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LINEAR RECURRENCES ASSOCIATED TO RAYS IN PASCAL'S TRIANGLE AND COMBINATORIAL IDENTITIES

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ABSTRACT. Our main purpose is to describe the recurrence relation associated to the sum of diagonal elements laying along a finite ray crossing Pascal's triangle. We precise the generating function of the sequence of described sums. We also answer a question of Horadam posed in his paper [*Chebyshev and Pell connections*, Fibonacci Quart. **43** (2005), 108–121]. Further, using Morgan-Voyce sequence, we establish the nice identity $F_{n+1} - iF_n = i^n \sum_{k=0}^{n} \binom{n+k}{2k} (-2-i)^k$ of Fibonacci numbers, where i is the imaginary unit. Finally, connections to continued fractions, bivariate polynomials and finite differences are given.

1. Introduction

In this paper, first we assimilate the elements of Pascal's triangle to the lattice $\mathbb{Z} \times \mathbb{Z}$ by the map $(n,k) \mapsto \binom{n}{k}$ with the convention $\binom{n}{k} = 0$ for $k > n$ or $k < 0$.

Let $n \in \mathbb{N}$, moreover let $r \in \mathbb{N}^+$, $q \in \mathbb{Z}$ with $q + r > 0$. Then the grid point $(n, 0)$ and the direction (r, q) define a *diagonal ray* of Pascal's triangle which is

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containing the elements

$$T^{(r,q)}(n,k) = \binom{n-qk}{rk} x^{n-(q+r)k} y^{rk}, \qquad k = 0, \dots, \left\lfloor \frac{n}{r+q} \right\rfloor, \tag{1}$$

where x and y are nonzero real parameters. The set of these rays, when n runs across the set of natural numbers, forms the *field* of direction (r,q) (of Pascal' triangle).

For $r \geq 2$, we even define the *intermediate rays* of order p ($p = 1, 2, \dots, r-1$) by

$$T^{(r,q,p)}(n,k) = \binom{n-qk}{p+rk} x^{n-p-(q+r)k} y^{p+rk}, \qquad k = 0, \dots, \left\lfloor \frac{n-p}{r+q} \right\rfloor. \tag{2}$$

Similarly, we may have the corresponding *intermediate fields* of order p. Note, that $n < p$ would mean an empty ray, therefore, according to the conventional interpretation of the binomial coefficients, we consider this case as a ray with one 0 element. Allowing the case $p = 0$, it corresponds to the main ray given by relation (1).

Now, for a fixed direction (r,q) and a fixed value of p, define the sequence

$$\left(T_n^{(r,q,p)}\right)_{n \in \mathbb{N}}.$$

The general term constitutes the sum of elements laying on the corresponding ray, i.e.

$$T_{n+1}^{(r,q,p)} := \sum_{k=0}^{\lfloor (n-p)/(q+r) \rfloor} T^{(r,q,p)}(n,k) = \sum_{k \geq 0} \binom{n-qk}{p+rk} x^{n-p-(q+r)k} y^{p+rk}, \tag{3}$$

with the convention $T_0 = 0$.

Notice that (because a sum over empty set is zero)

$$(T_0 =)\, T_1 = \cdots = T_p = 0, \tag{4}$$

and

$$T_j = \binom{j-1}{p} x^{j-p-1} y^p, \qquad p+1 \leq j \leq r+q+p-1. \tag{5}$$

Especially, for $p = 0$, we obtain

$$T_{n+1}^{(r,q)} := \sum_{k=0}^{\lfloor n/(q+r) \rfloor} T^{(r,q)}(n,k) = \sum_{k \geq 0} \binom{n-qk}{rk} x^{n-(q+r)k} y^{rk}. \tag{6}$$

