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Counting on Continued Fractions

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Introduction

You might be surprised to learn that the finite continued fraction

$$3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292}}}} \quad \text{and its reversal} \quad 292+\cfrac{1}{1+\cfrac{1}{15+\cfrac{1}{7+\cfrac{1}{3}}}}$$

have the same numerator. These fractions simplify to $\frac{103993}{33102}$ and $\frac{103993}{355}$ respectively. In this paper, we provide a combinatorial interpretation for the numerators and denominators of continued fractions which makes this reversal phenomenon easy to see. Through the use of counting arguments, we illustrate how this and other important identities involving continued fractions can be easily visualized, derived, and remembered.

We begin by defining some basic terminology. Given an infinite sequence of integers $a_0 \geq 0, a_1 \geq 1, a_2 \geq 1, \ldots$, let $[a_0, a_1, \ldots, a_n]$ denote the finite continued fraction

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_n}}}}$$

The *infinite* continued fraction $[a_0, a_1, a_2, \ldots]$ is the limit of $[a_0, a_1, \ldots, a_n]$ as $n \to \infty$. This limit always exists and is some irrational number α **[3]**. The rational number $r_n := [a_0, a_1, \ldots, a_n]$ is a fraction p_n/q_n in lowest terms, called the n-th *convergent* of α. It is well-known that p_n and q_n satisfy the recurrences

$$p_n = a_n p_{n-1} + p_{n-2}$$
$$q_n = a_n q_{n-1} + q_{n-2}$$

for $n \geq 2$, with initial conditions $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $q_0 = 1$, $q_1 = a_1$.

Now let's do some combinatorics. For a given continued fraction $[a_0, a_1, a_2, \ldots]$, consider the following tiling problem. Let P_n count the number of ways to tile a $1 \times (n+1)$ board with dominoes and stackable square tiles. All cells (numbered $0, 1, \ldots, n$) must be covered by a tile. Nothing can be stacked on top of a domino, but cell number i may be covered by a stack of as many as a_i square tiles, $i = 0, \ldots, n$.

FIGURE 1 shows an empty board with the *height conditions* $a_0, a_1, \ldots, a_n$ indicated. FIGURE 2 gives an example of a valid tiling for a 1×12 board with height conditions $5, 10, 3, 1, 4, 8, 2, 7, 7, 4, 2, 3$.

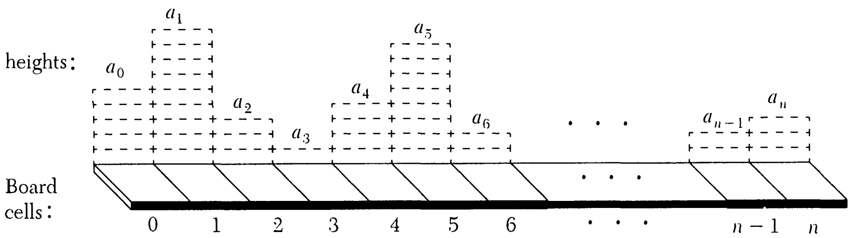

FIGURE 1
An empty $1 \times (n+1)$ board.

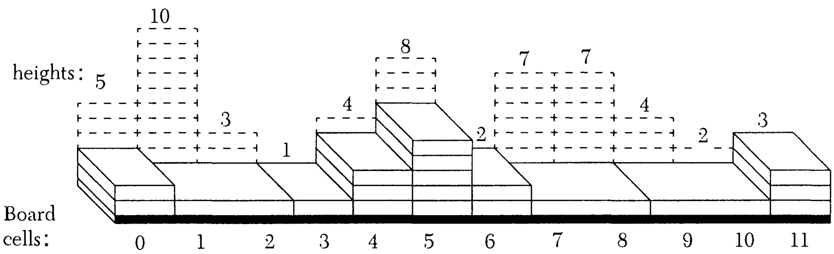

FIGURE 2
A tiling satisfying the height conditions $5, 10, 3, 1, 4, 8, 2, 7, 7, 4, 2, 3$.

For $n \geq 2$, we show

$$P_n = a_n P_{n-1} + P_{n-2}.$$

This follows from the observation that a tiling either ends with a stack of square tiles or a single domino. In the first case, there are a_n choices for the stack size and P_{n-1} ways to tile cells 0 through $n-1$. In the second case, there is only one choice for the last domino, and there are P_{n-2} ways to tile cells 0 through $n-2$. Using FIGURE 3 one can check that $P_0 = a_0$ and $P_1 = a_0 a_1 + 1$. Since P_n and p_n satisfy the same recurrence and initial conditions, we have $P_n = p_n$.

