\{x\} - \{y\}).$$

Our assumptions on k imply

$$\|K_1\|_2 \leqslant CN^{-1}\alpha^{3/2}, \qquad \|\partial_x K_1\|_2 \leqslant CN^{-1}\alpha^{3/2}.$$

Here (and in the future) C denotes a constant, different each time it appears. We also write

$$\beta = N/\alpha$$

so our assumption $\alpha^2/N \to 0$ is equivalent to $\alpha/\beta \to 0$. The above inequalities and Lemma 2.1(ii) (with ∂_y replaced by ∂_x) give

$$s_n(A_1) \leqslant Cn^{-1/2}\beta^{-1}\alpha^{1/2}, \qquad s_n(A_1) \leqslant Cn^{-3/2}\beta^{-1}\alpha^{5/2}$$

(where the second inequality holds for $n \geqslant 2$). We apply the first inequality if $n < \alpha$ and the second if $n \geqslant \alpha$ to deduce

$$\|A_1\|_1 = \sum s_n(A_1) \leqslant C\beta^{-1}\alpha.$$

As for A_2, right multiplication by the unitary operator $f(x) \to \beta^{-1/2} f(\beta x)$ from $L_2(0, N)$ to $L_2(0, \alpha)$ yields the operator with kernel

$$\beta^{-1/2} K_2(x, \beta^{-1} y).$$

This is constant in the y-intervals $[j, j+1)$ and so A_2 has the same s numbers as the operators from $L_2(Z_N)$ to $L_2(0, \alpha)$ with kernel

$$\beta^{-1/2}K_2(x, \beta^{-1}j) = \beta^{-1/2}[k(x-\beta^{-1}j) - k(\{x\} - \beta^{-1}j)].$$

Since $x - \{x\} \leqslant \beta^{-1}$ this kernel is bounded by a constant times

$$\beta^{-3/2}(1 + |x - \beta^{-1}j|)^{-1}$$

and it follows that its L_2 norm is at most $C\beta^{-3/2}N^{1/2} = C\beta^{-1}\alpha^{1/2}$.

Hence by Lemma 2.1(i)

$$s_n(A_2) \leqslant Cn^{-1/2}\beta^{-1}\alpha^{1/2}. \tag{2.2}$$

Next, we have

$$\beta^{-1/2}\Delta_j K_2(x, \beta^{-1}j) = -\beta^{-1/2}\int_{\{x\}}^{x}\int_{\beta^{-1}j}^{\beta^{-1}(j+1)} k''(s-t)\,dt\,ds.$$

This is bounded by a constant times

$$\beta^{-5/2}(1 + |x - \beta^{-1}j|)^{-1}$$

and thus has L_2 norm at most a constant times $\beta^{-5/2}N^{1/2}$. Consequently, by Lemma 2.1(iii)

$$s_n(A_2) \leqslant Cn^{-3/2}\beta^{-5/2}N^{3/2} = Cn^{-3/2}\beta^{-1}\alpha^{3/2} \tag{2.3}$$

for $n \geqslant 3$. We apply (2.2) if $n < \alpha$ and (2.3) if $n \geqslant \alpha$ to deduce

$$\|A_2\|_1 = \sum s_n(A_2) \leqslant C\beta^{-1}\alpha.$$

Thus $\|A\|_1 \leqslant C\beta^{-1}\alpha \to 0$, and Lemma 2.3 is established.

LEMMA 2.4. *Assume σ satisfies the hypothesis of Theorem* II *and in addition $1-\sigma$ vanishes outside a bounded set. If σ is defined by* (1.5) *and if $\alpha^2/N \to 0$, then* (1.6) *holds.*

Proof. We shall apply Lemma 2.1 with $k = (\sigma - 1)\hat{}$, the hypotheses of the lemma easily being seen to hold. We continue to write $\beta = N/\alpha$. The operator with kernel $k(\{x\} - \{y\})$ is unitarily equivalent to the operator on $L_2(0, N)$ with kernel

$$\beta^{-1}k(\{\beta^{-1}x\} - \{\beta^{-1}y\})$$

which in turn has the same nonzero spectrum as the matrix with i, j entry

$$\beta^{-1}k(\beta^{-1}(i-j)), \qquad i, j = 0, ..., N-1.$$

This is just the Toeplitz matrix $T_N(\phi - 1)$ since for large β the function $\sigma(\beta\theta) - 1$ is supported in $|\theta| < \pi$ and so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [\sigma(\beta\theta) - 1]\, e^{-ij\theta}\, d\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\sigma(\beta\theta) - 1]\, e^{-ij\theta}\, d\theta$$
$$= \beta^{-1}k(\beta^{-1}j).$$

Thus if T is the operator with kernel $k(\{x\} - \{y\})$, then

$$\det T_N(\phi) = \det(I + T)$$

and, by Lemma 2.2, $\|T - W_\alpha(\sigma - 1)\|_1 \to 0$. Equivalently

$$\|I + T - W_\alpha(\sigma)\|_1 \to 0.$$

Now under the assumptions of Theorem II the operators $W_\alpha(\sigma)$ are invertible and $\|W_\alpha(\sigma)^{-1}\| = O(1)$ as $\alpha \to \infty$. (See [6, Chap. IV, Sect. 4] for the Toeplitz analog.) We deduce that

$$\|W_\alpha(\sigma)^{-1}(I + T) - I\|_1 \to 0,$$

so $\det[W_\alpha(\sigma)^{-1}(I + T)] \to 1$ and (1.6) follows.

