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Differential and difference operators having orthogonal polynomials with two linear perturbations as eigenfunctions

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Abstract

In this paper we consider the polynomials $\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty}$, orthogonal with respect to a certain symmetric bilinear form of Sobolev type. These polynomials are the result of two linear perturbations to the orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$, eigenfunctions of a linear differential or difference operator $\mathbf{L}$. We show that the polynomials $\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty}$ are eigenfunctions of one or more linear differential or difference operators (possibly of infinite order) of the form $\mathbf{L}+\mu\mathbf{A}+\nu\mathbf{B}+\mu\nu\mathbf{C}$.

Keywords: Differential operators; Difference operators; Orthogonal polynomials

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1. Introduction

Let $\{P_n(x)\}_{n=0}^{\infty}$ with $\deg[P_n(x)]=n$ be a system of orthogonal polynomials relative to a positive-definite moment functional σ and let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of real numbers with $\lambda_0=0$ and $\{\lambda_n\}_{n=1}^{\infty}$ not all equal to zero such that $\{P_n(x)\}_{n=0}^{\infty}$ are eigenfunctions of a linear differential or difference operator $\mathbf{L}$ of the form

$$\mathbf{L}:=\sum_{i=1}^{\infty} e_i(x)\mathfrak{D}_x^i \tag{1}$$

with eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. Here $\{e_i(x)\}_{i=1}^{\infty}$ is a sequence of polynomials with $\deg[e_i(x)]\leqslant i$ for all $i\in\{1,2,3,\ldots\}$. $\mathfrak{D}_x y(x)$ may be read as the derivative $\mathbf{D}y(x)=\mathrm{d}y(x)/\mathrm{d}x$, the forward difference $\Delta y(x)=y(x+1)-y(x)$ or backward difference $\nabla y(x)=y(x)-y(x-1)$ and $\mathfrak{D}_x^i y(x)=\mathfrak{D}_x(\mathfrak{D}_x^{i-1}y(x))$, $\mathscr{D}_x^0 y(x)=y(x)$. Let ϕ be the symmetric bilinear form of Sobolev type defined by

$$\phi(p,q)=\langle\sigma,pq\rangle+\mu p^{(l_1)}(c_1)q^{(l_1)}(c_1),$$

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where $\mu>0$ and c_1 are real constants, $l_1 \in \{0,1,2,\dots\}$, p and q are any polynomials and the notation

$$p^{(l_1)}(x)=\mathfrak{D}_x^{l_1}p(x)$$

is used. In [3, Section 4.1] (see also [6]) it was shown that if $P_n^{(l_1)}(c_1)\neq 0$ for all $n\in\{l_1,l_1+1,l_1+2,\dots\}$, then the corresponding orthogonal polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$, which we call linear perturbations of $\{P_n(x)\}_{n=0}^{\infty}$ of the class l_1 at c_1 with parameter μ, are eigenfunctions of one (or more if $l_1>0$) linear operators of the form $\mathbf{L}+\mu\mathbf{A}$ with

$$\mathbf{A}:=\sum_{i=1}^{\infty}a_i(x)\mathfrak{D}_x^i \tag{2}$$

and eigenvalues $\{\lambda_n+\mu\alpha_n\}_{n=0}^{\infty}$. Here $\{a_i(x)\}_{i=1}^{\infty}$ is a sequence of polynomials with $\deg[a_i(x)]\leqslant i$ for all $i\in\{1,2,3,\dots\}$, $\alpha_0=0$ and (if $l_1>0$) the numbers $\{\alpha_n\}_{n=1}^{l_1}$ can be chosen arbitrarily. The operator $\mathbf{A}$ and the numbers $\{\alpha_n\}_{n=l_1+1}^{\infty}$ are uniquely determined, when $\{\alpha_n\}_{n=1}^{l_1}$ are chosen.

In this paper we consider polynomials $\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty}$, orthogonal with respect to the symmetric bilinear form of Sobolev type defined by

$$\psi(p,q)=\langle\sigma,pq\rangle+\mu p^{(l_1)}(c_1)q^{(l_1)}(c_1)+\nu p^{(l_2)}(c_2)q^{(l_2)}(c_2), \tag{3}$$

where $\mu>0,\nu>0$, c_1 and c_2 are real constants, l_1 and l_2 are nonnegative integers, and p and q are any polynomials. We will show that if $P_n^{(l_1)}(c_1)\neq 0$ for all $n\in\{l_1,l_1+1,l_1+2,\dots\}$ and $P_n^{(l_2)}(c_2)\neq 0$ for all $n\in\{l_2,l_2+1,l_2+2,\dots\}$ and some other conditions are satisfied, the polynomials $\{P_n^{\mu,\nu}(x)\}_{n=0}^{\infty}$ are eigenfunctions of one (or more if $\min(l_1,l_2)>0$) linear differential or difference operators of form

