BROWNIAN MOTION

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Brownian motion

- Brownian motion plays a central role in
  - probability theory,
  - the theory of stochastic processes,
  - physics,
  - economics and finance,
  - . . .

Definition of Brownian motion

- A stochastic process $W = (W_t)_{t \geq 0}$ is called (***standard***) ***Brownian motion*** or a ***Wiener process*** if the following conditions are satisfied:
  - $W_0 = 0$ (the process **starts at zero**);
  - it has **stationary, independent increments**;
  - for every $t > 0$, W_t has a **Gaussian** $\mathcal{N}(0, t)$ **distribution**;
  - it has **continuous sample paths** (no “jumps”).

Figure 1.3.1

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A brief history of Brownian motion

- Brownian motion is named after the biologist Robert **Brown** whose research dates to the 1820s.
- Early in the last century, **Louis Bachelier** (1900), Albert Einstein (1905) and Norbert Wiener (1923) began developing the mathematical theory of Brownian motion.
- The construction of Bachelier (1900) was erroneous but captured many of the essential properties of the process. Wiener (1923) was the first to put Brownian motion on a firm mathematical basis.

Distribution a of Brownian motion

- The finite-dimensional distributions of Brownian motion are **multivariate Gaussian**;

➥ **W is a Gaussian process.**

- Brownian motion has independent Gaussian increments.

Distribution a of Brownian motion

- The random variables

$$W_t - W_s \qquad \text{and} \qquad W_{t-s}$$

have a Gaussian $\mathcal{N}(0,\, t-s)$ distribution for $s < t$.

➥ This follows from the stationarity of the increments.

➥ $W_t - W_s$ has the same distribution as $W_{t-s} - W_0 = W_{t-s}$ which is normal with zero mean and variance $t - s$.

Distribution a of Brownian motion

- The variance of $W_t - W_s$ is proportional to the length of the interval $[s, t]$.

➥ The larger the interval, the larger the fluctuations of Brownian motion on this interval.

Figure 1.3.2

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Remark

- The distribution identity

$$W_t - W_s \stackrel{d}{=} W_{t-s}$$

does not imply pathwise identity: in general,

$$W_t(\omega) - W_s(\omega) \neq W_{t-s}(\omega)\,.$$

Brownian motion vs. Poisson process

- The definitions of Brownian motion and the Poisson process coincide insofar that they are processes with stationary, independent increments.
- The crucial difference is the kind of distribution of the increments.
- The requirement of the Poisson distribution makes the sample path pure jumps functions, whereas the Gaussian assumption makes the sample paths continuous.

Expectation and covariance functions of Brownian motion

- The Brownian motion has **expectation function**

$$\mu_W(t) = \mathbb{E}(W_t) = 0 \qquad t \geq 0\,.$$

- Since the increments $W_s - W_0 = W_s$ and $W_t - W_s$ are independent for $s < t$, it has **covariance function**

$$\begin{aligned} cov_W(s,t) &= \mathbb{E}\big[[(W_t - W_s) + W_s]\,W_s\big] \\ &= \mathbb{E}\big[(W_t - W_s)\,W_s\big] + \mathbb{E}\big[(W_s)^2\big] \qquad\qquad 0 \leq s < t \\ &= \mathbb{E}(W_t - W_s)\,\mathbb{E}(W_s) + s \\ &= 0 + s = s\,. \end{aligned}$$

An alternative definition of Brownian motion

- Since a Gaussian process is characterized by its expectation and covariance functions, we can give an **alternative definition.**
- Brownian motion is a Gaussian process with

$$\mu_W(t) = \mathbb{E}(W_t) = 0$$

and

$$cov_W(s,t) = \min(s,t)\,.$$

➥ What can we say about variance?

Path properties

- In what follows, **we fix one sample path** $W_t(\omega)$, $t \geq 0$ (which is a function of t), and consider its properties.

- From the definition of Brownian motion, we know that **its sample paths are continuous.**

- **Brownian paths are extremely irregular**: they oscillate wildly.

  - The main reason is that the increments of W are independent. In particular, increments of Brownian motion on adjacent intervals are independent whatever the length of the intervals.

  - The continuity of the paths is rather surprisingly!

Question

How irregular is a Brownian sample path?

- In order to answer this question, we need to introduce a class of stochastic processes which contains Brownian motion as a special case.

Self-similar stochastic processes

- A stochastic process $X = (X_t)_{t \geq 0}$ is H***-self-similar***, for some $H > 0$, if its finite-dimensional distributions satisfy the condition

$$\left(T^H X_{t_1},\, \ldots,\, T^H X_{t_n}\right) \stackrel{d}{=} \left(X_{T\,t_1},\, \ldots,\, X_{T\,t_n}\right), \qquad (1)$$

  for every $T > 0$, any choice of $t_i \geq 0$, $i = 1, \ldots, n$, and $n \geq 1$.

