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Scale invariant operators and combinatorial expansions

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ABSTRACT

Scale invariance is a property shared by the operational operators xD, Dx and a whole class of linear operators. We give a complete characterization of this class and derive some of the common properties of its members. As an application, we show that a number of classical combinatorial results, such as Boole's additive formula or the Akiyama–Tanigawa transformation, can be derived in this setting.

1. Introduction

Let $p(x)$ and $q(x)$ be two polynomials. The proof of a formula such as

$$p\big(q(x)\big) = \sum_{k\geqslant 0} \frac{(q(x)-x)^k}{k!} p^{(k)}(x) \tag{1}$$

does not require any analytic apparatus. One can show it using only formal mathematics. More precisely, it can be viewed as an expansion of the composition operator $C_q p(x) = p(q(x))$:

$$C_q = \sum_{k\geqslant 0} \frac{(q(x)-x)^k}{k!} D^k. \tag{2}$$

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Specializing to $q(x) = x + a$, we have the ubiquitous shift operator

$$E^a = \sum_{k \geqslant 0} \frac{a^k}{k!} D^k. \tag{3}$$

The study of the class of the linear operators that commute with E^a is the subject of the Umbral Calculus [10,16]. These operators are essentially formal power series equipped with the Cauchy product (discrete convolution)

$$\left(\sum_{n \geqslant 0} a_n t^n\right)\left(\sum_{n \geqslant 0} b_n t^n\right) = \sum_{n \geqslant 0}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) t^n. \tag{4}$$

The choice $q(x) = ax$ seems to have attracted less attention. It corresponds to applying the scale operator $Sc^a p(x) = p(ax)$. We deduce from (1) that this operator can be expanded as

$$Sc^a = \sum_{k \geqslant 0} (a-1)^k \frac{x^k}{k!} D^k. \tag{5}$$

The linear operators that commute with the scale operators are the main topic of this paper. Notable operators of this type are xD and Dx. We shall see that the scale invariant operators are also essentially formal power series, this time equipped with the product

$$\left(\sum_{n \geqslant 0} a_n t^n\right) \odot \left(\sum_{n \geqslant 0} b_n t^n\right) = \sum_{n \geqslant 0} a_n b_n t^n. \tag{6}$$

The shift operators and the scale operators exhaust the class of the automorphisms acting on the polynomials. Indeed, every automorphism is of the form $T_{a,b} p(x) = p(ax + b)$, that is, a shift followed by a scaling. Together, the shift invariant operators and the scale invariant operators are a gold mine for finding formulas and expansions in numerical and combinatorial analysis.

In Section 2, we tackle a classical elementary result as a motivation and show how one is gradually led to a scale invariance situation. Section 3 is devoted to the systematic characterization of the scale invariant operators and some of their properties. All the expansions are carried in terms of the differential operator D, even though this can be done with other operators. A matrix formulation of the theory has been left out. In Section 4, we apply our results to some well-known examples borrowed from combinatorics.

The level of the exposition is quite elementary and no specialized background is assumed.

2. Boole's additive formula

In [11], a proof of Boole's additive formula

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^n = n! \tag{7}$$

is given as a consequence of Lagrange's interpolation theorem. As a motivation for the theory to come in the next section, let us derive this formula using simple operator methods. Recalling that

$$(xD)^n x^k = k^n x^k, \tag{8}$$

where $D = \frac{d}{dx}$, we have

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^n x^k &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(xD)^n x^k\\ &= (xD)^n\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}x^k\\ &= (xD)^n(x-1)^n.\end{aligned}$$

We used the binomial theorem to get the last equality. Hence our problem amounts to proving

$$e_1(xD)^n(x-1)^n = n!, \tag{9}$$

where e_1 is the evaluation functional at $x=1$ ($e_a p(x) = p(a)$) [15]. In order to do so, let us use the familiar expansion (see for instance the classic book of Riordan [14, p. 218, (34)])

$$(xD)^n = \sum_{k=0}^{n} S(n,k)x^k D^k \tag{10}$$

where $S(n,k)$ are the Stirling numbers of the second kind. Apply this to the polynomial $(x-1)^n$:

$$\begin{aligned}(xD)^n(x-1)^n &= \sum_{k=0}^{n} S(n,k)x^k D^k(x-1)^n\\ &= \sum_{k=0}^{n} S(n,k)x^k(n)_k(x-1)^{n-k}\\ &= S(n,n)x^n n! + \sum_{k=0}^{n-1} S(n,k)x^k(n)_k(x-1)^{n-k}.\end{aligned}$$

The result follows upon replacing x by 1 and using the fact that $S(n,n)=1$.