$$\mathbf{L}+\mu\mathbf{A}+\nu\mathbf{B}+\mu\nu\mathbf{C} \tag{4}$$

with eigenvalues

$$\{\lambda_n+\mu\alpha_n+\nu\beta_n+\mu\nu\gamma_n\}_{n=0}^{\infty}. \tag{5}$$

Here the operators $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ have similar forms as (2), $\alpha_0=\beta_0=\gamma_0=0$ and the numbers $\{\alpha_n\}_{n=1}^{l_1}$ (if $l_1>0$), $\{\beta_n\}_{n=1}^{l_2}$ (if $l_2>0$) and $\{\gamma_n\}_{n=1}^{\min\{l_1,l_2\}}$ (if $\min\{l_1,l_2\}>0$) can be chosen arbitrarily. The operators $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ and the numbers $\{\alpha_n\}_{n=l_1+1}^{\infty}$, $\{\beta_n\}_{n=l_2+1}^{\infty}$ and $\{\gamma_n\}_{n=\min\{l_1,l_2\}+1}^{\infty}$ are uniquely determined, when $\{\alpha_n\}_{n=1}^{l_1}$, $\{\beta_n\}_{n=1}^{l_2}$ and $\{\gamma_n\}_{n=1}^{\min\{l_1,l_2\}}$ are fixed.

In the last part of the paper we will give applications of the results in some concrete situations, where the polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are classical orthogonal polynomials, eigenfunctions of a second-order differential or difference operator $\mathbf{L}$.

2. Orthogonal polynomials with two linear perturbations

We construct polynomials orthogonal with respect to (3) by adding to $\{P_n(x)\}_{n=0}^{\infty}$ successively a linear perturbation of the class l_1 at c_1 with parameter μ and a linear perturbation of the class l_2 at c_2 with parameter ν. If we write

$$K_n^{(r,s)}(x,y)=\sum_{i=0}^{n}\frac{P_i^{(r)}(x)P_i^{(s)}(y)}{\langle\sigma,P_i^2\rangle},\quad n,r,s\in\{0,1,2,\dots\},$$

then (see [1, 2]) the polynomials $\{P_n^\mu(x)\}_{n=0}^\infty$ can be written as

$$P_n^\mu(x) := P_n(x) + \mu Q_n(x) = (1 + \mu K_{n-1}^{(l_1,l_1)}(c_1,c_1))P_n(x) - \mu P_n^{(l_1)}(c_1)K_{n-1}^{(0,l_1)}(x,c_1).$$

For the kernel

$$G_n(x,y;\mu) = \sum_{i=0}^{n} \frac{P_i^\mu(x)P_i^\mu(y)}{\phi(P_i^\mu,P_i^\mu)}, \tag{6}$$

by using Proposition 3.2 in [6], the following formula is derived

$$G_n^{(r,s)}(x,y;\mu) = K_n^{(r,s)}(x,y) - \mu\frac{K_n^{(r,l_1)}(x,c_1)K_n^{(l_1,s)}(c_1,y)}{1+\mu K_n^{(l_1,l_1)}(c_1,c_1)}, \quad n,r,s \in \{0,1,2,\ldots\}. \tag{7}$$

Applying the second perturbation we obtain

$$\begin{aligned}
P_n^{\mu,\nu}(x) &= P_n^\mu(x) + \nu Q_n^*(x;\mu) \\
&= (1+\nu G_{n-1}^{(l_2,l_2)}(c_2,c_2;\mu))P_n^\mu(x) - \nu P_n^{\mu(l_2)}(c_2)G_{n-1}^{(0,l_2)}(x,c_2;\mu) \\
&= \left[1 + \nu K_{n-1}^{(l_2,l_2)}(c_2,c_2) - \mu\nu\frac{(K_{n-1}^{(l_1,l_2)}(c_1,c_2))^2}{1+\mu K_{n-1}^{(l_1,l_1)}(c_1,c_1)}\right] \\
&\quad \times[(1+\mu K_{n-1}^{(l_1,l_1)}(c_1,c_1))P_n(x) - \mu P_n^{(l_1)}(c_1)K_{n-1}^{(0,l_1)}(x,c_1)] \\
&\quad -\nu[(1+\mu K_{n-1}^{(l_1,l_1)}(c_1,c_1))P_n^{(l_2)}(c_2) - \mu P_n^{(l_1)}(c_1)K_{n-1}^{(l_1,l_2)}(c_1,c_2)] \\
&\quad \times\left[K_{n-1}^{(0,l_2)}(x,c_2) - \mu\frac{K_{n-1}^{(0,l_1)}(x,c_1)K_{n-1}^{(l_1 l_2)}(c_1,c_2)}{1+\mu K_{n-1}^{(l_1,l_1)}(c_1,c_1)}\right].
\end{aligned}$$