LEMMA 2.5. *If ϕ satisfies the hypothesis of Theorem* I *and in addition is continuous, then the Hankel operator $H(\phi)$, having matrix*

$$(\phi_{i+j+1}), \qquad i, j \geqslant 0.$$

is trace class. If σ satisfies the hypothesis of Theorem II *and in addition is continuous, then the Hankel operator $H(\sigma)$, having kernel*

$$k(x + y), \qquad x, y > 0,$$

is trace class.

Proof. We shall consider only the second statement, the proof of the first being much easier. We may replace σ by $\sigma - 1$ since the δ summand of k is irrelevant to the Hankel operator.

First we consider the special case where σ, as well as being continuous, satisfies

$$\sigma'(\xi_{r+}) = \sigma'(\xi_{r-}) \tag{2.4}$$

for each r. This implies that σ' is absolutely continuous.

If k vanishes outside a bounded set, say $|x| \leqslant M$, then

$$k(x+y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi_{[0,M]}(x)\, e^{i\xi(x+y)} \chi_{[0,M]}(y)\, \sigma(\xi)\, d\xi.$$

For each fixed ξ the integrand is the kernel of a rank one operator of norm M. Hence in this case

$$\|H(\sigma)\|_1 \leqslant (M/2\pi)\, \|\sigma\|_1 < \infty.$$

Let L be a C^∞ function with compact support and equal to 1 on $[0, 1]$. Let $\check{f}$ denote the inverse Fourier transform. Write

$$\sigma = \hat{h} * \sigma + (\sigma - \hat{h} * \sigma).$$

Since $(\hat{h} * \sigma)^{\check{}} = hk$ has compact support $H(\hat{h} * \sigma)$ is trace class. As for the second term it is easy to check that it satisfies the same conditions assumed about σ, but in addition its inverse Fourier transform vanishes on the interval $[0, 1]$. Thus we have shown that we may assume to begin with that $k(z) = 0$ for $z \in (0, 1)$.

To estimate the quantities $s_n(H(\sigma))$ we write

$$H(\sigma) = H_1 + H_2,$$

where H_1, H_2 have kernels

$$k(x+y)\, \chi_{[1, n^{1/2}]}(y), \qquad k(x+y)\, \chi_{[n^{1/2}, \infty)}(y),$$

respectively. To estimate $s_n(H_1)$ we apply the second assertion of Lemma 2.1(ii) to deduce that for $n \geqslant 3$

$$s_n(H_1) \leqslant C n^{-3/2} \left\{ \int_1^\infty x\, |k''(x)|^2\, dx \right\}^{1/2}.$$

It is an elementary exercise that $\xi^2 \sigma''(\xi) \in L_2$ and $\sigma \in L_2$ imply $(\xi^2 \sigma(\xi))'' \in L_2$. (We do have $\sigma \in L_2$ under our assumption that k vanish near 0 since $\sigma'' \in L_2$ implies $x^2 k(x) \in L_2$ and so also $k(x) \in L_2$.) Hence $x^2 k''(x) \in L_2$ and the integral above is finite. Thus $\|H_1\|_1 = O(1)$ as $n \to \infty$. To estimate $s_n(H_2)$ we use Lemma 2.1(i) and find

$$s_n(H_2) \leqslant C n^{-1/2} \left\{ \int_{n^{1/2}}^\infty x\, |k(x)|^2\, dx \right\}^{1/2}.$$

Since $x^2k(x) \in L_2$

$$\int_{n^{1/2}}^{\infty} x\,|k(x)|^2\,dx \leqslant n^{-3/2} \int_{n^{1/2}}^{\infty} x^4\,|k(x)|^2\,dx = O(n^{-3/2}).$$

Hence also $\|H_2\|_1 = O(1)$ and Lemma 2.5 is established in the case where (2.4) is satisfied.

To remove this restriction we need consider only the special case

$$\begin{aligned} \sigma(\xi) &= e^{-a\xi}, \qquad \xi > 0, \\ &= e^{b\xi}, \qquad \xi < 0, \end{aligned} \tag{2.5}$$

where a and b are positive constants; for any σ under consideration is the sum of one satisfying (2.4) and a linear combination of translates of functions of this particular form. We make two observations. First, the argument used above showed that the Hankel operator was trace class if the kernel $k(x)$ satisfied

$$\int_0^{\infty} x\,|k''(x)|^2\,dx < \infty, \qquad \int_0^{\infty} x^4\,|k(x)|^2\,dx < \infty. \tag{2.6}$$

Second, a Hankel operator is trace class if its kernel is of the form

$$k(x) = \frac{1}{2\pi} \int_0^{\infty} e^{-xt} u(t)\,dt, \tag{2.7}$$

where

$$\int_0^{\infty} t^{-1}\,|u(t)|\,dt < \infty. \tag{2.8}$$

(For $e^{-(x+y)t}$ is the kernel of a rank one operator of norm t^{-1}.) Now for σ given by (2.5) the corresponding k is given by (2.7) with

$$u(t) = i(e^{-iat} - e^{-ibt}).$$

If we write

$$u = u\chi_{[0,1]} + u\chi_{[1,\infty)} = u_1 + u_2,$$

then the corresponding kernel k_1 gives rise to a trace class Hankel operator because (2.8) is satisfied for u_1; and the corresponding kernel k_2 is given by

$$k_2(x) = i \left[\frac{e^{-(x+ia)}}{x+ia} - \frac{e^{-(x+ib)}}{x+ib} \right]$$

and so gives rise to a trace class Hankel operator since (2.6) is satisfied. This completes the proof of Lemma 2.5.