Observe that none of the properties of the Stirling numbers were used. Only the form of the expansion (10) is necessary. We shall return to this point.

Note also in passing that if we apply the operator $(xD)^m$ to the polynomial $(x-1)^n$ and then replace x by 1, we get the classical identity [6, p. 67, (49)]

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^m = n!S(m,n). \tag{11}$$

The right hand side of (11) is equal to 0 for $0 \leqslant m < n$.

From (8), we deduce by linearity that for any polynomial $p(x)$ we have the well-known operational formula

$$p(xD)x^k = p(k)x^k \tag{12}$$

and, for arbitrary scalars a and b,

$$p(a+bxD)x^k = p(a+bk)x^k. \tag{13}$$

Hence

$$p(a+bxD)(x-1)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} p(a+bk)x^k. \tag{14}$$

On the other hand, for a given integer $r \geqslant 0$, we have

$$\begin{aligned}\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (a+kb)^r &= \sum_{m=0}^{r} \binom{r}{m} a^{r-m} b^m \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^m \\ &= n! \sum_{m=n}^{r} \binom{r}{m} a^{r-m} b^m S(m,n)\end{aligned}$$

by (11). In particular,

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (a+kb)^n = b^n n!.$$

Thus, if $p(x)$ is of degree n, we get

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} p(a+bk) = p^{(n)}(a+bx) \tag{15}$$

which is the main result of [11].

What happens if we use the operator Dx instead of xD? The pendant of (8) is obviously

$$(Dx)^n x^k = (k+1)^n x^k. \tag{16}$$

The same argument as above, and the expansion [14, p. 221, (47a)]

$$(Dx)^n = \sum_{k=0}^{n} S(n+1,k+1) x^k D^k \tag{17}$$

give the formula (contained in (15))

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (k+1)^n = n!. \tag{18}$$

The operators $(xD)^n$ and $(Dx)^n$ share a common property of scale invariance. This is the subject of Section 3. Some examples are given in Section 4.

3. Scale invariant operators

In the sequel, $\mathbb{K}$ will be a field of characteristic 0 (in practice, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\mathbf{P} = \mathbb{K}[x]$ the algebra of polynomials in the variable x over $\mathbb{K}$. Let $\mathcal{A}$ be the algebra of all linear operators acting on $\mathbf{P}$ over the same field.

Given a scalar a, the scale operator Sc^a is defined by

$$Sc^a p(x) = p(ax) \tag{19}$$

where $p(x)$ is a polynomial. It is easy to see that the class of scale operators Sc^a $(a \neq 0)$ is a multiplicative abelian group isomorphic to the group of shift operators (and to the group $(\mathbb{K}^*, \cdot)$). A particularly important scale operator is the linear functional $Sc^0 = e_0$:

$$Sc^0 p(x) = p(0). \tag{20}$$

An operator A in $\mathcal{A}$ is said to be scale invariant if it commutes with all scale operators. In symbols

$$ASc^a = Sc^a A \tag{21}$$

for all a in $\mathbb{K}$. As noted earlier, examples of scale invariant operators are xD and Dx.

We now characterize these operators.

Proposition 1. *A linear operator A is scale invariant if and only if there exists a sequence of scalars a_n such that*

$$Ax^n = a_n x^n, \tag{22}$$

for $n = 0, 1, 2, \ldots$.

Proof. Suppose A is defined by $Ax^n = a_n x^n$, where a_n is a given sequence of scalars $(n = 0, 1, \ldots)$, and let h be a parameter in the field $\mathbb{K}$. Then we have

$$ASc^h x^n = Ah^n x^n = h^n Ax^n = h^n a_n x^n$$

and

$$Sc^h Ax^n = Sc^h a_n x^n = a_n Sc^h x^n = a_n h^n x^n.$$

This proves that A is scale invariant. To prove the converse statement, let A be scale invariant and put $Ax^n = p_n(x)$. Then

$$Sc^h Ax^n = Sc^h p_n(x) = p_n(hx),$$

while

$$ASc^h x^n = Ah^n x^n = h^n Ax^n = h^n p_n(x).$$

Hence, we have the equality

$$p_n(hx) = h^n p_n(x).$$

Put $x = 1$ and then replace h by x:

$$p_n(x) = p_n(1)x^n = a_n x^n$$

with $a_n = p_n(1)$. The result follows by simple linearity (replace x^n by an arbitrary polynomial). □

Remark. A careful look at this proof shows that our definition of scale invariance can be relaxed. If the operator A commutes with Sc^h for some $h \neq 0$ and $|h| \neq 1$, then A is scale invariant.