Hence

$$\begin{aligned}
P_n^{\mu,\nu}(x) &= P_n(x) + \mu\begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_1)}(x,c_1) \\ P_n^{(l_1)}(c_1) & K_{n-1}^{(l_1,l_1)}(c_1,c_1)\end{vmatrix} + \nu\begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_2)}(x,c_2) \\ P_n^{(l_2)}(c_2) & K_{n-1}^{(l_2,l_2)}(c_2,c_2)\end{vmatrix} \\
&\quad +\mu\nu\begin{vmatrix} P_n(x) & K_{n-1}^{(0,l_1)}(x,c_1) & K_{n-1}^{(0,l_2)}(x,c_2) \\ P_n^{(l_1)}(c_1) & K_{n-1}^{(l_1,l_1)}(c_1,c_1) & K_{n-1}^{(l_1,l_2)}(c_1,c_2) \\ P_n^{(l_2)}(c_2) & K_{n-1}^{(l_2,l_1)}(c_2,c_1) & K_{n-1}^{(l_2,l_2)}(c_2,c_2)\end{vmatrix} \\
&:= P_n(x) + \mu Q_n(x) + \nu R_n(x) + \mu\nu S_n(x), \quad n \in \{0,1,2,\ldots\}.
\end{aligned}$$

Note that $Q_n(x)=0$ for all $n\in\{1,2,\ldots,l_1\}$, $R_n(x)=0$ for all $n\in\{1,2,\ldots,l_2\}$ and $S_n(x)=0$ for all $n\in\{1,2,\ldots\max\{l_1,l_2\}\}$. If we assume that $P_{l_1}^{(l_1)}(c_1)\neq 0$, then we have $\deg[Q_n(x)]=n$ for all $n\in\{l_1+1,l_1+2,l_1+3,\ldots\}$ and if we assume that $P_{l_2}^{(l_2)}(c_2)\neq 0$, then we have $\deg[R_n(x)]=n$ for all $n\in\{l_2+1,l_2+2,l_2+3,\ldots\}$. Moreover, by the Cauchy–Schwarz inequality, it follows that $\deg[S_n(x)]=n$ for all $n\in\{\max\{l_1,l_2\}+1,\max\{l_1,l_2\}+2,\ldots\}$ unless $l_1=l_2$. If $l_1=l_2$, then $S_{l_1+1}(x)=0$ and $\deg[S_n(x)]=n$ for all $n\in\{l_1+2,l_1+3,l_1+4,\ldots\}$ unless $c_1=c_2$. In the following we will assume that $(c_1,l_1)\neq(c_2,l_2)$.

3. The operators

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of orthogonal polynomials relative to a positive-definite moment functional σ and let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of real numbers with $\lambda_0=0$ and $\{\lambda_n\}_{n=1}^{\infty}$ not all equal to zero such that $\{P_n(x)\}_{n=0}^{\infty}$ are eigenfunctions of a linear operator $\mathbf{L}$ of the form (1) with eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. We will construct operators, linear in μ and v, for which the polynomials $P_n^{\mu,v}(x)$, found in the preceding section, are eigenfunctions and with a sequence of eigenvalues of the form (5). The general idea of this construction is as follows. By applying the (c_1,l_1) perturbation first and the (c_2,l_2) perturbation afterwards we obtain a set V of sequences of eigenvalues of the form (5), where each element of V corresponds to a linear differential operator, depending linearly on v. In the second construction we apply the (c_2,l_2) perturbation first and the (c_1,l_1) perturbation afterwards. We then obtain a set W of sequences of eigenvalues of the form (5), where each element of W corresponds to a linear differential operator, depending linearly on μ. By Lemma 1 in [3] it is clear that a sequence of polynomials and a sequence of eigenvalues uniquely determine a linear differential operator, possibly of infinite order. Thus to each element of $V\cap W$ corresponds a linear differential operator of the form (4). Since we are only interested in elements of $V\cap W$, we will in the cases that in both constructions certain values of α_n,β_n or γ_n can be chosen arbitrarily, take them to be the same in both constructions. Without loss of the generality we may assume that

$$l_1\leqslant l_2.$$

3.1. *The first construction*

In [3, Section 4.1] it was shown that if

$$P_n^{(l_1)}(c_1)\neq 0 \quad \text{for all } n\in\{l_1,l_1+1,l_1+2,\ldots\} \tag{8}$$

holds, then the polynomials $\{P_n(x)+\mu Q_n(x)\}_{n=0}^{\infty}$, orthogonal with respect to

$$\phi(p,q)=\langle\sigma,pq\rangle+\mu p^{(l_1)}(c_1)q^{(l_1)}(c_1),$$

are eigenfunctions of an operator $\mathbf{L}+\mu\mathbf{A}$ with eigenvalues $\{\lambda_n+\mu\alpha_n\}_{n=0}^{\infty}$, where $\alpha_0=0,\{\alpha_n\}_{n=1}^{l_1}$ are arbitrary (if $l_1>0$) and for $n>l_1$

$$\alpha_n=\alpha_{l_1}+\sum_{j=l_1+1}^{n}(\lambda_j-\lambda_{j-1})K_{j-1}^{(l_1,l_1)}(c_1,c_1). \tag{9}$$