3. The Main Lemma

Given a function ϕ defined on the unit circle $T(\phi)$ and $H(\phi)$ denote, respectively, the semi-infinite Toeplitz and Hankel matrices

$$(\phi_{i-j}), \qquad (\phi_{i+j+1}), \qquad i, j = 0, 1, \ldots$$

thought of as acting on L_2 of the nonnegative integers Z^+. As usual we shall occasionally identify $T_N(\phi)$ with $P_N T(\phi) P_N$, where

$$P_N(a_0, a_1, \ldots) = (a_0, \ldots, a_{N-1}, 0, \ldots).$$

We define the function ϕ^t by

$$\phi^t(e^{i\theta}) = \phi(e^{-i\theta}).$$

(This was denoted by $\tilde{\phi}$ in earlier papers. Since the tilde is standard notation for the conjugate function, which will arise later in this paper, a change in notation seems in order.) If we define the operator Q_N by

$$Q_N(a_0, a_1, \ldots) = (a_{N-1}, \ldots, a_0, 0, \ldots),$$

then it is easy to check that

$$T_N(\phi^t) = Q_N T(\phi) Q_N. \tag{3.1}$$

Also easily checked are the identities

$$T(\phi\psi) - T(\phi)\, T(\psi) = H(\phi)\, H(\psi^t) \tag{3.2}$$

$$T_N(\phi\psi) - T_N(\phi)\, T_N(\psi) = P_N H(\phi)\, H(\psi^t)\, P_N + Q_N H(\phi^t)\, H(\psi)\, Q_N. \tag{3.3}$$

In this last identity we identify $L_2(Z_N)$ with $P_N L_2(Z^+)$, and shall continue to do this from time to time.

In the main lemma we are given a pair of functions ϕ, ψ (which may depend on N) and associate with this pair the operators on $L_2(Z^+)$

$$A_N(\phi, \psi) = T_N(\psi)^{-1}\, T_N(\phi)^{-1}\, P_N H(\phi)\, H(\psi^t)\, P_N, \qquad B_N(\phi, \psi) = A_N(\phi^t, \psi^t),$$

$$A(\phi, \psi) = T(\psi)^{-1}\, T(\phi)^{-1}\, H(\phi)\, H(\psi^t), \qquad B(\phi, \psi) = A(\phi^t, \psi^t).$$

It is tacitly assumed in the statement of the lemma that the operators $T(\phi)$,

$T(\psi)$, $T_N(\phi)$, $T_N(\psi)$ are invertible for large N. Moreover, we shall no longer display the dependence of the operators A_N, B_N, A, B on ϕ and ψ.

MAIN LEMMA. *Assume that the pair ϕ, ψ satisfies the following three conditions:*

(i) *ϕ and ψ are nonzero and piecewise C^2, have continuously defined argument, and have no common point of discontinuity.*

(ii) *The functions ϕ, ψ as well as the operators $T(\phi)^{-1}$, $T(\psi)^{-1}$, $T_N(\phi\psi)^{-1}$ are uniformly bounded.*

(iii) *As $N \to \infty$ the trace norms of each of the operators*

$$A_N - A, \qquad B_N - B, \qquad P_N A Q_N B P_N$$

tends to zero.

Then if (1.3) holds for each of the functions ϕ and ψ it holds also for $\phi\psi$.

Proof. We note first that the operators $H(\phi)\,H(\tilde{\psi})$, $H(\tilde{\phi})\,H(\psi)$ are trace class. Let f, g be continuous piecewise C^2 functions with sum 1 such that $f\phi$ and $g\psi$ belong to C^2. By (3.2)

$$\begin{aligned} H(\phi)\,H(\tilde{\psi}) &= T(\phi\psi) - T(\phi)\,T(\psi) \\ &= T(\phi f \psi) + T(\phi g \psi) - T(\phi)\,T(f)\,T(\psi) - T(\phi)\,T(g)\,T(\psi). \end{aligned}$$

Two more applications of (3.2) give

$$T(\phi f \psi) - T(\phi)\,T(f)\,T(\psi) = H(\phi)\,H(\tilde{f})\,T(\psi) + H(\phi f)\,H(\tilde{\psi}).$$

Since f and ϕf are continuous and piecewise C^2, Lemma 2.5 tells us that $H(\tilde{f})$, $H(\phi f)$ are trace class. Similarly,

$$T(\phi g \psi) - T(\phi)\,T(g)\,T(\psi)$$

is trace class and so also is $H(\phi)\,H(\tilde{\psi})$. An analogous argument applies to $H(\tilde{\phi})\,H(\psi)$.

Multiplying identity (3.3) on the left by $T_N(\psi)^{-1}\,T_N(\phi)^{-1}$ and using (3.1) show that

$$T_N(\psi)^{-1}\,T_N(\phi)^{-1}\,T_N(\phi\psi) = I_N + A_N + Q_N B_N Q_N,$$

where I_N denotes the identity operator on $L_2(Z_N)$. Now

$$\begin{aligned} I_N + A_N + Q_N B_N Q_N - P_N(I + A)\,Q_N(I + B)\,Q_N \\ = P_N(A_N - A)\,P_N + Q_N(B_N - B)\,Q_N + P_N A Q_N B Q_N \end{aligned}$$

and by assumption (iii) the trace norm of this tends to zero. Since also the operators

$$T_N(\psi)^{-1}\, T_N(\phi)^{-1}\, T_N(\phi\psi)$$

have uniformly bounded inverses, by assumption (ii), the same is true of $I_N + A_N + Q_N B_N Q_N$ and so we can deduce as at the end of the proof of Lemma 2.4 that

$$\det(I_N + A_N + Q_N B_N Q_N) \sim \det P_N(I+A)\, Q_N(I+B)\, Q_N.$$

This may be rewritten as

$$\frac{\det T_N(\phi\psi)}{\det T_N(\phi) \det T_N(\psi)} \sim \det P_N(I+A)\, P_N \det Q_N(I+B)\, Q_N.$$

Here we have used the facts that A and B are trace class (since, as was established earlier, $H(\phi)\, H(\psi^t)$ and $H(\phi^t)\, H(\psi)$ are) and that $Q_N(I+B)\, Q_N$ and $P_N(I+B)\, P_N$ have the same determinant.