Remark

- Note that self-similarity is a distributional, not a pathwise property.
- In (1), one must not replace "$\stackrel{d}{=}$" with "$=$".
- Roughly speaking, self-similarity means that the properly scaled patterns of a sample path in any small or large time interval have ***similar shape***, but they are ***not identical.***

Non-Differentiability

- The sample paths of a self-similar process are nowhere differentiable.

Non-Differentiability

- Brownian motion is $\frac{1}{2}$-***self-similar***:

$$\left(T^{1/2}W_{t_1},\,\ldots,\,T^{1/2}W_{t_n}\right) \stackrel{d}{=} \left(W_{T\,t_1},\,\ldots,\,W_{T\,t_n}\right), \qquad (2)$$

for every $T > 0$, any choice of $t_i \geq 0$, $i = 1, \ldots, n$, and $n \geq 1$.

➥ Its **sample paths** are **nowhere differentiable.**

Self-similarity

- The left-hand side of (2) are Gaussian vectors, therefore in order to check the distributional identity it suffices to verify that they have the same expectation and covariance matrix.

Differentiability of a function

The graph of a differentiable function is ***smooth***.

If the limit

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists and is finite for some $x_0 \in (0, t)$, say, then we may write for small Δx

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + h(x_0, \Delta x)\Delta x\,,$$

where $h(x_0, \Delta x) \to 0$ as $\Delta x \to 0$.

Differentiability of a function

Hence, in a small neighborhood of x_0, the function f is roughly linear (as a function of Δx). This explains its smoothness.

Alternatively, differentiability of f at x_0 implies that there exists a unique tangent to the curve of the function at this point.

Figure 1.3.3

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Question

How is the graph of a nowhere differentiable function?

- The graph of such a function changes its shape in the neighborhood of any point in a completely non-predictable manner.
- It is very difficult to imagine a nowhere differentiable function, nevertheless, Brownian motion is considered as a very good approximation to many real-life phenomena.

Self-Similarity

- The self-similarity property of Brownian motion has an interesting consequence for the simulation of its sample paths.
- In order to simulate a path on $[0, T]$ it suffices to simulate one path on $[0, 1]$, then scale the time interval by the factor T and the sample path by the factor $T^{1/2}$.

Non-Differentiability

The existence of nowhere differentiable continuous functions was discovered in the 19th century.

One such function was constructed by Weierstrass. It was considered as a curiosity, far away from any practical application.

Brownian motion is a process with nowhere differentiable sample paths. Currently it is considered as one of those processes which have a multitude of applications in very different fields.

- One of them is stochastic calculus and it is used, in particular, in mathematical finance.

Unbounded Variation

A further indication of the irregularity of Brownian sample paths is the concept of unbounded variation.

- Brownian sample paths do not have bounded variation on any finite interval $[0, T]$.
- This means that

$$\sup_{\tau} \sum_{i=1}^{n} \left| W_{t_i}(\omega) - W_{t_{i-1}}(\omega) \right| = \infty$$

where the *supremum* is taken over all possible partitions $\tau \ : \ 0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$.

Supremum

The notation

$$\sup_n a_n = a$$

means the *supremum* of a sequence of real numbers a_n.

If $a \in \mathbb{R}$, $a \geq a_n$ for all n and for every $\varepsilon > 0$ there exists a k such that $a - \varepsilon < a_k$.

If $a = \infty$, then for every $M > 0$ there exists a k such that $a_k > M$.

For a finite index set I,

$$\sup_{n \in I} a_n = \max_{n \in I} a_n \, .$$

Observation

- The unbounded variation and non-differentiability of Brownian motion sample paths are major reasons for the failure of classical integration methods, when applied to these paths, and for the introduction of stochastic calculus.

Processes derived from Brownian motion

- Various Gaussian and non-Gaussian stochastic processes of practical relevance can be derived from Brownian motion.
- We introduce some of those processes which will find further interesting applications in finance.
- As before, $W = (W_t)_{t \geq 0}$ denotes Brownian motion.

Brownian bridge

- Consider the process

$$X_t = W_t - t\,W_1 \qquad 0 \le t \le 1\,.$$

- Obviously,

$$X_0 = W_0 - 0 \cdot W_1 = 0$$

and

$$X_1 = W_1 - 1 \cdot W_1 = 0\,.$$

➥ For this simple reason, the process X is called (**standard**) ***Brownian bridge*** or ***tied down Brownian motion***.

Figure 1.3.4

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Brownian bridge

- Using the formula for linear transformations of Gaussian random vectors, one can show that the finite-dimensional distributions of X are Gaussian.

  (Exercise)

Linear transformation of Gaussian random vectors

Let $\mathbf{X} = (X_1, \ldots, X_n)$ have a Gaussian $\mathcal{N}(\mu, \Sigma)$ distribution and A be an $m \times n$ matrix.

Then $A\,\mathbf{X}'$ has a Gaussian $\mathcal{N}(A\,\mu', A\,\Sigma\,A')$.

Note that $\mu \in \mathbb{R}^n$ and Σ is an $n \times n$ matrix.