The following consequences are immediate.

Corollary 2. *Any two scale invariant operators commute.*

Corollary 3. *A scale invariant operator $Ax^n = a_n x^n$ is invertible if and only if $a_n \neq 0$ for all n.*

Corollary 4. *The class of all scale invariant operators is a commutative subalgebra of $\mathcal{A}$.*

Corollary 5. *The algebra of scale invariant operators is isomorphic onto the algebra $(\mathcal{S}, +, \odot)$ of scalars sequences, where $\odot$ is the Hadamard product* [2, p. 85, Problem 30] $(a_n) \odot (b_n) = (a_n b_n)$.

Other characterizations of the scale invariant operators are given in the following proposition.

Proposition 6. *Let A be a linear operator. The following statements are equivalent*:

1. *A is scale invariant.*
2. *The operators A and xD commute*: $AxD = xDA$.
3. *The operators A and Dx commute*: $ADx = DxA$.

Proof. Since the operators xD and Dx are both scale invariant, they commute with all scale invariant operators. Let us show that if xD and A commute, then A must be scale invariant. Put

$$Ax^n = p_n(x) = \sum_{k \geqslant 0} c_{nk} \frac{x^k}{k!}$$

for $n = 0, 1, 2, \ldots$. Now

$$AxDx^n = np_n(x) = \sum_{k \geqslant 0} nc_{nk} \frac{x^k}{k!} \tag{23}$$

and

$$xDAx^n = xDp_n(x) = \sum_{k \geqslant 0} kc_{nk} \frac{x^k}{k!}. \tag{24}$$

Equating the extreme right hand sides of (23) and (24), we get

$$nc_{nk} = kc_{nk}$$

for all $n \geqslant 0$ and $k \geqslant 0$. But this is possible only if $c_{nk} = 0$ whenever $n \neq k$, that is if $p_n(x) = c_{nn} x^n$. Now, since x^n is a base for the vector space $\mathbf{P}$, the result remains true for any polynomial $p(x)$ and therefore A is scale invariant. The proof of the last part is identical to the above one. $\square$

Notations. We shall frequently use the convenient umbral notation $\underline{a}^n = a_n$ [10, p. 199]. If A is a scale invariant operator with $Ax^n = a_n x^n$, then we simply write $A = Sc^{\underline{a}}$ and $Sc^{\underline{a}} p(x) = p(\underline{a}x)$.

Remark. The action of the operator $Sc^{\underline{a}}$ on x^n may be viewed as the Hadamard product of the sequence a_n with the sequence x^n. If L is the linear functional defined by $Lx^n = a_n$, and if we equip the class of linear functionals with this product $(L \odot M)x^n = (Lx^n)(Mx^n)$, making it into an algebra, then a scale invariant operator is essentially a linear functional (the two algebras are isomorphic). On the other hand, every shift invariant operator P may be defined by the discrete convolution $P\frac{x^n}{n!} = a_n * \frac{x^n}{n!} = \sum_{k=0}^{n} a_k \frac{x^{n-k}}{(n-k)!}$. Furthermore, we have $L = Sc^0 P$ (here $L\frac{x^n}{n!} = a_n$). Hence, if we equip the linear functionals with the underlying discrete convolution $(L * M)x^n = Lx^n * Mx^n$, we see that a shift invariant operator is essentially a linear functional [15].

The polynomial $(x-1)^n$ played a prominent role in the derivation of Boole's formula. We have

$$Sc^{\underline{a}}(x-1)^n = (\underline{a}x-1)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k x^k. \tag{25}$$

In particular,

$$e_1 Sc^{\underline{a}}(x-1)^n = (\underline{a}-1)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k = \Delta^n a_0,$$

where Δ is the difference operator: $\Delta a_n = a_{n+1} - a_n$. Hence,

$$(\underline{a}-1)^n = \Delta^n a_0. \tag{26}$$

We will also write

$$\bar{a}_n = \Delta^n a_0. \tag{27}$$

The following well-known result (binomial transform) will be used repeatedly.