Apart from the trivial case $q = 0$, there is some interest to illustrate such sequences by examples. Here the field associated to Morgan-Voyce sequence $(M_n)_n$:

$$T^{(2,-1)}(n,k) = \binom{n+k}{2k} \longrightarrow M_{n+1} = \sum_{k=0}^{n} \binom{n+k}{2k},$$

and

$$T^{(2,-1,1)}(n,k) = \binom{n+k}{1+2k}.$$

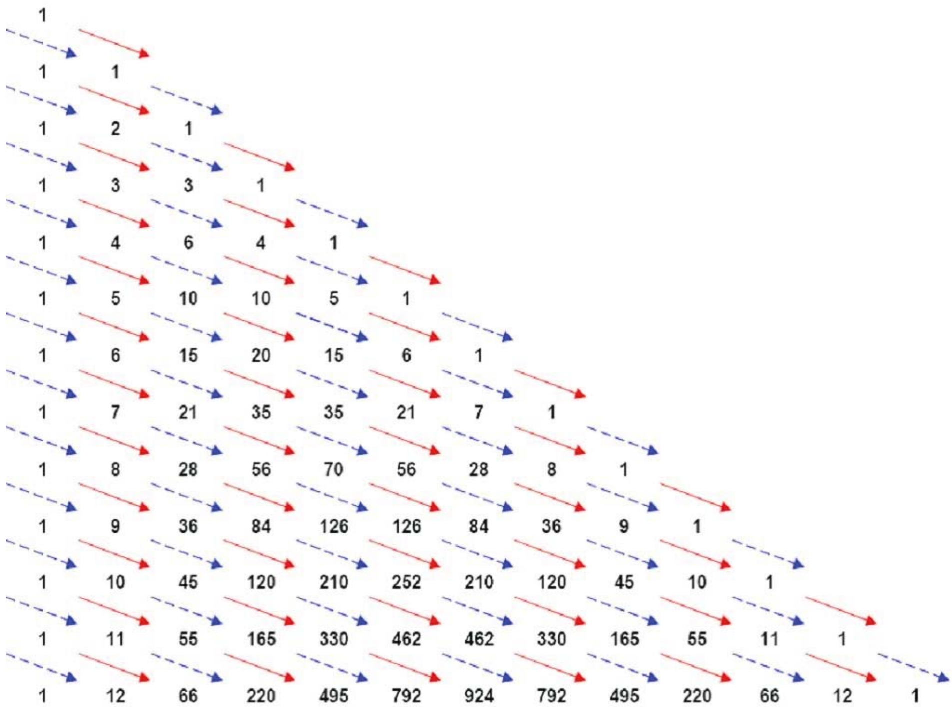

FIGURE 1. Illustration of the field of direction $(2,-1)$ and its correspondent intermediate field

The principal aim of this work is to determine the linear recurrence sequence associates to the general field (of Pascal' triangle) of direction (r,q) with the real parameters x and y.

2. Classical examples

In this section, we present some classical examples disseminated throughout the literature.

I. Case $(r,q) = (1,1)$.

This direction is the most known of the rays of Pascal's triangle, and it deals with the sequence $U_{n+1} = T_n^{(1,1)}$ given by

$$U_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k} y^k \qquad \text{with the convention} \quad U_0 = 0.$$

• For $(x, y) = (1, 1)$, we obtain the Fibonacci sequence

$$(F_n)_{n\geq 0} = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$$

listed in OEIS [15] as A000045. Thus

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}, \quad \text{satisfying} \quad \begin{cases} F_0 = 0, \quad F_1 = 1, \\ F_n = F_{n-1} + F_{n-2}, \quad (n \geq 2). \end{cases}$$

• For $(x, y) = (2, 1)$, we have the Pell sequence

$$(P_n)_{n\geq 0} = (0, 1, 2, 5, 12, 29, 70, \ldots)$$

listed in OEIS as A000129, and

$$P_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k} \binom{n-k}{k}, \quad \text{satisfying} \quad \begin{cases} P_0 = 0, \quad P_1 = 1, \\ P_n = 2P_{n-1} + P_{n-2}, \quad (n \geq 2). \end{cases}$$