Now $A_N = P_N A_N P_N$ and so from hypothesis (iii) we have

$$\|P_N A P_N - A\|_1 \to 0$$

and from hypothesis (ii)

$$\|(I+A)^{-1}\| = \|T(\phi\psi)^{-1}\, T(\phi)\, T(\psi)\| = O(1).$$

Therefore

$$\det P_N(I+A)\, P_N \sim \det(I+A)$$

and similarly

$$\det P_N(I+B)\, P_N \sim \det(I+B).$$

We have shown that under assumptions (i)–(iii)

$$\frac{\det T_N(\phi\psi)}{\det T_N(\phi) \det T_N(\psi)} \sim \det(I+A) \det(I+B).$$

The conclusion will therefore follow if we can show that

$$\det(I+A) \det(I+B) = (E(\phi\psi)/E(\phi)\, E(\psi)). \tag{3.4}$$

To prove this identity we may of course take ϕ and ψ to be fixed. Assume first that in addition ϕ and ψ are continuous. Then (1.3) holds for $\phi\psi$ as well as for ϕ and ψ by the classical result. But what we have already shown implies that for any fixed pair ϕ, ψ satisfying (i)–(iii) for which (1.3) holds for ϕ and ψ individually, (1.3) holds for $\phi\psi$ if and only if (3.4) holds. Hence it suffices to show, and we shall show, that (ii) and (iii) hold *automatically* if ϕ and ψ are fixed functions satisfying (i). (Note that they need not be continuous for this to be so, a fact which will be useful later.)

That (ii) holds follows from [6, Chap.. IV, Theorem 4.1]. As for (iii) observe that since A is compact and $Q_N \to 0$ weakly, $AQ_N \to 0$ strongly. Since B is trace class this implies $\|AQ_NB\|_1 \to 0$. (This would clearly be true if B had finite rank and the finite rank operators are $\| \ \|_1$-dense in the trace class operators.) So the last assertion of (iii) is easily established. As for the first (the second being similar), the identity

$$T_N(\phi)^{-1} P_N - T(\phi)^{-1}$$
$$= T_N(\phi)^{-1} P_N T(\phi)(I-P_N)\, T(\phi)^{-1} - (I-P_N)\, T(\phi)^{-1} \tag{3.5}$$

shows that $T_N(\phi)^{-1} P_N \to T(\phi)^{-1}$ strongly. This holds also with ϕ replaced by ψ and so, since $H(\phi)\, H(\psi^t)$ is trace class, $A_N(\phi, \psi)$ converges in trace norm to

$$T(\psi)^{-1}\, T(\phi)^{-1}\, H(\phi)\, H(\psi^t) = A(\phi, \psi).$$

So (3.4) is established if ϕ and ψ are continuous. In general there is a factorization

$$\phi(e^{i\theta}) = \phi^0(e^{i\theta}) \prod e^{\lambda_r u(\theta-\theta_r)}, \tag{3.6}$$

where $u(\theta)$ is the periodic function equal to $\pi - \theta$ for $0 < \theta < 2\pi$ and ϕ^0 is continuous and piecewise C^2. We define ϕ_s $(0 < s < 1)$ to be the function obtained by replacing each function $u(\theta)$ appearing in (3.5) with its convolution by the Poisson kernel P_s. We define ψ_s similarly. We know that (3.4) holds for ϕ_s, ψ_s and so it suffices to verify that each side is continuous at $s = 1$ (where we think of ϕ_1 and ψ_1 as ϕ and ψ).

For the left side we refer back to the argument given at the beginning of this proof that showed that $H(\phi)\, H(\psi^t)$ and $H(\phi^t)\, H(\psi)$, and then also A and B, are trace class. The various operators $H(\phi_s)$, $T(\psi_s)^{-1}$, etc., converge strongly to the corresponding operators with $s = 1$. It suffices to show, therefore, that

$$H(\phi_s f) \to H(\phi f), \qquad H(\psi_s g) \to H(\psi g)$$

in trace norm. We consider only the first. The function f could have been chosen so that (referring to (3.6)) $f\phi^0 \in C^2$ and f vanishes in the neighborhood of each θ_r. But then, since

$$P_s * u \to u$$

in C^2 outside any neighborhood of $\theta = 0$, we have that

$$f\phi_s \to f\phi$$

in C^2 of the entire circle. This guarantees that

$$\|H(f\phi_s) - H(f\phi)\|_1 \to 0$$

because of the general estimate $\|H(\phi)\|_1 \leqslant \sum |k| \, |\phi_k| \leqslant C \, \|\phi''\|_2$.

As for the right side of (3.4), it can be written as the exponential of

$$\sum_{k=-\infty}^{\infty} |k| \, (\log \phi)_k (\log \psi)_{-k}.$$

Factoring ϕ as $\phi^0(\phi/\phi^0)$, and factoring ψ similarly, results in a representation of the above series in the obvious way as the sum of four series, all convergent. Replacing ϕ, ψ by ϕ_s, ψ_s leaves one of these series unchanged, but has the effect of inserting the factor $s^{|k|}$ into the summands of two of the series and the factor $s^{2|k|}$ into the summand of the last. Thus, by Abel's theorem, the sum is continuous at $s = 1$.