Distribution, Expectation and Covariance of the Brownian bridge

- The Brownian bridge X is a Gaussian process.
- One can easily calculate the expectation and covariance functions of the Brownian bridge:

$$\mu_X(t) = \mathbb{E}(X_t) = 0 \qquad t \in [0,1]$$

and

$$cov_X(s,t) = \min(s,t) - s \cdot t \qquad s,t \in [0,1]\,.$$

- Since X is Gaussian, the Brownian bridge is characterized by these two functions.

Brownian motion with drift

- Let us consider the process

$$X_t = \mu\, t + \sigma\, W_t \qquad t \geq 0\,,$$

  for constants $\sigma > 0$ and $\mu \in \mathbb{R}$.

- X is a Gaussian process, with expectation and covariance

$$\mu_X(t) = \mathbb{E}(X_t) = \mu\, t \qquad t \geq 0$$

  and

$$cov_X(s,t) = \sigma^2 \min(s,t) \qquad s,t \geq 0\,,$$

  respectively.

The *drift*

- The expectation function $\mu_X(t) = \mu\, t$ essentially determines the characteristic shape of the sample paths.
- The term $\mu\, t$ is the **deterministic *drift*** of the process X.
- Therefore X is called ***Brownian motion with (linear) drift***.

Figure 1.3.6

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Prices of risky assets

With the fundamental discovery of Bachelier (1900) that prices of risky assets (stock indices, exchange rates, share prices, etc.) can be well described by Brownian motion, a new area of applications of stochastic processes was born.

However, Brownian motion, as a Gaussian process, may assume negative values, which is not a very desirable property of a price.

Geometric Brownian motion

- Consider the process

$$X_t = e^{\mu\, t + \sigma W_t} \qquad t \geq 0\,,$$

  i.e. it is the exponential of Brownian motion with drift.

- The process X is called ***geometric Brownian motion***.

➥ X is not a Gaussian process.

Expectation of a lognormal random variable

- Let $Z \sim \mathcal{N}(0,1)$, then

$$\mathbb{E}\left[e^{\lambda Z}\right] = e^{\lambda^2/2} \qquad \lambda \in \mathbb{R}\,. \tag{3}$$

Proof

$$\mathbb{E}\left[e^{\lambda Z}\right] = \frac{1}{(2\,\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{\lambda\, z} e^{-z^2/2}\, dz$$
$$= e^{\lambda^2/2} \frac{1}{(2\,\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{-(z-\lambda)^2/2}\, dz$$
$$= e^{\lambda^2/2}\,.$$

Here we used the fact that $\frac{1}{(2\,\pi)^{1/2}} e^{-(z-\lambda)^2}$ is the density of a Gaussian $\mathcal{N}(\lambda, 1)$ random variable.

Expectation function of geometric Brownian motion

- From (3) and the self-similarity of Brownian motion, it follows that

$$\begin{aligned} \mu_X(t) &= e^{\mu\, t}\, \mathbb{E}\left[e^{\sigma\, W_t}\right] \\ &= e^{\mu\, t}\, \mathbb{E}\left[e^{\sigma t^{1/2} W_1}\right] \\ &= e^{(\mu + 1/2\sigma^2)t}\,. \end{aligned} \tag{4}$$

Covariance function of geometric Brownian motion

- For $s \leq t$, $W_t - W_s$ and W_s are independent, and $W_t - W_s \stackrel{d}{=} W_{t-s}$.
- Hence

$$cov_X(s,t) = e^{(\mu + 1/2\,\sigma^2)(s+t)} \left(e^{\sigma^2 s} - 1 \right) . \qquad (5)$$

Proof

$$
\begin{aligned}
cov_X(s,t) &= \mathbb{E}\left[X_s X_t\right] - \mathbb{E}\left[X_s\right]\mathbb{E}\left[X_t\right] \\
&= e^{\mu(s+t)}\mathbb{E}\left[e^{\sigma(W_s+W_t)}\right] - e^{(\mu+1/2\,\sigma^2)(s+t)} \\
&= e^{\mu(s+t)}\mathbb{E}\left[e^{\sigma[(W_t-W_s)+2W_s]}\right] - e^{(\mu+1/2\,\sigma^2)(s+t)} \\
&= e^{\mu(s+t)}\mathbb{E}\left[e^{\sigma(W_t-W_s)}\right]\mathbb{E}\left[e^{2\sigma W_s}\right] - e^{(\mu+1/2\,\sigma^2)(s+t)} \\
&= e^{(\mu+1/2\,\sigma^2)(s+t)}\left(e^{\sigma^2 s} - 1\right) .
\end{aligned}
$$

Variance function of geometric Brownian motion

- In particular, geometric Brownian motion has variance function

$$\sigma_X^2(t) = cov_X(t,t) = e^{(2\mu+\sigma^2)t}\left(e^{\sigma^2 t} - 1\right) . \tag{6}$$

Figure 1.3.9

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