• For $(x, y) = (1, 2)$, it is the Jacobsthal sequence

$$(J_n)_{n\geq 0} = (0, 1, 1, 3, 5, 11, 21, \ldots)$$

listed in OEIS as A001045,

$$J_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{k} \binom{n-k}{k}, \quad \text{satisfying} \quad \begin{cases} J_0 = 0, \quad J_1 = 1, \\ J_n = J_{n-1} + 2J_{n-2}, \quad (n \geq 2). \end{cases}$$

• For $(x, y) = (3, 2)$, we get the Fermat sequence (see [7])

$$(\Phi_n)_{n\geq 0} = (0, 1, 3, 11, 39, 139, 495, \ldots)$$

listed in OEIS as A007482. Thus

$$\Phi_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} 3^{n-2k} 2^{k} \binom{n-k}{k}, \quad \text{satisfying} \quad \begin{cases} \Phi_0 = 0, \quad \Phi_1 = 1, \\ \Phi_n = 3\Phi_{n-1} + 2\Phi_{n-2}, \\ (n \geq 2). \end{cases}$$

Observe, that arbitrary real parameters x and y provide the sequence

$$\begin{cases} U_0 = 0, \quad U_1 = 1, \\ U_n = xU_{n-1} + yU_{n-2}, \quad (n \geq 2). \end{cases}$$

II. Case direction $(1, q)$.

This case is sometimes cited in the literature as the generalized q-Fibonacci sequence

$$U_{n+1}^{(q)} = \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \binom{n-qk}{k} x^{n-(q+1)k} y^k.$$

The sequence $\left(U_{n+1}^{(q)}\right)_n$, with suitable initial values, satisfy the linear recurrence

$$U_n^{(q)} = xU_{n-1}^{(q)} + yU_{n-q-1}^{(q)}, \qquad (n > q)\,.$$

3. Linear recurrence sequences linked to the elements of Pascal's triangle

Now we deal with the main result, establishing the linear recurrence sequence associated to any field of direction $(r, q) \in \mathbb{N}^+ \times \mathbb{Z}$ with real parameters x and y. First we prove a very useful lemma, and then we provide the main result of the present work.

Lemma 1. *Let a, b and r are non-negative integers satisfy the conditions $r \le a$. Then*

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} \binom{a-s}{b} = \binom{a-r}{b-r}.$$

Proof. It is easy to see that

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} \binom{a-s}{b} = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \binom{a-r+s}{b}.$$

By the Vandermonde identity $\binom{a-r+s}{b} = \sum\limits_{j=0}^{s} \binom{s}{j}\binom{a-r}{b-j}$ we obtain

$$\begin{aligned}
\sum_{s=0}^{r} (-1)^s \binom{r}{s} \binom{a-s}{b} &= \sum_{0 \le j \le s \le r} (-1)^{r-s} \binom{r}{s} \binom{s}{j} \binom{a-r}{b-j} \\
&= \sum_{j=0}^{r} (-1)^r \binom{a-r}{b-j} \sum_{s=j}^{r} (-1)^s \binom{r}{s} \binom{s}{j} \\
&= \sum_{j=0}^{r} (-1)^r \binom{a-r}{b-j} (-1)^j \binom{r}{j} \sum_{k=0}^{r-j} (-1)^k \binom{r-j}{k},
\end{aligned}$$

when all terms vanish unless $j = r$. □

Theorem 1. *The terms of the sequence $(T_n)_n$ given by*

$$T_{n+1} = \sum_{k=0}^{\lfloor (n-p)/(q+r) \rfloor} \binom{n-qk}{p+rk} x^{n-p-(q+r)k} y^{p+rk},$$

satisfy the linear recurrence relation

$$T_n - x\binom{r}{1}T_{n-1} + x^2\binom{r}{2}T_{n-2} + \cdots + (-1)^r x^r \binom{r}{r} T_{n-r} = y^r T_{n-r-q}. \tag{7}$$