This concludes the proof of (3.4) for fixed ϕ and ψ, and so also the proof of the main lemma.

4. Proof of Theorem I

It follows by induction from the main lemma (and the fact that its hypotheses (ii) and (iii) are redundant when ϕ and ψ are independent of N) that (1.3) holds for any product of functions, each satisfying the hypothesis of the theorem, if no two have a common point of discontinuity and if (1.3) holds for each. Since (1.3) holds for each factor in (3.6) it holds for the product.

5. Proof of Theorem II

There is a Wiener–Hopf analog of the main lemma:

LEMMA 5.1. *Let σ, τ be (fixed) functions satisfying the hypothesis of Theorem* II. *Assume they have no common point of discontinuity and that* (1.4) *holds for each of them. Then* (1.4) *holds also for $\sigma\tau$.*

We shall not give the proof of this since it is hardly different from that of the main lemma. Of course the second part of Lemma 2.5 is crucial here.

As in the proof of Theorem I, then, it suffices to prove (1.4) for some function σ with a single jump discontinuity, which may be assumed to occur at $\xi = 0$. We write

$$\lambda = \frac{1}{2\pi} \log \frac{\sigma(0+)}{\sigma(0-)}.$$

This constant must of course be allowed to be arbitrary but otherwise we may place on σ any restrictions we desire. In fact we shall assume that $\sigma = 1$ outside some bounded set, that σ is constant on each of the intervals $(-\eta, 0)$ and $(0, \eta)$ for some $\eta > 0$, and that σ is positive and belongs to C^∞ on $(-\infty, 0) \cup (0, \infty)$. We retain the notation of Sections 1 and 2, defining ϕ by (1.5) and assuming that $\alpha^2/N \to 0$ as $\alpha \to \infty$. By Lemma 2.4

$$\det T_N(\phi) \sim \det W_\alpha(\sigma).$$

Hence (1.4) for this σ will be a consequence of Lemmas 5.2 and 5.3.

LEMMA 5.2. *Formula* (1.3) *holds for* ϕ *if* $N = O(\alpha^{3-\delta})$ *for some* $\delta > 0$.

LEMMA 5.3. $G(\phi)^N N^{\lambda^2} E(\phi) \sim G(\sigma)^\alpha \alpha^{\lambda^2} E(\sigma)$.

The first of these is the crucial one and will be proved, as mentioned in the Introduction, by applying the main lemma twice. Here Lemma 2.2 will be used several times in the verification of hypothesis (iii).

In one application of the main lemma the pair of functions will be

$$e^{\lambda u(\theta)}, \qquad \phi(e^{i\theta})\, e^{-\lambda u(\theta)},$$

where as before $u(\theta)$ is the periodic function equal to $\pi - \theta$ for $0 < \theta < 2\pi$. In the other application the functions are the Wiener–Hopf factors of the right-hand side of the above equation. For a general ψ these are given by

$$\psi_\pm = \exp \tfrac{1}{2}[(\log \psi) \pm i(\log \psi)^\sim],$$

where $^\sim$ denotes conjugate function

$$\tilde{f}(e^{i\theta}) = \frac{1}{2\pi} \int f(e^{i\zeta}) \cot \frac{1}{2} (\theta - \zeta)\, d\zeta.$$

We shall write

$$\psi(e^{i\theta}) = \phi(e^{i\theta})\, e^{-\lambda u(\theta)} = \sigma(\beta\theta)\, e^{-\lambda u(\theta)}$$

and begin by estimating the Fourier coefficients of ψ, $\psi_\pm$ and their differences

$$\Delta\psi_k = \psi_{k+1} - \psi_k,$$

etc. Recall that we write β for N/α so that $\beta \to \infty$.

SUBLEMMA. *We have the estimates*

$$|\phi_k| \leqslant C \min((1+|k|)^{-1}, \beta(1+|k|)^{-2}), \qquad |\Delta\psi_k| \leqslant C((1+|k|)^{-2})$$

$$|(\psi_\pm)_k| \leqslant C \min(\log\beta(1+|k|)^{-1}, \beta(1+|k|)^{-2}), \quad |\Delta(\psi_\pm)_k| \leqslant C((1+|k|)^{-2}),$$

and similar estimates hold for ψ^{-1}, $\psi_\pm^{-1}$.

Proof. We consider the case of ψ_+, the others being either analogous or simpler. Since $\|\psi_+\|_\infty = O(1)$ we may assume $k \neq 0$. The estimate will be immediate once we show

$$\|\psi'_+\|_1 = O(\log\beta), \qquad \|\psi''_+\|_1 = O(\beta), \qquad \|(1-e^{i\theta})\psi''_+\|_1 = O(1). \quad (5.1)$$

We have

$$\frac{\psi'_+}{\psi_+} = \frac{\psi'}{\psi} + \left(\frac{\psi'}{\psi}\right)^{\sim}, \tag{5.2}$$

where we have used the fact that $\tilde{}$ commutes with differentiation. Since

$$\frac{\psi'}{\psi} = \beta\frac{\sigma'(\beta\theta)}{\sigma(\beta\theta)} - \lambda \tag{5.3}$$

and

$$\left\|\beta\frac{\sigma'(\beta\theta)}{\sigma(\beta\theta)}\right\|_1 = \left\|\frac{\sigma'(\theta)}{\sigma(\theta)}\right\|_1 < \infty \tag{5.4}$$

we have

$$\left\|\frac{\psi'}{\psi}\right\|_1 = O(1).$$

Next,

$$\left(\frac{\psi'}{\psi}\right)^{\sim} = \left[\beta\frac{\sigma'(\beta\theta)}{\sigma(\beta\theta)}\right]^{\sim}.$$