Similarly, if

$$P_n^{(l_2)}(c_2)+\mu Q_n^{(l_2)}(c_2)\neq 0 \quad \text{for all } n\in\{l_2,l_2+1,l_2+2,\ldots\} \tag{10}$$

holds, then the polynomials

$$P_n^{\mu,v}(x)=(P_n(x)+\mu Q_n(x))+v(R_n(x)+\mu S_n(x)),\quad n\in\{0,1,2,\ldots\},$$

orthogonal with respect to (3) are eigenfunctions of an operator $(\mathbf{L}+\mu\mathbf{A})+v\mathbf{B}(\mu)$ with eigenvalues $\{\lambda_n+\mu\alpha_n+v\beta_n(\mu)\}_{n=0}^{\infty}$, where $\beta_0(\mu)=0,\{\beta_n(\mu)\}_{n=1}^{l_2}$ can be chosen arbitrarily (if $l_2>0$), and for $n > l_2$

$$\beta_n(\mu) = \beta_{l_2}(\mu) + \sum_{j=l_2+1}^{n} (\lambda_j + \mu\alpha_j - \lambda_{j-1} - \mu\alpha_{j-1}) G_{j-1}^{(l_2,l_2)}(c_2, c_2; \mu),$$

where $G_n(x, y; \mu)$ is given by (6). We choose $\beta_j(\mu) = \beta_j + \mu\gamma_j$ for $j \in \{1, 2, \ldots, l_2\}$ (if $l_2 > 0$), hence linear in μ. We find for $n > l_2$

$$\beta_n(\mu) = \beta_{l_2} + \mu\gamma_{l_2} + \sum_{j=l_2+1}^{n} (\lambda_j - \lambda_{j-1}) K_{j-1}^{(l_2,l_2)}(c_2, c_2)$$
$$+\mu \sum_{j=l_2+1}^{n} (\lambda_j - \lambda_{j-1}) [K_{j-1}^{(l_1,l_1)}(c_1, c_1) K_{j-1}^{(l_2,l_2)}(c_2, c_2) - (K_{j-1}^{(l_1,l_2)}(c_1, c_2))^2].$$

It follows that the eigenvalues of the operator $\mathbf{L} + \mu\mathbf{A} + v\mathbf{B}(\mu)$, which is linear in v, can be written as

$$\lambda_n + \mu\alpha_n + v\beta_n + \mu v\gamma_n,$$

where $\{\alpha_n\}_{n=1}^{l_1}$ are arbitrary (if $l_1 > 0$), α_n is given by (9) for $n > l_1$, $\{\beta_n\}_{n=1}^{l_2}$ are arbitrary (if $l_2 > 0$) for $n > l_2$,

$$\beta_n = \beta_{l_2} + \sum_{j=l_2+1}^{n} (\lambda_j - \lambda_{j-1}) K_{j-1}^{(l_2,l_2)}(c_2, c_2), \tag{11}$$

$\{\gamma_n\}_{n=1}^{l_2}$ are arbitrary (if $l_2 > 0$), and for $n > l_2$

$$\gamma_n = \gamma_{l_2} + \sum_{j=l_2+1}^{n} (\lambda_j - \lambda_{j-1}) [K_{j-1}^{(l_1,l_1)}(c_1, c_1) K_{j-1}^{(l_2,l_2)}(c_2, c_2) - (K_{j-1}^{(l_1,l_2)}(c_1, c_2))^2]. \tag{12}$$

3.2. The second construction

If

$$P_n^{(l_2)}(c_2) \neq 0 \quad \text{for all } n \in \{l_2, l_2 + 1, l_2 + 2, \ldots\} \tag{13}$$

holds, then the polynomials $\{P_n(x) + vR_n(x)\}_{n=0}^{\infty}$, orthogonal with respect to

$$\phi^*(p, q) = \langle \sigma, pq \rangle + vp^{(l_2)}(c_2) q^{(l_2)}(c_2),$$

are eigenfunctions of an operator $\mathbf{L} + v\mathbf{B}^*$ with eigenvalues $\{\lambda_n + v\beta_n\}_{n=0}^{\infty}$, where $\beta_0 = 0$, $\{\beta_n\}_{n=1}^{l_2}$ are arbitrary (if $l_2 > 0$) and for $n > l_2$ we have (11). Similarly if

$$P_n^{(l_1)}(c_1) + vR_n^{(l_1)}(c_1) \neq 0 \quad \text{for all } n \in \{l_1, l_1 + 1, l_1 + 2, \ldots\} \tag{14}$$