Proof. For $n \geq \max(r+q, r)$, let us compute $\sum_{j=0}^{r} (-1)^j x^j \binom{r}{j} T_{n-j}$.

$$\begin{aligned}
\sum_{j=0}^{r} (-x)^j \binom{r}{j} T_{n-j} &= \sum_{k\geq 0}\sum_{j\geq 0} (-x)^j \binom{r}{j}\binom{n-j-1-qk}{p+rk} x^{n-j-1-p-(q+r)k} y^{rk+p} \\
&= \sum_{k\geq 0} x^{n-1-p-(q+r)k} y^{rk+p} \sum_{j\geq 0} (-1)^j \binom{r}{j}\binom{(n-qk-1)-j}{p+rk} \\
&= y^p x^{n-p-1} \sum_{j\geq 0} (-1)^j \binom{r}{j}\binom{n-1-j}{p} \\
&\quad + \sum_{k\geq 1} x^{n-1-p-(q+r)k} y^{rk+p} \sum_{j\geq 0} (-1)^j \binom{r}{j}\binom{n-qk-1-j}{p+rk}.
\end{aligned}$$

Now apply Lemma 1 and finish the proof by

$$\sum_{j=0}^{r} (-x)^j \binom{r}{j} T_{n-j} = 0 + \sum_{k\geq 1} x^{n-1-p-(q+r)k} y^{rk+p} \binom{n-qk-1-r}{p+rk-r}.$$

and by

$$y^r \sum_{k\geq 0} x^{n-1-p-(q+r)(k+1)} y^{rk+p} \binom{n-(q+r)-qk-1}{p+rk} = y^r T_{n-r-q}.$$

□

4. Corresponding generating function

In this section, we set $s = r - p$ and

$$A(k,s) := \sum_{j=0}^{k} (-1)^j \binom{p+s}{k-j}\binom{p+j}{j}.$$

Lemma 2. *For $k \geq 2$, we have the following*

(1) $A(k,s) = A(k,s-1) + A(k-1,s-1)$;

(2) $A(k,k-1) = 0$;

(3) $A(k,k) = 1$.

Proof.

(1) A simple computation gives the result.

(2) It is relation 3.23 in Gould's book [6].

(3) From the recurrence relation, we have
$A(k,k) = A(k,k-1) + A(k-1,k-1)$,
thus $A(k,k) = 0 + A(k-1,k-1) = A(k-2,k-2) = \cdots = A(0,0) = 1$.

□

Lemma 3. *We have for $s \geq k$*

$$A(k,s) = \binom{s-1}{k}.$$

Proof. Using the precedent Lemma and proceeding by induction over s, for $s \geq k$, we get the result. □

Theorem 2. *The generating function associated to the sum of elements lying along a finite ray crossing Pascal's triangle is given by*

$$T(z) := \sum_{n\geq 0} T_{n+1}^{(r,q,p)} z^n = \frac{y^p z^{p+1} (1-xz)^{r-p-1}}{(1-xz)^r - y^r z^{q+r}}.$$

Proof. For simplicity, set $U_n = T_{n+1}$, this gives (see relations (4) and (5)) $U_{-1} = U_0 = \cdots = U_{p-1} = 0$ and

$$U_j = \binom{j}{p} x^{j-p} y^p, \qquad p \leq j \leq r+q-1.$$

Thus, Theorem 1 implies

$$U_n = \sum_{j=1}^{r+q} (-1)^{j-1} a_j U_{n-j},$$

where

$$a_j = \begin{cases} \binom{r}{j} x^j, & 1 \leq j \leq r, \\ 0, & r+1 \leq j \leq r+q-1, \\ (-1)^{j+1} y^{j-q}, & j = r+q. \end{cases}$$