Lemma 7. *Let $a_n(x)$ and $\bar{a}_n(x)$ be two sequences of polynomials and put $a_n = a_n(1)$, $\bar{a}_n = \bar{a}_n(1)$. Then we have the inversion formulas*:

$$a_n(x) = \sum_{k=0}^{n} \binom{n}{k} \bar{a}_k(x) x^{n-k} \quad \Longleftrightarrow \quad \bar{a}_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k(x) x^{n-k}. \tag{28}$$

In particular,

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \bar{a}_k \quad \Longleftrightarrow \quad \bar{a}_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k. \tag{29}$$

Proof. There exists a number of proofs of (28). We give one based on the discrete convolution (the Cauchy product) of two sequences u_n and v_n: $u_n * v_n = \sum_{k=0}^{n} u_k v_{n-k}$ and the fact that the inverse with respect to $*$ of the sequence $\frac{x^n}{n!}$ is $\frac{(-x)^n}{n!}$. Now the first relation in (28) can be written as $\frac{1}{n!}a_n(x) = \frac{1}{n!}\bar{a}_n(x) * \frac{x^n}{n!}$ while the second one can be written as $\frac{1}{n!}\bar{a}_n(x) = \frac{1}{n!}a_n(x) * \frac{(-x)^n}{n!}$. The result is apparent. Put $x = 1$ in (28) to get (29). $\square$

We give our first expansion theorem.

Theorem 8. *Every scale invariant operator $Sc^{\underline{a}}$ can be uniquely expanded as*

$$Sc^{\underline{a}} = \sum_{k \geqslant 0} \bar{a}_k \frac{x^k}{k!} D^k \tag{30}$$

where $\bar{a}_k = \Delta^k a_0$.

Proof. It suffices to prove that the right hand side of (30), when applied to the monomial x^n, gives $a_n x^n$. We have

$$\sum_{k \geqslant 0} \bar{a}_k \frac{x^k}{k!} D^k x^n = \sum_{k \geqslant 0} \binom{n}{k} \bar{a}_k x^n = x^n \sum_{k \geqslant 0} \binom{n}{k} \bar{a}_k = a_n x^n$$

by the inversion formulas (29). $\square$

Example. Let $p(x)$ be an arbitrary polynomial and c a scalar. Then, from (30), we get

$$p(cx) = \sum_{k \geqslant 0} (c-1)^k \frac{x^k}{k!} p^{(k)}(x). \tag{31}$$

Taylor's formula is obtained as a particular case:

$$p(0) = \sum_{k \geqslant 0} \frac{(-x)^k}{k!} p^{(k)}(x). \tag{32}$$

Note also that if we formally replace c by x in (31), we get

$$p\left(x^2\right) = \sum_{k \geqslant 0} \left(x^2 - x\right)^k \frac{1}{k!} p^{(k)}(x). \tag{33}$$

This is a special case of (1).

Corollary 9. *Let L be a linear functional defined on* $\mathbf{P}$ *and put $Lx^n = a_n$. Then $L = e_1 Sc^{\underline{a}}$ and it can be uniquely expanded as*

$$L = e_0 \sum_{k \geqslant 0} \frac{a_k}{k!} D^k \tag{34}$$

$$= e_1 \sum_{k \geqslant 0} \frac{\bar{a}_k}{k!} D^k. \tag{35}$$

Proof. The first expansion (34) is a classical result [15]. It suffices to apply both members to x^n and then extend by linearity to all polynomials. The second expansion (35) is a direct consequence of (30):

$$Lx^n = a_n = e_1 Sc^{\underline{a}} x^n = e_1 \sum_{k\geqslant 0} \bar{a}_k \frac{x^k}{k!} D^k x^n$$

$$= e_1 \sum_{k\geqslant 0} \frac{\bar{a}_k}{k!} D^k x^n.$$

The result follows for an arbitrary polynomial. □

Example. As a simple example, consider the definite integral $Lf(x) = \int_0^1 f(x)\,dx$. We have $a_n = Lx^n = \frac{1}{n+1}$ and a trite calculation (using (29)) gives $\bar{a}_n = \frac{(-1)^n}{n+1}$. Using both expansions (34) and (35), we get the identity

$$\sum_{n\geqslant 0} \frac{1}{(n+1)!} f^{(n)}(0) = \sum_{n\geqslant 0} \frac{(-1)^n}{(n+1)!} f^{(n)}(1), \tag{36}$$

valid at least for all polynomials $f(x)$. This identity may be used for instance to accelerate the convergence of a series. As an illustration, take $f(x) = \frac{1}{1+x}$. Then we get (we omit the details) a classical result:

$$\ln 2 = \sum_{n\geqslant 0} \frac{(-1)^n}{n+1} = \sum_{n\geqslant 0} \frac{1}{(n+1)2^{n+1}}.$$

The expansions developed in this section can be used to derive numerous well-known identities. We give a sample in the next section. Furthermore, the form of expansion (30) (in terms of the powers D^k) is not peculiar to scale invariant operators. In fact, every linear operator in $\mathcal{A}$ can be expanded this way as we show now (for the sake of completeness).