holds, then the polynomials

$$P_n^{\mu,v}(x) = (P_n(x) + vR_n(x)) + \mu(Q_n(x) + vS_n(x)), \quad n \in \{0, 1, 2, \ldots\},$$

orthogonal with respect to (3), are eigenfunctions of an operator $(\mathbf{L}+v\mathbf{B}^*)+\mu\mathbf{A}^*(v)$ with eigenvalues $\{\lambda_n+v\beta_n+\mu\alpha_n^*(v)\}_{n=0}^{\infty}$, where $\alpha_0^*(v)=0, \{\alpha_n^*(v)\}_{n=1}^{l_1}$ can be chosen arbitrarily (if $l_1>0$), and for $n>l_1$

$$\alpha_n^*(v)=\alpha_{l_1}^*(v)+\sum_{j=l_1+1}^{n}(\lambda_j+v\beta_j-\lambda_{j-1}-v\beta_{j-1})G_{j-1}^{*(l_1,l_1)}(c_1,c_1;v),$$

where $G_n^*(x,y;v)$ is given by

$$G_n^*(x,y;v)=\sum_{i=0}^{n}\frac{P_i^{*v}(x)P_i^{*v}(y)}{\phi^*(P_i^{*v},P_i^{*v})},$$

with $P_n^{*v}(x)=P_n(x)+vR_n(x)$. For $G_n^*(x,y;v)$ a formula analogous to (7) holds. We choose $\alpha_j^*(v)=\alpha_j+v\gamma_j$ for $j\in\{1,2,\dots,l_1\}$ (if $l_1>0$), hence linear in v. We find for $n\in\{l_1+1,l_1+2,\dots,l_2\}$ (if $l_1<l_2$)

$$\alpha_n^*(v)=\alpha_{l_1}+v\gamma_{l_1}+\sum_{j=l_1+1}^{n}(\lambda_j+v\beta_j-\lambda_{j-1}-v\beta_{j-1})K_{j-1}^{(l_1,l_1)}(c_1,c_1)$$

and for $n>l_2$

$$\alpha_n^*(v)=\alpha_{l_1}+v\gamma_{l_2}+\sum_{j=l_1+1}^{n}(\lambda_j-\lambda_{j-1})K_{j-1}^{(l_1,l_1)}(c_1,c_1)$$
$$+v\sum_{j=l_2+1}^{n}(\lambda_j-\lambda_{j-1})[K_{j-1}^{(l_1,l_1)}(c_1,c_1)K_{j-1}^{(l_2,l_2)}(c_2,c_2)-(K_{j-1}^{(l_1,l_2)}(c_1,c_2))^2].$$

It follows that the eigenvalues of the operator $(\mathbf{L}+v\mathbf{B}^*)+\mu\mathbf{A}^*(v)$, which is linear in μ, can be written as

$$\lambda_n+\mu\alpha_n+v\beta_n+\mu v\gamma_n,$$

where $\{\alpha_n\}_{n=1}^{l_1}$ are arbitrary (if $l_1>0$), α_n is given by (9) for $n>l_1, \{\beta_n\}_{n=1}^{l_2}$ are arbitrary (if $l_2>0$), β_n is given by (11) for $n>l_2, \{\gamma_n\}_{n=1}^{l_1}$ are arbitrary (if $l_1>0$), in the case that $l_1<l_2$ for $n\in\{l_1+1,l_1+2,\dots,l_2\}$ the values of γ_n are given by

$$\gamma_n=\gamma_{l_1}+\sum_{j=l_1+1}^{n}(\beta_j-\beta_{j-1})K_{j-1}^{(l_1,l_1)}(c_1,c_1) \tag{15}$$

and for $n>l_2$ they are given by (12).

3.3. Conclusion

Let the conditions (8), (13), (10) and (14) hold. We see that if the arbitrary values $\{\alpha_k\}_{k=1}^{l_1}$ and $\{\beta_k\}_{k=1}^{l_2}$ are chosen the same in both constructions, then all the other values of α_k and β_k remain the same in both constructions. In the first construction the values $\{\gamma_k\}_{k=1}^{l_2}$ were arbitrary, in the second $\{\gamma_k\}_{k=1}^{l_1}$, whereas if $l_1<l_2$ the values $\{\gamma_k\}_{k=l_1+1}^{l_2}$ were given by (15). In both constructions the

higher values of γ_k are given by (12). It is clear that the second construction imposes the strongest conditions. We may conclude that if the eigenvalues are taken as in the second construction, then the corresponding differential operator will depend linearly on μ and ν, hence it will be of the form (4).