Now the support of $\sigma'(\beta\theta)$ is contained in an interval about 0 whose length tends to 0 as $\beta \to \infty$ and

$$\frac{1}{2}\cot\frac{1}{2}(\theta-\zeta)-\frac{1}{\theta-\zeta}$$

is bounded for $|\theta| \leq \pi$ and $|\zeta|$ small. Because of this, and (5.3) and (5.4), the L_1 norm of $(\psi'/\psi)^{\tilde{}}$ is at most a constant times the L_1 norm of

$$\frac{1}{\pi}\int \beta\frac{\sigma'(\beta\zeta)}{\sigma(\beta\zeta)}\frac{d\zeta}{\theta-\zeta}. \tag{5.5}$$

This equals $\beta\tau(\beta\theta)$ when τ is the Hilbert transform of σ'/σ,

$$\tau(\theta)=\frac{1}{\pi}\int\frac{\sigma'(\zeta)}{\sigma(\zeta)}\frac{d\zeta}{\theta-\zeta}.$$

Since σ'/σ belongs to C^2 and has compact support it is trivial that

$$|\tau(\theta)| \leqslant C(1+|\theta|)^{-1}$$

and so

$$\int_{-\pi}^{\pi}|\beta\tau(\beta\theta)|\,d\theta=\int_{-\pi\beta}^{\pi\beta}|\tau(\theta)|\,d\theta \leqslant C\log\beta.$$

This establishes the first assertion of (5.1) since $\|\psi_+\|_\infty = O(1)$.

For the second we consider the effect of differentiating each term of (5.2). Identity (5.4) becomes

$$\left\|\beta\frac{d}{d\theta}\frac{\sigma'(\beta\theta)}{\sigma(\beta\theta)}\right\|_1=\beta\left\|\frac{d}{d\theta}\frac{\sigma'(\theta)}{\sigma(\theta)}\right\|_1$$

and the integral (5.5) is replaced by

$$\frac{1}{\pi}\int\beta\frac{d}{d\zeta}\frac{\sigma'(\beta\zeta)}{\sigma(\beta\zeta)}\frac{d\zeta}{\theta-\zeta}=\beta^2\tau'(\beta\theta), \tag{5.6}$$

where

$$\tau'(\theta)=\frac{1}{\pi}\int\frac{d}{d\zeta}\frac{\sigma'(\zeta)}{\sigma(\zeta)}\frac{d\zeta}{\theta-\zeta}.$$

Now $\tau' \in L_1$ since

$$\int\frac{d}{d\zeta}\frac{\sigma'(\zeta)}{\sigma(\zeta)}\,d\zeta=0$$

and so

$$\tau'(\theta) = \frac{1}{\pi}\int \frac{d}{d\zeta}\frac{\sigma'(\zeta)}{\sigma(\zeta)}\left[\frac{1}{\theta-\zeta} - \frac{1}{\theta}\right] d\zeta$$

$$= \frac{\theta^{-1}}{\pi}\int \frac{d}{d\zeta}\frac{\sigma'(\zeta)}{\sigma(\zeta)}\frac{\zeta\, d\zeta}{\theta-\zeta} = O(|\theta|^{-2}).$$

Hence (5.6) has L_1 norm $O(\beta)$ and we have shown that

$$\left\| \left(\frac{\psi'_+}{\psi}\right)' \right\|_1 = O(\beta).$$

This gives

$$\left\| \frac{\psi''_+}{\psi} \right\|_1 \leqslant \left\| \left(\frac{\psi'_+}{\psi}\right)^2 \right\|_1 + O(\beta) = \left\| \frac{\psi'_+}{\psi} \right\|_2^2 + O(\beta).$$

This is $O(\beta)$ since

$$\left\| \beta \frac{\sigma'(\beta\theta)}{\sigma(\beta\theta)} \right\|_2^2 \leqslant \int_{-\infty}^{\infty} \beta \left| \frac{\sigma'(\theta)}{\sigma(\theta)} \right|^2 d\theta.$$

Hence the second estimate of (5.2) is established.

For the last, the reader can easily check that introduction of the factor $1 - e^{i\theta}$ has the effect of removing the factor β from each of the immediately preceding estimates. This completes the proof of the sublemma.

Proof of Lemma 5.2. We show first that the hypotheses of the main lemma are satisfied by the pair ψ_+, ψ_-. The first is immediate since $\psi_\pm$ are the Wiener–Hopf factors of the positive C^∞ function ψ. For the second,

$$|\psi_\pm| = \psi^{1/2}$$

so the functions $\psi_\pm$ are uniformly bounded. Moreover

$$T(\psi_\pm)^{-1} = T(\psi_\pm^{-1})$$

so these are uniformly bounded and

$$T_N(\psi_+\psi_-)^{-1} = T_N(\psi)^{-1}$$

which is uniformly bounded since ψ is positive and bounded away from zero.