$$\begin{aligned} T(z) &= \sum_{n\geq 0} U_n z^n \\ &= \sum_{n=0}^{r+q-1} U_n z^n + \sum_{n\geq r+q} U_n z^n \\ &= \sum_{n=0}^{r+q-1} U_n z^n + \sum_{n\geq r+q} \left(\sum_{j=1}^{r+q} (-1)^{j-1} a_j U_{n-j} \right) z^n \\ &= \sum_{n=0}^{r+q-1} U_n z^n + \sum_{j=1}^{r+q} (-1)^{j-1} a_j z^j \left(T(z) - \sum_{k=0}^{r+q-1-j} U_k z^k \right), \end{aligned}$$

thus

$$\left(\sum_{j=0}^{r+q} (-1)^j a_j z^j \right) T(z) = \sum_{k=0}^{r+q-1} U_k z^k + \sum_{k=1}^{r+q-1} \left(\sum_{j=1}^{k} (-1)^{j-1} a_j U_{k-j} \right) z^k$$

which gives

$$T(z) = \frac{\sum_{k=1}^{r+q-1}\left(\sum_{j=1}^{k}(-1)^{j-1}a_jU_{k-j}\right)z^k}{\sum_{j=0}^{r+q}(-1)^j a_j z^j} := \frac{A(z)}{B(z)},$$

with $a_0 = 1$.

We have

$$\begin{aligned} B(z) &= \sum_{j=0}^{r}(-1)^j x^j\binom{r}{j}z^j + (-1)^{r+q+1}(-1)^{r+q}y^r z^{r+q} \\ &= (1-xz)^r - y^r z^{r+q}. \end{aligned}$$

and

$$\begin{aligned} A(z) &= \sum_{k=0}^{r}\left(\sum_{j=0}^{k}(-x)^j\binom{r}{j}U_{k-j}\right)z^k \\ &= \sum_{k=0}^{r}\left(\sum_{j=p}^{k}(-x)^{k-j}\binom{r}{k-j}U_j\right)z^k \\ &= \sum_{p\le j\le k\le r}(-1)^{k-j}x^{k-j}\binom{r}{k-j}\binom{j}{p}x^{j-p}y^p z^k \\ &= \sum_{0\le j\le k\le r-p}(-1)^{k-j}x^k\binom{r}{k-j}\binom{j+p}{p}y^p z^{k+p} \\ &= (yz)^p\sum_{k=0}^{r-p}(-1)^k\sum_{j=0}^{k}(-1)^j\binom{r}{k-j}\binom{p+j}{j}(xz)^k \\ &= y^p z^p\sum_{k=0}^{r-p}(-1)^k\binom{r-p-1}{k}(xz)^k. \end{aligned}$$

by Lemma 3, we get

$$A(z) = y^p z^p (1-xz)^{r-p-1}.$$

We conclude by shifting n to $n+1$ for T_{n+1} which transform z^p to z^{p+1} in $A(z)$. □

Exploiting the works of Sprugnoli [17] and Chu and Vicenti [5], we can deduce the generating function of the sequence $(T_n)_n$ according to relation (3) directly, using Riordan rays without use the linear recurrence expression.

5. Morgan-Voyce and quasi Morgan-Voyce sequences

Classically, *Morgan-Voyce sequence* is defined by

$$\begin{cases} M_1 = 1, \quad M_2 = 1 + t + s, \\ M_n = (2+t)\, M_{n-1} - M_{n-2}, \quad (n \geq 3). \end{cases} \tag{8}$$

(Notice the shift of the initial conditions.)

It was introduced by Morgan-Voyce [13] and studied by Swamy [18] and Swamy and Bhattacharyya [19]. Later, André-Jeannin [2] gave the general term according to Pascal triangle and received

$$M_{n+1} = \sum_{k=0}^{n} M\,(n,k) \qquad \text{with} \quad M\,(n,k) = \binom{n+k}{2k} t^k + s \binom{n+k}{1+2k} t^k.$$

Since the sum of two sequences satisfying the same linear recurrence is also satisfies the recurrence relation, we obtain the same result using Theorem 1 when $r = 2$, $q = -1$, $x = 1$ and $y = \sqrt{t}$.