4. General form of the operators

Let $\mathscr{P}$ denote the vector space of polynomials with real coefficients. For $k \in \{1,2,3,\ldots\}$ we define the operator $\mathbf{J}_k$ on $\mathscr{P}$ by

$$\mathbf{J}_k P_n(x) = \delta_{n,k} P_n(x) \quad \text{for all } n \in \{0,1,2,\ldots\}$$

and the operator $\mathbf{K}_k$ on $\mathscr{P}$ by

$$\mathbf{K}_k P_n(x) = 0 \quad \text{for all } n \in \{0,1,\ldots,k-1\},$$

$$\mathbf{K}_k P_n(x) = P_n(x) \quad \text{for all } n \in \{k,k+1,k+2,\ldots\}.$$

Further we define the operator $\mathbf{M}_k$ on $\mathscr{P}$ by

$$\mathbf{M}_k P_n(x) = 0 \quad \text{for all } n \in \{0,1,\ldots,k-1\},$$

$$\mathbf{M}_k Q_n(x) = \delta_{n,k} Q_n(x) \quad \text{for all } n \in \{k,k+1,k+2,\ldots\}.$$

and the operator $\mathbf{N}_k$ on $\mathscr{P}$ by

$$\mathbf{N}_k P_n(x) = 0 \quad \text{for all } n \in \{0,1,\ldots,k-1\},$$

$$\mathbf{N}_k R_n(x) = R_n(x) \quad \text{for all } n \in \{k,k+1,k+2,\ldots\}.$$

Let $l_1 \leqslant l_2$. The operator $\mathbf{A}$ can be put in the form

$$\mathbf{A} = \mathbf{A}_0 + \sum_{k=1}^{l_1-1} \alpha_k \mathbf{J}_k + \alpha_{l_1} \mathbf{K}_{l_1},$$

where $\{\alpha_n\}_{n=1}^{l_1}$ are arbitrary (if $l_1 > 0$) and the operator $\mathbf{A}_0$ is uniquely determined by

$$\mathbf{A}_0 P_n(x) = 0 \quad \text{for all } n \in \{0,1,\ldots,l_1\},$$

$$\mathbf{A}_0 Q_n(x) = \alpha_n^0 Q_n(x) \quad \text{for all } n \in \{l_1+1, l_1+2, l_1+3,\ldots\},$$

where for $n > l_1$

$$\alpha_n^0 = \sum_{j=l_1+1}^{n} (\lambda_j - \lambda_{j-1}) K_{j-1}^{(l_1,l_1)}(c_1,c_1).$$

The operator $\mathbf{B}$ can be put in the form

$$\mathbf{B} = \mathbf{B}_0 + \sum_{k=1}^{l_2-1} \beta_k \mathbf{J}_k + \beta_{l_2} \mathbf{K}_{l_2},$$

where $\{\beta_n\}_{n=1}^{l_2}$ are arbitrary (if $l_2 > 0$) and the operator $\mathbf{B}_0$ is uniquely determined by

$$\mathbf{B}_0 P_n(x) = 0 \quad \text{for all } n \in \{0, 1, \ldots, l_2\},$$

$$\mathbf{B}_0 R_n(x) = \beta_n^0 R_n(x) \quad \text{for all } n \in \{l_2+1, l_2+2, l_2+3, \ldots\},$$

where for $n > l_2$

$$\beta_n^0 = \sum_{j=l_2+1}^{n} (\lambda_j - \lambda_{j-1}) K_{j-1}^{(l_2,l_2)}(c_2, c_2).$$

The operator $\mathbf{C}$ can be put in the form

$$\mathbf{C} = \mathbf{C}_0 + \sum_{k=1}^{l_1-1} \gamma_k \mathbf{J}_k + \gamma_{l_1}(\mathbf{K}_{l_1} - \mathbf{K}_{l_2+1}) + \sum_{k=l_1+1}^{l_2-1} \gamma_k^0 \mathbf{M}_k + \gamma_{l_2}^0 \mathbf{N}_{l_2},$$

where $\{\gamma_n\}_{n=1}^{l_1}$ are arbitrary (if $l_1 > 0$) and in the case that $l_1 < l_2$ for $n \in \{l_1+1, l_1+2, \ldots, l_2\}$

$$\gamma_n^0 = \sum_{j=l_1+1}^{n} (\beta_j - \beta_{j-1}) K_{j-1}^{(l_1,l_1)}(c_1, c_1).$$

If $l_2 > l_1$ the operator $\mathbf{C}_0$ is uniquely determined by

$$\mathbf{C}_0 P_n(x) = 0 \quad \text{for all } n \in \{0, 1, \ldots, l_2\},$$
$$\mathbf{C}_0 S_n(x) = \gamma_n^0 S_n(x) \quad \text{for all } n \in \{l_2+1, l_2+2, l_2+3, \ldots\},$$

where for $n > l_2$

$$\gamma_n^0 = \sum_{j=l_2+1}^{n} (\lambda_j - \lambda_{j-1})[K_{j-1}^{(l_1,l_1)}(c_1, c_1) K_{j-1}^{(l_2,l_2)}(c_2, c_2) - (K_{j-1}^{(l_1,l_2)}(c_1, c_2))^2]. \tag{16}$$

If $l_1 = l_2$ the operator $\mathbf{C}_0$ is uniquely determined by

$$\mathbf{C}_0 P_n(x) = 0 \quad \text{for all } n \in \{0, 1, \ldots, l_1+1\},$$

$$\mathbf{C}_0 S_n(x) = \gamma_n^0 S_n(x) \quad \text{for all } n \in \{l_1+2, l_1+3, l_1+4, \ldots\},$$

where for $n > l_1 + 1$ (16) holds.