For the last notice that $B = B_N = 0$ so we need prove only that

$$\|A_N - A\|_1 \to 0. \tag{5.7}$$

If we apply ientity (3.5) to ψ_+ and to ψ_- and multiply the results we find that

$$T_N(\psi_-)^{-1} T_N(\psi_+)^{-1} = T(\psi_-)^{-1} T(\psi_+)^{-1} + U_N(I-P_N) T(\psi_-)^{-1} + V_N(I-P_N) T(\psi_+)^{-1},$$

where $\|U_N\| = O(1)$, $\|V_N\| = O(1)$. Since

$$A_N = T_N(\psi_-)^{-1} T_N(\psi_+)^{-1} P_N H(\psi_+) H(\psi^t_-) P_N,$$
$$A = T(\psi_-)^{-1} T(\psi_+)^{-1} H(\psi_+) H(\psi^t_-)$$

(5.7) will follow if we can show

$$\|(I-P_N) H(\psi_+) H(\psi^t_-)\|_1 \to 0, \qquad \|H(\psi_+) H(\psi^t_-)(I-P_N)\|_1 \to 0,$$
$$\|(I-P_N) T(\psi_\pm)^{-1} H(\psi_+) H(\psi^t_-)\|_1 \to 0. \tag{5.8}$$

First, consider the operator

$$(I-P_N) H(\psi_+) H(\psi^t_-). \tag{5.9}$$

By the sublemma,

$$H(\psi_+)_{i,j} = O(\log\beta(i+j+1)^{-1}), \qquad H(\psi^t_-)_{i,j} = O(\beta(i+j+1)^{-2}) \tag{5.10}$$

and a little computation gives

$$(H(\psi_+) H(\psi^t_-))_{i,j} = O(\beta\log\beta(i+j+1)^{-2}\log(i+j+1))$$
$$= O(\beta\log\beta(i+j+1)^{-2+\varepsilon}),$$

where $\varepsilon > 0$ is arbitrary. We also have from the sublemma

$$\Delta_j H(\psi^t_-)_{i,j} = O((i+j+1)^{-2})$$

and this gives

$$\Delta_j (H(\psi_+) H(\psi^t_-))_{i,j} = O(\log\beta(i+j+1)^{-2+\varepsilon}).$$

If we use the fact that β is at most a power of N we see by an easy computation that for the operator (5.9) we have, in the notation of Lemma 2.2,

$$p_n = O\left(\frac{\beta N^\varepsilon}{N+n}\right) = O(\beta n^{-1} N^\varepsilon), \qquad q_n = O\left(\frac{N^\varepsilon}{N+n}\right).$$

If in the statement of that lemma we choose

$$m = [n^{1/2}(N+n)^{1/2}\beta^{1/2}]$$

we obtain the estimate

$$s_n = O(n^{-1}(N+n)^{-1/2}\beta^{1/2}N^{\varepsilon})$$

for $n \geqslant 4$ and so

$$\sum_{n=4}^{\infty} s_n = O(N^{-1/2+\varepsilon}\beta^{1/2}). \tag{5.11}$$

Our hypothesis $N = O(\alpha^{3-\delta})$ for some $\delta > 0$ is equivalent to

$$\beta = O(N^{2/3-\delta}) \tag{5.12}$$

for some $\delta > 0$ and so certainly (5.11) is $O(1)$. Since also

$$\sum_{n=1}^{3} s_n = O(p_0) = O(\beta N^{-1+\varepsilon})$$

we have shown that the first assertion of (5.8) is true.

The second operator is the adjoint of the first so the second assertion follows from the first. The last is just a little different. Here the sublemma tells us that $T(\psi_\pm)^{-1}$, which is the same as $T(\psi_\pm^{-1})$, has i, j entry

$$O(\log\beta(|i-j|+1)^{-1}).$$

Combining this with the first estimate of (5.10) gives

$$(T(\psi_\pm)^{-1}H(\psi_+))_{i,j} = O(\log\beta(i+j+1)^{-1+\varepsilon})$$

and we proceed as before.

So (5.8), and with it the hypotheses of the main lemma for ψ_+, ψ_-, is verified. Since, as was already noted, (1.3) is an identity for Wiener–Hopf factors, we now know that (1.3) holds for ψ.

It remains only to show that the main lemma is applicable to the pair

$$e^{\lambda u}, \qquad \psi$$

since (1.3) holds for $e^{\lambda u}$ and we have just shown that it holds for ψ. Hypotheses (i) and (ii) are easily verified and it remains to check the three parts of (iii). For the first part (the second being entirely analogous), the sublemma gives the estimates

$$H(\psi')_{i,j} = O(\beta(i+j+1)^{-2}), \qquad \Delta_j H(\psi')_{i,j} = O((i+j+1)^{-1})$$

and trivially

$$H(e^{\lambda u})_{i,j} = O((i+j+1)^{-1}).$$

From these we obtain

$$\begin{aligned} (H(e^{\lambda u})\,H(\psi'))_{i,j} &= O(\beta(i+j+1)^{-2+\varepsilon}), \\ \Delta_j(H(e^{\lambda u})\,H(\psi'))_{i,j} &= O((i+j+1)^{-2+\varepsilon}). \end{aligned} \tag{5.13}$$

These are the same estimates we used above for the pair ψ_+, ψ_- (in fact they are better since $\log\beta$ does not appear) and so the analog of the first part of (5.8) holds in the same way. The other two are handled similarly.

It remains to verify that

$$\|P_N A Q_N B P_N\|_1 \to 0.$$

We shall show first that the entries of

$$A = T(\psi)^{-1}\,T(e^{\lambda u})^{-1}\,H(e^{\lambda u})\,H(\psi')$$

satisfy estimates very similar to those for $H(e^{\lambda u})\,H(\psi')$. We know that for the Wiener–Hopf factors $\psi_\pm$,

$$T(\psi_\pm^{-1})_{i,j} = O(\log\beta(|i-j|+1)^{-1})$$

and it follows that

$$T(\psi)_{i,j}^{-1} = (T(\psi_+^{-1})\,T(\psi_-^{-1}))_{i,j} = O(\log\beta)^2(|i-j|+1)^{-1+\varepsilon}). \tag{5.14}$$