Theorem 10 *(General Expansion Theorem). Let A be a linear operator and put $Ax^n = a_n(x)$, $n = 0, 1, \ldots$. Then A can be uniquely expanded as*

$$A = \sum_{k\geqslant 0} \frac{1}{k!} \bar{a}_k(x) D^k \tag{37}$$

where $\bar{a}_n(x)$ is a sequence of polynomials defined by

$$\bar{a}_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k(x) x^{n-k}. \tag{38}$$

Proof. Apply (37) to x^n and use (28). □

Remark. The expansion (37) is implicit in (28). It is a particular case of an apparently more general one, known as the Kurbanov–Maksimov theorem [7]. We say apparently because the latter can be derived from (37).

Corollary 11. *The vector space $(\mathcal{A}, +)$ is isomorphic to the vector space of generating functions of sequences of polynomials. This isomorphism sends the operator*

$$A = \sum_{n \geqslant 0} \bar{a}_n(x) \frac{1}{n!} D^n$$

onto the generating function

$$\bar{a}(x,t) = \sum_{n \geqslant 0} \bar{a}_n(x) \frac{t^n}{n!}. \tag{39}$$

We omit the obvious proof.

The generating function $\bar{a}(x,t)$ is called the indicator of the operator A.

Our next result gives a simple device to find $\bar{a}(x,t)$ and hence the expansion of A.

Proposition 12. *Let A be a linear operator. Then the indicator $\bar{a}(x,t)$ of A is given by the operational formula*

$$\bar{a}(x,t) = e^{-xt} A e^{xt}. \tag{40}$$

Proof. This is just (38) expressed via generating functions:

$$\begin{aligned} e^{-xt} A e^{xt} &= \left(\sum_{n \geqslant 0} \frac{(-x)^n}{n!} t^n \right) \left(\sum_{n \geqslant 0} a_n(x) \frac{t^n}{n!} \right) \\ &= \sum_{n \geqslant 0} \left(\frac{(-x)^n}{n!} * \frac{1}{n!} a_n(x) \right) \frac{t^n}{n!} \\ &= \sum_{n \geqslant 0} \bar{a}_n(x) \frac{t^n}{n!} = \bar{a}(x,t). \qquad \square \end{aligned}$$

Example. Apply (40) to Sc^a:

$$e^{-xt} Sc^a e^{xt} = e^{-xt} e^{axt} = e^{(a-1)xt}.$$

In particular, for $a = 0$, the indicator of Sc^0 is $e^{-xt} = \sum_{k \geqslant 0} \frac{(-x)^k}{k!} t^k$, in accordance with (32).

4. Some examples

In this section, we show how our simple minded operator technique can be used to easily derive well-known results and even extend them somewhat. The list is by no means exhaustive and is only meant to show the power of the method.

Example *(Euler's transformation).* Apply the expansion formula (30) to the formal power series

$$\frac{1}{1-xt} = \sum_{n \geqslant 0} x^n t^n. \tag{41}$$

We obtain

$$\sum_{n\geqslant 0} a_n x^n t^n = \sum_{k\geqslant 0} \left(\Delta^k a_0\right) \frac{(xt)^k}{(1-xt)^{k+1}}.$$

Put $x = 1$:

$$\sum_{n\geqslant 0} a_n t^n = \sum_{k\geqslant 0} \left(\Delta^k a_0\right) \frac{t^k}{(1-t)^{k+1}}. \tag{42}$$

Note that if $a_n = p(n)$ is a polynomial in n, the right hand side of (42) reduces to a finite sum. In particular, if $p(n) = n^r$, we have an expression of the Eulerian polynomials [1], [4, Lemma 2.7], [8].

These polynomials are sometimes given by the generating function [5]

$$E_n(x) = (1-x)^{n+1} \sum_{k\geqslant 0} (k+1)^n x^k.$$

Hence the operational formula

$$E_n(x) = (1-x)^{n+1} (Dx)^n (1-x)^{-1}.$$

Replace the operator $(Dx)^n$ by an arbitrary scale invariant operator $Sc^{\underline{a}}$:

$$(1-x)^{n+1} Sc^{\underline{a}} (1-x)^{-1} = \sum_{k\geqslant 0} \left(\Delta^k a_0\right) x^k (1-x)^{n-k}.$$

Truncating the right hand side at n, we get a generalization of the Eulerian polynomials:

$$E_n(x; \underline{a}) = \sum_{k=0}^{n} \left(\Delta^k a_0\right) x^k (1-x)^{n-k}. \tag{43}$$

Convergence questions depend on the choice of the sequence a_n.