Remark. If for a certain value $n = n_0$ one of the conditions (10) and (14) is not satisfied, then it is possible that some more freedom is available in the choice of the eigenvalues for $n = n_0$ in the first or second construction. However, if (8) and (13) are both satisfied, this will only lead to a more general operator of the form (4), if (10) and (14) both fail for the same value $n = n_0$, due to the fact that the eigenvalues must be the same in both constructions.

5. Applications

5.1. Sobolev-type Laguerre polynomials

Consider the Sobolev-type Laguerre polynomials $\{L_n^{\alpha,M,N}(x;k,l)\}_{n=0}^{\infty}$, which are orthogonal with respect to the inner product

$$\langle f,g\rangle=\frac{1}{\Gamma(\alpha+1)}\int_0^{\infty}f(x)g(x)x^{\alpha}e^{-x}\,dx+M\mathbf{D}^k f(0)\mathbf{D}^k g(0)+N\mathbf{D}^l f(0)\mathbf{D}^l g(0),$$

with $M>0$, $N>0$, $\alpha>-1$, $k,l\in\{0,1,2,\dots\}$, $k<l$. By the results of Section 3 it follows that there exist linear differential operators $\mathbf{A}^{(\alpha,k,l)},\mathbf{B}^{(\alpha,k,l)},\mathbf{C}^{(\alpha,k,l)}$ and numbers $\{\alpha_n^{(\alpha,k,l)}\}_{n=0}^{\infty},\{\beta_n^{(\alpha,k,l)}\}_{n=0}^{\infty}$, $\{\gamma_n^{(\alpha,k,l)}\}_{n=0}^{\infty}$ such that for $n=0,1,2\dots$ the polynomials $\{L_n^{(\alpha,M,N)}(x;k,l)\}_{n=0}^{\infty}$ are solutions of the differential equation

$$\begin{aligned}&[(\mathbf{L}^{(\alpha)}-n\mathbf{I})+M(\mathbf{A}^{(\alpha,k,l)}-\alpha_n^{(\alpha,k,l)}\mathbf{I})\\&\quad+N(\mathbf{B}^{(\alpha,k,l)}-\beta_n^{(\alpha,k,l)}\mathbf{I})+MN(\mathbf{C}^{(\alpha,k,l)}-\gamma_n^{(\alpha,k,l)}\mathbf{I})]y(x)=0.\end{aligned}$$

Here $\mathbf{L}^{(\alpha)}$ is given by

$$\mathbf{L}^{(\alpha)}:=-x\mathbf{D}^2-(\alpha+1-x)\mathbf{D}$$

and

$$\mathbf{A}^{(\alpha,k,l)}=\sum_{i=1}^{\infty}a_i(x;\alpha,k,l)\mathbf{D}^i,\quad \mathbf{B}^{(\alpha,k,l)}=\sum_{i=1}^{\infty}b_i(x;\alpha,k,l)\mathbf{D}^i,\quad \mathbf{C}^{(\alpha,k,l)}=\sum_{i=1}^{\infty}c_i(x;\alpha,k,l)\mathbf{D}^i.$$

We have to take $\alpha_0^{(\alpha,k,l)}=\beta_0^{(\alpha,k,l)}=\gamma_0^{(\alpha,k,l)}=0$ and the values $\{\alpha_n^{(\alpha,k,l)}\}_{n=1}^{k}$ (if $k>0$), $\{\beta_n^{(\alpha,k,l)}\}_{n=1}^{l}$, $\{\gamma_n^{(\alpha,k,l)}\}_{n=0}^{k}$ (if $k>0$) can be chosen arbitrarily. The form of the operators can be determined by means of Section 4. Further we state the following conjecture.

Conjecture 1. *In the case that all the values* $\{\alpha_n^{(\alpha,k,l)}\}_{n=1}^{k}$ (*if* $k>0$), $\{\beta_n^{(\alpha,k,l)}\}_{n=1}^{l}$, $\{\gamma_n^{(\alpha,k,l)}\}_{n=1}^{k}$ (*if* $k>0$) *are taken to be* 0 *and* α *is a nonnegative integer, the corresponding linear differential operator*

$$\mathbf{L}^{(\alpha)}+M\mathbf{A}_0^{(\alpha,k,l)}+N\mathbf{B}_0^{(\alpha,k,l)}+MN\mathbf{C}_0^{(\alpha,k,l)}$$

is of finite order $4\alpha+6+4k+4l$. *In all the other cases the differential equation is of infinite order.*

In [10] this conjecture has been proved when $k=0$, $l=1$. One of the problems there was to prove the existence of the linear differential operators. This is now a consequence of the results of [3] and of this paper. Here the conditions (8), (13) and (10) are always satisfied. Condition (14) becomes

$$\binom{n+\alpha}{n}\left[1-\frac{N}{\alpha+1}\binom{n+\alpha+1}{n-2}\right]\neq 0,\quad n\in\{2,3,4,\dots\}.$$

This is always true if $N > \alpha + 1$; for smaller values of N this may not be true for one specific value of n.