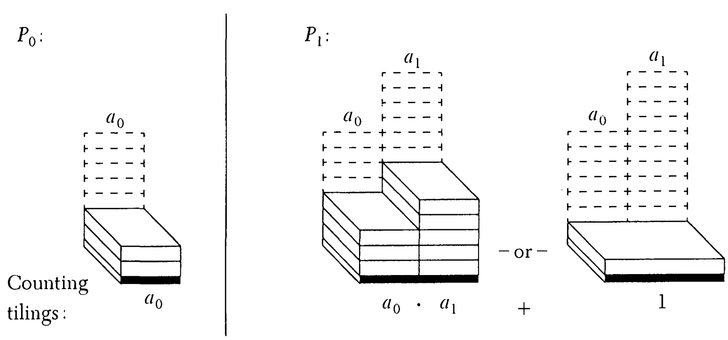

FIGURE 3
Verifying the initial conditions for the recurrence relation $P_n = a_n P_{n-1} + P_{n-2}$.

Removing cell 0 from the previous board, let Q_n count the number of ways to tile the $1 \times n$ board with dominoes and stackable square tiles, where the ith cell may be covered by a stack of as many as a_i square tiles, $i = 1, \ldots, n$. By the same reasoning as before, (and letting $Q_0 = 1$ denote the "empty" tiling) we see that $Q_n = q_n$.

To illustrate, consider the continued fraction representation for π, which begins $[3, 7, 15, 1, 292, \ldots]$. See FIGURE 4. If we count the number of ways to tile cells 0, 1, and 2, we get $p_2 = 333$. Counting the number of ways to tile only cells 1 and 2 easily gives us $q_2 = 106$. This produces the π approximation $r_2 = 333/106$. The reader should verify that tiling cells 0 through 3 produces $r_3 = 355/113$.

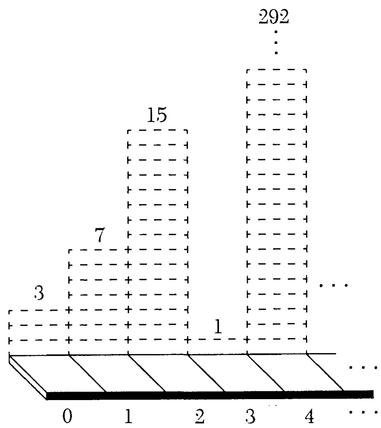

FIGURE 4
The beginning of the π board.

When $a_i = 1$ for all $i \geq 0$, it is well-known that the nth convergent p_n/q_n is the ratio of two consecutive Fibonacci numbers. Specifically, if we define $f_0 = 1$, $f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$, then $p_n = f_{n+1}$ and $q_n = f_n$. You may recall that the Fibonacci number f_n counts the number of ways to tile a $1 \times n$ board with 1×1 squares and 1×2 dominoes. So the continued fraction tiling problem generalizes the tiling interpretation of Fibonacci numbers [**1**, **2**].

Identities

Armed with our tiling interpretation, many well-known continued fraction identities can be explained combinatorially. We begin with the reversal identity.

THEOREM 1. *If* $[a_0, a_1, \ldots, a_{n-1}, a_n] = p_n/q_n$, *then* $[a_n, a_{n-1}, \ldots, a_1, a_0] = p_n/p_{n-1}$.

Proof. Although one can easily prove this by induction, the theorem is nearly obvious when viewed combinatorially. To understand the common numerator, we see that the number of ways to tile the board with height conditions $a_n, a_{n-1}, \ldots, a_1, a_0$ is the same as the number of ways to tile the board with height conditions $a_0, a_1, \ldots, a_{n-1}, a_n$. The denominator of $[a_n, a_{n-1}, \ldots, a_1, a_0]$ is the number of ways to tile the board with height conditions $[a_{n-1}, \ldots, a_1, a_0]$, which by reversal is p_{n-1}.

The next few identities are useful for measuring the rate of convergence of convergents.

THEOREM 2. *The difference between consecutive convergents of* $[a_0, a_1, a_2, \ldots]$ *is:* $r_n - r_{n-1} = (-1)^{n-1}/q_n q_{n-1}$. *Equivalently, after multiplying both sides by* $q_n q_{n-1}$, *we have*

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

Proof. Given height conditions $a_0, a_1, \ldots, a_n$, let $\mathscr{P}_n$ denote the set of all tilings on cells $0, \ldots, n$ and let $\mathscr{Q}_n$ denote the set of all tilings on cells $1, \ldots, n$. Note that these sets have sizes $|\mathscr{P}_n| = p_n$ and $|\mathscr{Q}_n| = q_n$.

We will construct an *almost one-to-one correspondence* between the sets $\mathscr{P}_n \times \mathscr{Q}_{n-1}$ and $\mathscr{P}_{n-1} \times \mathscr{Q}_n$. Consider $(S, T) \in \mathscr{P}_n \times \mathscr{Q}_{n-1}$. For $i \geq 1$, we say (S, T) has a *fault* at cell i if both S and T have tiles that end at i. We say (S, T) has a fault at cell 0 if S has a square at cell 0. For instance, in FIGURE 5, there are faults at cells 0, 3, 5, and 6.

If (S, T) has a fault, construct (S', T') by swapping the "tails" of S and T after the rightmost fault. See FIGURES 5 and 6. Note that $(S', T') \in \mathscr{P}_{n-1} \times \mathscr{Q}_n$. Since (S', T') has the same rightmost fault as (S, T), this procedure is completely reversible.