Example *(Bernstein polynomials).* Apply both sides of (30) to the polynomial $(1-x)^n$ (see (25)):

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k x^k &= \sum_{k\geqslant 0} \bar{a}_k \frac{x^k}{k!} D^k (1-x)^n \\ &= \sum_{k\geqslant 0} \binom{n}{k} (-1)^k \bar{a}_k x^k (1-x)^{n-k}.\end{aligned}$$

Put $b_n = (-1)^n a_n$ and $\bar{b}_n = (-1)^n \bar{a}_n$. We obtain

$$\sum_{k=0}^{n} \binom{n}{k} b_k x^k = \sum_{k\geqslant 0} \binom{n}{k} \bar{b}_k x^k (1-x)^{n-k}, \tag{44}$$

valid for any sequence of scalars b_n.

Example *(The Borel transform and Dobinski formula).* Apply both sides of (30) to the exponential function e^x and then multiply by e^{-x} (see (40)). We get the Borel transform [12, pp. 55–56]

$$e^{-x}\sum_{k\geqslant 0} a_k \frac{x^k}{k!} = \sum_{k\geqslant 0}\left(\Delta^k a_0\right)\frac{x^k}{k!}. \tag{45}$$

Take $a_k = a(k) = k^n$. In this case, using (8), we see we are expanding the operator $(xD)^n$ and, specializing to (10), we get Dobinski's formula for the exponential polynomials [15, p. 66]

$$\varphi_n(x) = e^{-x}\sum_{k\geqslant 0}\frac{k^n}{k!}x^k = \sum_{k\geqslant 0} S(n,k)x^k. \tag{46}$$

Example *(Automorphisms of* **P***).* The automorphisms of **P** are all of the form $T_{a,b}p(x) = p(ax+b)$, where a and b are scalars, $a \neq 0$. To see this, let T be an automorphism. Then, obviously, $\deg Tx^n = \deg(Tx)^n = n\deg Tx$. Since T is surjective, we must have $\deg(Tx) \leqslant 1$, for otherwise x would not be the image of any polynomial. Similarly, since T is injective, we must have $\deg Tx \geqslant 1$. Hence $\deg Tx = 1$ and $Tx = ax+b$, $a \neq 0$. In other words, $T_{a,b}$ is the product of a shift operator and a scale operator: $T_{a,b} = Sc^a E^b$. The indicator (40) of $T_{a,b}$ is given by

$$\bar{a}(x,t) = e^{-xt}T_{a,b}e^{xt} = e^{-xt}e^{(ax+b)t} = e^{((a-1)x+b)t}$$

and therefore, using (37), we have

$$T_{a,b} = \sum_{k\geqslant 0}\frac{((a-1)x+b)^k}{k!}D^k. \tag{47}$$

This expansion remains valid if $a = 0$, in which case $T_{0,b}$ is the evaluation functional $T_{0,b}p(x) = Sc^0 E^b p(x) = p(b)$.

Now, if a_n and b_n are two arbitrary sequences of scalars, we may define the operator $T_{\underline{a},\underline{b}}p(x) = p(\underline{a}x+\underline{b})$. This is the product of a scale invariant operator with a shift invariant operator and we have the expansion

$$T_{\underline{a},\underline{b}} = \sum_{k\geqslant 0}\frac{((\underline{a}-1)x+\underline{b})^k}{k!}D^k. \tag{48}$$

Now, apply both sides of (48) to x^n and put $x = 1$. We get, after some rearrangement

$$\sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k} = \sum_{k=0}^{n}\binom{n}{k}\left(\Delta^k a_0\right)\sum_{j=0}^{n-k}\binom{n-k}{j}b_j. \tag{49}$$

Put $c_r = \sum_{j=0}^{r}\binom{r}{j}b_j$ and use the inversion formula (29). We obtain the reciprocity formula (see [3, p. 221, (1.8)]) between two arbitrary sequences a_n and c_n:

$$\sum_{k=0}^{n}\binom{n}{k}a_{n-k}\left(\Delta^k c_0\right) = \sum_{k=0}^{n}\binom{n}{k}c_{n-k}\left(\Delta^k a_0\right). \tag{50}$$