The case of one perturbation ($M = 0,\ N > 0$ or $M > 0,\ N = 0$) has been treated in [4].

5.2. Sobolev-type Meixner polynomials

In [5] Sobolev-type Meixner polynomials are considered, orthogonal with respect to the inner product

$$\langle f, g\rangle = (1-c)^{\beta} \sum_{x=0}^{\infty} \frac{(\beta)_x c^x}{x!} f(x)g(x) + \mu f(0)g(0) + \nu \Delta f(0)\Delta g(0), \tag{17}$$

$\beta > 0,\ 0 < c < 1,\ \mu \geqslant 0,\ \nu \geqslant 0$. It was shown that the polynomials $\{M_n^{\mu,\nu}(x;\beta,c)\}_{n=0}^{\infty}$, which are orthogonal with respect to (17), are eigenfunctions of a difference operator of the form $\mathbf{L} + \mu\mathbf{A}$ in the case $\mu > 0,\ \nu = 0$ and of a difference operator of the form $\mathbf{L} + \nu\mathbf{B}$ in the case $\mu = 0,\ \nu > 0$. In both cases these difference operators are of infinite order. By the results of this paper we may conclude that the polynomials $\{M_n^{\mu,\nu}(x;\beta,c)\}_{n=0}^{\infty}$ are eigenfunctions of linear difference operators of the form $\mathbf{L} + \mu\mathbf{A} + \nu\mathbf{B} + \mu\nu\mathbf{C}$. The operators $\mathbf{A}$ and $\mathbf{B}$ were shown to be of infinite order in all the cases. As in the preceding example the conditions (8), (13) and (10) are always satisfied and condition (14) may fail for one value of n. Similarly linear difference operators can be investigated having the polynomials, orthogonal with respect to inner product

$$\langle f, g\rangle = (1-c)^{\beta} \sum_{x=0}^{\infty} \frac{(\beta)_x c^x}{x!} f(x)g(x) + \mu \Delta^k f(0)\Delta^k g(0) + \nu \Delta^l f(0)\Delta^l g(0),$$

$\beta > 0,\ 0 < c < 1,\ \mu \geqslant 0,\ \nu \geqslant 0,\ k, l \in \{0, 1, 2, \ldots\},\ k < l$, as eigenfunctions.

5.3. Jacobi type polynomials

In different papers (see [7, 9]) the problem is considered of finding differential equations for the generalized Jacobi polynomials (Jacobi type polynomials) $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$, which are introduced in [12] and are orthogonal with respect to the inner product

$$\langle f, g\rangle = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx$$
$$+ Mf(-1)g(-1) + Nf(1)g(1),$$

where $\alpha > -1,\ \beta > -1,\ M \geqslant 0,\ N \geqslant 0$. From the results of this paper we may conclude that there exist unique linear differential operators $\mathbf{A}^{\alpha,\beta}$, $\mathbf{B}^{\alpha,\beta}$ and $\mathbf{C}^{\alpha,\beta}$ and numbers $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ such that the polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$ are eigenfunctions of the operator $\mathbf{L}^{\alpha,\beta} + M\mathbf{A}^{\alpha,\beta} + N\mathbf{B}^{\alpha,\beta} + MN\mathbf{C}^{\alpha,\beta}$ with eigenvalues $\{\lambda_n + M\alpha_n + N\beta_n + MN\gamma_n\}_{n=0}^{\infty}$. Here

$$\mathbf{L}^{\alpha,\beta} = (x^2 - 1)\mathbf{D}^2 + (\alpha - \beta + (\alpha + \beta + 2)x)\mathbf{D}$$

and $\lambda_n = n(n + \alpha + \beta + 1)$ for $n \in \{0, 1, 2, \ldots\}$. The conditions (8), (13), (10) and (14) are always satisfied in this context. Very recently Koekoek and Koekoek [10] succeeded in showing that in the case that α and β are both nonnegative integers the operator $\mathbf{A}^{\alpha,\beta}$ is of finite order $2\beta+4$, the operator $\mathbf{B}^{\alpha,\beta}$ is of finite order $2\alpha+4$ and the operator $\mathbf{C}^{\alpha,\beta}$ is of finite order $2\alpha+2\beta+6$. Note that in the case that $\alpha=\beta$ and $M=N$, a simpler operator is known (see [8]).

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