Horadam [8] introduced the *quasi Morgan-Voyce sequence* (with a simple extension according to the initial conditions, without loss of generality, we can omit it) as follows. Let

$$\begin{cases} D_1 = 1, \quad D_2 = 1 + t + s, \\ D_n = (2+t)\, D_{n-1} + D_{n-2}, \quad (n \geq 3). \end{cases} \tag{9}$$

He asked the following question [9]:

Can we find, if it exists, a formula $d\,(n,k)$ involving binomial coefficients analogous to that for $M\,(n,k)$, i.e.: $D_n = \sum_{k=0}^{n} d\,(n,k)\, t^k$?

For real parameters the response is negative, because by Theorem 1, we are able to determine all recurrence relations corresponding to the binomial coefficients, and in this case the unique possibility holds for $r = 2$, $q = -1$. It gives

$$\begin{cases} T_1 = 1, \quad T_2 = 1 + t + s, \\ T_n = \left(2x + y^2\right) T_{n-1} - x^2 T_{n-2}, \quad (n \geq 3). \end{cases}$$

Therefore by the identification $-x^2 = 1$, we get a contradiction for real parameters.

6. Morgan-Voyce sequence and Fibonacci and Lucas numbers

Now, let us recall the relations (1) and (9) to deduce remarkable identities for Fibonacci and Lucas numbers. Note that $r = 2$ and $q = -1$. Assuming $t = -1$ and $s = 1$, the identification

$$\begin{cases} 2x + y^2 = 2 + t \\ -x^2 = 1 \end{cases}$$

gives $x = i$, $y = \sqrt{1-2i}$, where i is the imaginary unit. Hence by (1) and Theorem 1, $T_n = T_{n-1} + T_{n-2}$ follows, further

$$T_{n+1} = \sum_{k=0}^{n} \binom{n+k}{2k} i^{n-k} \left(\sqrt{1-2i}\right)^{2k} = i^n \sum_{k=0}^{n} \binom{n+k}{2k} (-2-i)^k .$$

Consequently, $T_1 = 1$ and $T_2 = 1 - i$ show the connection $T_{n+1} = F_{n+1} - iF_n$ with Fibonacci numbers. Thus

$$F_{n+1} = \operatorname{Re}\left(i^n \sum_{k=0}^{n} \binom{n+k}{2k} (-2-i)^k \right) = \operatorname{Re}\left(i^n \sum_{k=0}^{n} \binom{n+k}{2k} \sqrt{5}^k e^{ik \arctan \frac{1}{2}} \right),$$

which is equivalent to

$$F_{n+1} = \sum_{0 \le 2j < k \le n} (-1)^{n+k+j} \, 4^j \binom{n+k}{2k} \binom{k}{2j}.$$

In order to gain similar result for Lucas numbers we exploit the fact that Lucas numbers satisfy the equality $L_n = F_{n+1} + F_{n-1}$. Thus we obtain

$$L_n - iL_{n-1} = i^n \sum_{k=0}^{n-2} \binom{n-2+k}{2k} \frac{2k\,(2n-1)\,(-2-i)^k}{(n-k)\,(n-k-1)} + (1-2i)^{n-1}\,(1 + (2n-3)i)\,.$$