Example *(The Akiyama–Tanigawa transformation).* In [9], the operator $A = 1-(1-x)D$ (and its powers) is the defining operator of the Akiyama–Tanigawa transformation. Theorem 2.1 of [9] states that

$$A^n = \sum_{k=0}^{n} (-1)^k S(n+1, k+1)(1-x)^k D^k \tag{51}$$

where $S(n,k)$ are the Stirling numbers of the second kind. The proof of this theorem was given by induction. Now, observe that A can be written in the alternative form $A = D(x-1)$ and that (51) is formally identical to (17) where x is replaced by $x-1$. We proceed to "explain" this. The operator $D(x-1)$ is a "shifted" scale invariant operator in the following sense: $(D(x-1))(x-1)^k = (k+1)(x-1)^k$ and, more generally, $(D(x-1))^n(x-1)^k = (k+1)^n(x-1)^k$. Now, using the shift operator E ($E^a p(x) = p(x+a)$), we have the equalities

$$\big(D(x-1)\big)^n (x-1)^k = (k+1)^n (x-1)^k = E^{-1}(k+1)^n x^k = E^{-1}(Dx)^n x^k.$$

On the other hand,

$$\big(D(x-1)\big)^n (x-1)^k = \big(D(x-1)\big)^n E^{-1} x^k.$$

We have thus the following characterization of $(D(x-1))^n$:

$$\big(D(x-1)\big)^n = E^{-1}(Dx)^n E. \tag{52}$$

We now proceed to prove (51). The indicator of $(D(x-1))^n = E^{-1}(Dx)^n E$ is given by

$$\begin{aligned}
\bar{a}(x,t) &= e^{-xt} E^{-1}(Dx)^n E e^{xt} = e^{-xt} E^{-1}(Dx)^n e^{(x+1)t} \\
&= e^t e^{-xt} E^{-1} \sum_{k=0}^{n} S(n+1,k+1) x^k D^k e^{xt} \\
&= e^t e^{-xt} E^{-1} \sum_{k=0}^{n} S(n+1,k+1) x^k t^k e^{xt} \\
&= e^t e^{-xt} \sum_{k=0}^{n} S(n+1,k+1)(x-1)^k t^k e^{(x-1)t} \\
&= \sum_{k=0}^{n} S(n+1,k+1)(x-1)^k t^k.
\end{aligned}$$

This is the indicator of (51). Note that we used (17).

Example *(A generalization of the Akiyama–Tanigawa transformation).* One can obviously use the slightly more general operator $A = D(ax+b)$ where a and b are scalars, $a \neq 0$. We have

$$\big(D(ax+b)\big)^n (ax+b)^k = a^n (k+1)^n (ax+b)^k \tag{53}$$

where n is a non-negative integer. We see that this operator merely "scales" the base $(ax+b)^k$. The same argument as in the previous example gives

$$\big(D(ax+b)\big)^n = a^n Sc^a E^b (Dx)^n E^{-b} Sc^{1/a}. \tag{54}$$

The indicator of this operator is computed in a similar way and we have the expansion

$$\big(D(ax+b)\big)^n = \sum_{k=0}^{n} S(n+1,k+1)a^{n-k}(ax+b)^k D^k. \tag{55}$$

Since the operator $(D(ax+b))^n$ is obviously invertible (it sends a base to a base), we compute its inverse. From (53), we have

$$\big[\big(D(ax+b)\big)^n\big]^{-1}(ax+b)^k = \frac{1}{a^n(k+1)^n}(ax+b)^k.$$

A glance at (54) shows that finding an expression of the inverse operator reduces to finding an expression of the inverse $(x^{-1}D^{-1})^n$ of $(Dx)^n$. The indicator of this inverse operator is given by

$$\begin{aligned} e^{-xt}\big((Dx)^n\big)^{-1}e^{xt} &= e^{-xt}\sum_{k\geqslant 0}\frac{1}{k!(k+1)^n}x^k t^k \\ &= \sum_{k\geqslant 0}\left(\frac{1}{k!(k+1)^n} * \frac{(-1)^k}{k!}\right)x^k t^k \\ &= \sum_{k\geqslant 0}\left(\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{1}{(i+1)^n}\right)\frac{x^k}{k!}t^k. \end{aligned}$$

Hence,

$$\big((Dx)^n\big)^{-1} = \sum_{k\geqslant 0}\left(\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{1}{(i+1)^n}\right)\frac{x^k}{k!}D^k \tag{56}$$

as one should expect.