As for $T(e^{\lambda u})^{-1}$ the Wiener–Hopf factorization is given by

$$e^{\lambda u(\theta)} = (1-e^{i\theta})^{i\lambda}(1-e^{-i\theta})^{-i\lambda}$$

with $|\arg(1-e^{\pm i\theta})| < \pi/2$ for $\theta \neq 0$. The Fourier coefficients of the inverses of these factors are

$$\frac{\Gamma(|k| \pm i\lambda)}{\Gamma(|k|+1)\,\Gamma(\pm i\lambda)} \sim \frac{|k|^{\pm i\lambda - 1}}{\Gamma(\pm i\lambda)} = O(|k|^{-1})$$

and so we can conclude that

$$T(e^{\lambda u})_{i,j}^{-1} = O((|i-j|+1)^{-1+\varepsilon}).$$

Combining this with (5.14) and the estimates (5.13) yields for the entries of A

$$A_{i,j} = O(\beta(\log\beta)^2(i+j+1)^{-2+\varepsilon}) = O(\beta N^\varepsilon(i+j+1)^{-2+\varepsilon})$$
$$\Delta_j A_{i,j} = O((\log\beta)^2(i+j+1)^{-2+\varepsilon}) = O(N^\varepsilon(i+j+1)^{-2+\varepsilon})$$

and the estimates for the entries of B are the same.

An easy computation then shows that for $i, j < N$

$$(\mathrm{A}Q_N B)_{i,j} = O(\beta^2 N^\varepsilon[(i+N)^{-2}(j+1)^{-1} + (i+1)^{-1}(j+N)^{-2}])$$

and there is a similar estimate for $\Delta_j(\mathrm{A}Q_N B)_{i,j}$ with β^2 replaced by β. From these we find easily

$$\|(\mathrm{A}Q_N B)_{i,j}\|_2 = O(\beta^2 N^{-3/2+\varepsilon}), \qquad \|\Delta_j(\mathrm{A}Q_N B)_{i,j}\|_2 = O(\beta^{-1}N^{-3/2+\varepsilon}).$$

We now use Lemma 2.1. It follows from Lemma 2.1(i) that for the operator $P_N A Q_N B P_N$ we have

$$s_n = O(\beta^2 n^{-1/2} N^{-3/2+\varepsilon})$$

and from (iii) that for $n \geqslant 1$

$$s_n = O(\beta n^{-3/2} N^{-1/2+\varepsilon}).$$

If we use the first estimate for $n < N/\beta$ and the second for $n \geqslant N/\beta$ we obtain

$$\|P_N A Q_N B P_N\|_1 = O(\beta^{3/2} N^{-1+\varepsilon})$$

and this is $o(1)$ because of (5.12). This completes the proof of Lemma 5.2.

Proof of Lemma 5.3. This is a routine and not unpleasant computation. First,

$$G(\phi)^N = \exp\left\{\frac{N}{2\pi}\int \log\sigma(\beta\theta)\, d\theta\right\} = \exp\left\{\frac{N/\beta}{2\pi}\int \log\sigma(\theta)\, d\theta\right\} = G(\sigma)^\alpha$$

exactly. Next, since

$$(\log\phi)_k = \beta^{-1}(\log\sigma)\hat{}\,(\beta^{-1}k)$$

and since $\log\sigma$ is real, we have

$$\log E(\phi) = \sum_{k=1}^{\infty} \{k\beta^{-2}\,|(\log\sigma)\hat{}\,(\beta^{-1}k)|^2 - k^{-1}\lambda^2\}. \tag{5.15}$$

Now

$$|(\log \sigma)\hat{}(z)|^2 = z^{-2}\lambda^2 + O(z^{-3})$$

as $z \to \infty$ and from this it is easy to verify that the Riemann sums for

$$\int_1^\infty \{z\,|(\log \sigma)\hat{}(z)|^2 - z^{-1}\lambda^2\}\,dz$$

with subinterval length β^{-1} converge to this integral as $\beta \to \infty$. The part of the sum of the series in (5.15) corresponding to $k \geqslant \beta$ is just such a Riemann sum. Thus

$$\sum_{k \geqslant \beta} \to \int_1^\infty \{z\,|(\log \sigma)\hat{}(z)|^2 - z^{-1}\lambda^2\}\,dz. \tag{5.16}$$

As for the rest of the terms we have analogously

$$\sum_{k<\beta} k\beta^{-2}\,|(\log \sigma)\hat{}(\beta^{-1}k)|^2 \to \int_0^1 z\,|(\log \sigma)\hat{}(z)|^2\,dz,$$

and also

$$-\sum_{k<\beta} k^{-1}\lambda^2 = -(\log \beta + \gamma + o(1))\,\lambda^2.$$

Thus

$$\sum_{k<\beta} = \int_0^1 z\,|(\log \sigma)\hat{}(z)|^2\,dz - (\log \beta + \gamma + o(1))\,\lambda^2.$$

If we combine this with (5.16) and use $\beta = N/\alpha$ we see that we have shown

$$\log \left(\frac{N}{\alpha}\right)^{\lambda^2} E(\phi) = \int_1^\infty \{z\,|(\log \sigma)\hat{}(z)|^2 - z^{-1}\lambda^2\}\,dz$$
$$+ \int_0^1 z\,|(\log \sigma)\hat{}(z)|^2\,dz - \gamma\lambda^2 + o(1).$$

The representation

$$\gamma = \int_0^1 \frac{1-e^{-z}}{z}\,dz - \int_1^\infty \frac{e^{-z}}{z}\,dz$$

shows that the right side equals

$$\int_0^\infty \left\{z\,|(\log \sigma)\hat{}(z)|^2 - \frac{1-e^{-z}}{z}\,\gamma^2\right\} dz + o(1) = \log E(\sigma) + o(1).$$

This completes the proof of Lemma 5.3 and so of Theorem II.

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