7. Continued fraction expansions of the ratios of the sequence

Consider the sequence $\{u_n\}_{n \ge 0}$ satisfying the relation $u_n = xu_{n-1} + yu_{n-2}$ $(n \ge 2)$ with initial values u_0 and u_1. Then by

$$\frac{u_n}{u_{n-1}} = x + \frac{y}{\frac{u_{n-1}}{u_{n-2}}}\,,$$

we have for $n \ge 3$

$$\begin{aligned} \frac{u_n}{u_{n-1}} &= x + \cfrac{y}{x + \ddots + \cfrac{y}{x + \cfrac{y}{\frac{u_2}{u_1}}}} \\ &= x + \underbrace{\frac{y}{x+} \cdots \frac{y}{+x}}_{n-3} + \frac{y}{u_2/u_1}\,. \end{aligned}$$

In particular, for $n \ge 3$

$$\frac{F_n}{F_{n-1}} = 1 + \underbrace{\frac{1}{1} \genfrac{}{}{0pt}{}{}{+\cdots+} \frac{1}{1} \genfrac{}{}{0pt}{}{}{+} \frac{1}{1}}_{n-2} = [\underbrace{1, 1, \dots, 1}_{n-1}],$$

$$\frac{P_n}{P_{n-1}} = 2 + \underbrace{\frac{1}{2} \genfrac{}{}{0pt}{}{}{+\cdots+} \frac{1}{2} \genfrac{}{}{0pt}{}{}{+} \frac{1}{2}}_{n-2} = [\underbrace{2, 2, \dots, 2}_{n-1}],$$

$$\frac{J_n}{J_{n-1}} = 1 + \underbrace{\frac{2}{1} \genfrac{}{}{0pt}{}{}{+\cdots+} \frac{2}{1} \genfrac{}{}{0pt}{}{}{+} \frac{2}{1}}_{n-2},$$

$$\frac{\Phi_n}{\Phi_{n-1}} = 3 + \underbrace{\frac{2}{3} \genfrac{}{}{0pt}{}{}{+\cdots+} \frac{2}{3} \genfrac{}{}{0pt}{}{}{+} \frac{2}{3}}_{n-2},$$

$$\frac{M_n}{M_{n-1}} = 2 + t - \underbrace{\frac{1}{2+t} \genfrac{}{}{0pt}{}{}{-\cdots-} \frac{1}{2+t}}_{n-3} \genfrac{}{}{0pt}{}{}{-} \frac{1}{1+t+s}$$

$$= [1+t, \underbrace{\underbrace{1, t}, \dots, \underbrace{1, t}}_{n-3}, 1, t+s].$$

The last transformation was done by the equivalence rule in [10: 2.3.23, p.35] and the formula $[\dots, a, -b, \gamma] = [\dots, a-1, 1, b-1, -\gamma]$ in [14: Section 6].

8. Bivariate Fibonacci polynomials

Consider the bivariate polynomials $u_n = u_n(x, y)$, defined by $u_n = xu_{n-1} + yu_{n-2}$ $(n \ge 2)$ with arbitrary real numbers u_0 and u_1. Then

$$u_n = u_n(x, y) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{nu_1 + (u_0 x - 2u_1)k}{n-k} \binom{n-k}{k} x^{n-2k-1} y^k \qquad (n \ge 1).$$

Its generating function is given by

$$\sum_{n=0}^{\infty} u_n(x, y) t^n = \frac{(1 - xt)u_0 + u_1 t}{1 - xt - yt^2}.$$

If x is replaced by x^k, and $y = 1$, $u_0 = -1$, $u_1 = x - 1$, then this identity matches that in [1: Fact 2]. If $u_0 = x$ and $u_1 = y$, then $u_n(x, y) = g_n(x, y)$ are called the bivariate Fibonacci polynomials of the first kind (see [1]). If $u_0 = y$ and $u_1 = x$, then $u_n(x, y) = f_n(x, y)$ are called the bivariate Fibonacci polynomials of the second kind ([1]). If $u_0 = 1$ and $u_1 = x$, then $u_n = U_{n+1}$ is the bivariate

Fibonacci polynomial. If $u_0 = 2$ and $u_1 = x$, then $u_n = V_n$ is the bivariate Lucas polynomial ([4]). In addition, if $y = 1$, they are called the Fibonacci polynomial $F_{n+1}(x)$ and the Lucas polynomial $L_n(x)$, respectively (see [11: Chs. 37 and 38]). Their generating functions are given by

$$\sum_{n=0}^{\infty} F_n(x)t^n = \frac{t}{1-t(x+t)} \qquad \text{and} \qquad \sum_{n=0}^{\infty} L_n(x)t^n = \frac{2-xt}{1-t(x+t)}\,,$$

respectively.