Example *(The Binomial Transform revisited).* In [13], one finds the following pair of inversion formulas:

$$t_n = \sum_{k=0}^{n}\binom{n+d-1}{n-k}b^{n-k}c^k a_k \iff a_n = c^{-n}\sum_{k=0}^{n}\binom{n+d-1}{n-k}(-1)^{n-k}b^{n-k}t_k. \tag{57}$$

The formulas are derived using complex integration (residues). It is possible to extend this result a bit further. Let us first derive (57) using our operator method. Introduce the convolution operator K_b defined by

$$K_b f(x) = \frac{xf(x) - bf(b)}{x-b}$$

where b is an arbitrary scalar. This operator can alternatively be defined by (hence the name)

$$K_b x^n = \sum_{k=0}^{n} b^k x^{n-k} = b^n * x^n.$$

Since the generating function of the polynomial sequence x^n is $\frac{1}{1-xt}$, we have

$$K_b \frac{1}{1-xt} = \frac{1}{1-bt}\frac{1}{1-xt}.$$

For an arbitrary positive integer k, the k-fold convolution operator K_b^k is defined (using induction) by

$$K_b^k \frac{1}{1-xt} = \left(\frac{1}{1-bt}\right)^k \frac{1}{1-xt}.$$

We formally extend this definition to an "α-fold" convolution (for arbitrary complex α) in the obvious way:

$$K_b^\alpha \frac{1}{1-xt} = \left(\frac{1}{1-bt}\right)^\alpha \frac{1}{1-xt}.$$

Now, consider the operator

$$T = Sc^{\underline{a}} E^b K_b^\alpha.$$

We have

$$\begin{aligned}
T\frac{1}{1-xt} &= Sc^{\underline{a}} E^b K_b^\alpha \frac{1}{1-xt} = \left(\frac{1}{1-bt}\right)^\alpha Sc^{\underline{a}} E^b \frac{1}{1-xt} \\
&= \left(\frac{1}{1-bt}\right)^\alpha Sc^{\underline{a}} \frac{1}{1-bt} \frac{1}{1-\frac{xt}{1-bt}} \\
&= \left(\frac{1}{1-bt}\right)^{\alpha+1} Sc^{\underline{a}} \sum_{k\geqslant 0} \frac{x^k t^k}{(1-bt)^k} \\
&= \sum_{k\geqslant 0} \frac{a_k x^k t^k}{(1-bt)^{k+\alpha+1}} = \sum_{k\geqslant 0} a_k x^k t^k \sum_{r\geqslant 0} \binom{-k-\alpha-1}{r} (-b)^r t^r \\
&= \sum_{n\geqslant 0} \left(\sum_{k\geqslant 0} \binom{-k-\alpha-1}{n-k} (-b)^{n-k} a_k x^k \right) t^n.
\end{aligned}$$

Using the identity $\binom{-k-\alpha-1}{n-k}(-1)^{n-k} = \binom{\alpha+n}{n-k}$, we finally get (isolating the coefficient of t^n in the last equality)

$$Tx^n = \sum_{k\geqslant 0} \binom{\alpha+n}{n-k} b^{n-k} a_k x^k.$$

This is precisely the left hand side of (57) with $\alpha = d-1$ and $x = c$. Now, obviously both operators E^b and K_b^α are invertible (with respective inverses E^{-b} and $K_b^{-\alpha}$), so that (57) reduces to the trivial equivalence:

$$T = Sc^{\underline{a}} E^b K_b^\alpha \quad \Longleftrightarrow \quad Sc^{\underline{a}} = T K_b^{-\alpha} E^{-b}. \tag{58}$$

Hence, (58) is nothing more than a special case of

$$T = Sc^{\underline{a}} P \quad \Longleftrightarrow \quad Sc^{\underline{a}} = T P^{-1}, \tag{59}$$

where P is an invertible linear operator. Now, if we put $Px^n = \sum_{k \geqslant 0} c_{nk} x^k$, $P^{-1} x^n = \sum_{k \geqslant 0} d_{nk} x^k$ and $Tx^n = t_n(x)$, then (59) amounts to

$$t_n(x) = \sum_{k \geqslant 0} c_{nk} a_k x^k \quad \Longleftrightarrow \quad a_n x^n = \sum_{k \geqslant 0} d_{nk} x^k \tag{60}$$

and the challenge is the effective computation of the so-called connection constants c_{nk} from d_{nk} and vice versa in this general setting [10, p. 202].

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