If $u_0 = 1$, $x = 1$, and y is replaced by x, then $u_n = J_{n+1}$ is the Jacobsthal polynomial:

$$\begin{aligned} J_{n+1} &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^k \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{\lfloor (n+1)/2 \rfloor + k}{\lfloor n/2 \rfloor - k} x^{\lfloor n/2 \rfloor - k} \end{aligned}$$

(see [11: Ch. 39, (39.2)]).

9. Combinatorial sums and finite differences

We consider combinatorial sums and finite differences introduced by Spivey [16]. Let $\{u_n\}$ be a generalized Fibonacci-type sequence defined by $u_n = xu_{n-1} + yu_{n-2}$ $(k \ge 2)$ with $u_0 = 0$ and $u_1 = 1$. Let x and y be fixed positive integers.

Theorem 3. *The ordinary power series generating function $U(z)$ of $\sum_{k=0}^{n} \binom{n}{k} c^k u_k$ is given by*

$$U(z) = \frac{cz}{1-(cx+2)z-(yc^2-xc-1)z^2}\,.$$

Namely,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} c^k u_k z^n = \frac{cz}{1-(xc+2)z-(yc^2-xc-1)z^2}\,.$$

Proof. Let $b_k = c^{k+1}u_{k+1}$ and $a_k = c^k u_k$. Let $h_n = \sum_{k=0}^{n} \binom{n}{k} b_k$ and $g_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. We have $\Delta b_k = b_{k+1} - b_k = c^{k+2}u_{k+2} - c^{k+1}u_{k+1} = (xc-1)b_k + yc^2 a_k$. Solving the system of recurrences $h_{n+1} - 2h_n - c = (xc-1)h_n + yc^2 g_n$ and $g_{n+1} - g_n = h_n$ for g_n, we obtain $g_{n+2} = (xc+2)g_{n+1} + (yc^2 - xc - 1)g_n + c$. Thus, the ordinary power series generating function $U(z)$ of g_n satisfies the equation $U(z) - (xc+2)zU(z) - (yc^2 - xc - 1)z^2U(z) = cz$. Solving for $U(z)$ completes the proof. $\square$

A few special cases of this theorem produce particularly clean results. Let $\Psi = (x + \sqrt{x^2+4y})/(2y)$ and $\bar{\Psi} = (x - \sqrt{x^2+4y})/(2y)$. Take c to be Ψ, $\bar{\Psi}$, $-2/x$ and $-1/x$ in Theorem 3.

Proposition 1.

$$\sum_{k=0}^{n} \binom{n}{k} \Psi^k u_k = \Psi(x\Psi+2)^{n-1} \qquad (n \geq 1)\,,$$

$$\sum_{k=0}^{n} \binom{n}{k} \bar{\Psi}^k u_k = \bar{\Psi}(x\bar{\Psi}+2)^{n-1} \qquad (n \geq 1)\,,$$

$$\sum_{k=0}^{n} \binom{n}{k} \left(-\frac{2}{x}\right)^k u_k = \begin{cases} -\frac{2}{x}\left(\frac{4y}{x^2}+1\right)^{(n-1)/2} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}\,, \end{cases}$$

$$\sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{x}\right)^k u_k = -\frac{u_n}{x^n} \qquad (n \geq 1)\,.$$

If $x = y = 1$, then Theorem 3 and four identities in Proposition 1 match [16: Theorem 3] and four identities in [16: Identity 4–7], respectively.

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