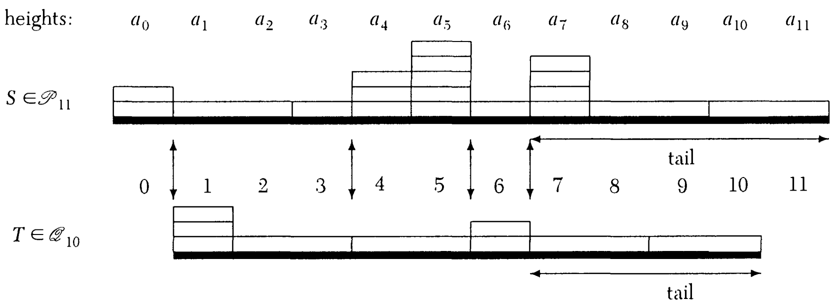

FIGURE 5
A pair of tilings with faults and tails indicated.

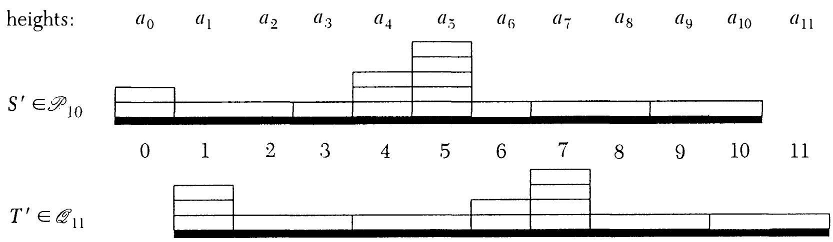

FIGURE 6
Result of swapping tails in FIGURE 5.

Notice when either S or T contains a square, (S, T) must have a fault. Thus the only fault-free pairs occur when S and T consist of all dominoes in *staggered formation* as illustrated in FIGURE 7. When n is odd (i.e., S and T both cover an even number of cells), there is precisely one fault-free element of $\mathscr{P}_n \times \mathscr{Q}_{n-1}$ and no fault-free elements of $\mathscr{P}_{n-1} \times \mathscr{Q}_n$. Therefore when n is odd, we have $|\mathscr{P}_n \times \mathscr{Q}_{n-1}| - |\mathscr{P}_{n-1} \times \mathscr{Q}_n| = 1$.

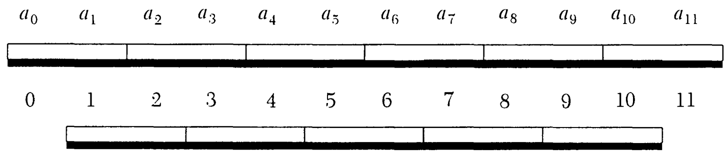

FIGURE 7
The fault-free pair consists of staggered dominoes.

Similarly when n is even, there are no fault-free elements of $\mathscr{P}_n \times \mathscr{Q}_{n-1}$ and exactly one fault-free element of $\mathscr{P}_{n-1} \times \mathscr{Q}_n$. Hence when n is even, $|\mathscr{P}_n \times \mathscr{Q}_{n-1}| - |\mathscr{P}_{n-1} \times \mathscr{Q}_n| = -1$. Treating the odd and even cases together, we obtain

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

The next identity shows that the even convergents are increasing, while the odd convergents are decreasing.

THEOREM 3. $r_n - r_{n-2} = (-1)^n a_n / q_n q_{n-2}$. *Equivalently, after multiplying both sides by* $q_n q_{n-2}$, *we have*

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n.$$

Proof. As in the last proof, we use tail swapping after the last fault to create a one-to-one correspondence between the "faulty" elements of $\mathscr{P}_n \times \mathscr{Q}_{n-2}$ and $\mathscr{P}_{n-2} \times \mathscr{Q}_n$. The proof is essentially given in FIGURES 8, 9, and 10.

The only unmatched elements are those that are fault-free. When n is odd, there are no fault-free elements of $\mathscr{P}_n \times \mathscr{Q}_{n-2}$, but there are precisely a_n fault-free

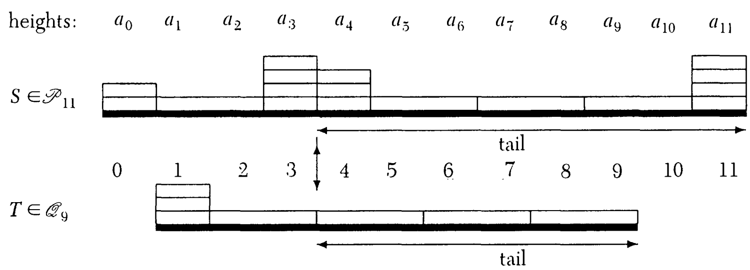

FIGURE 8
An element of $\mathscr{P}_{11}\mathscr{Q}_9$ with rightmost fault indicated.

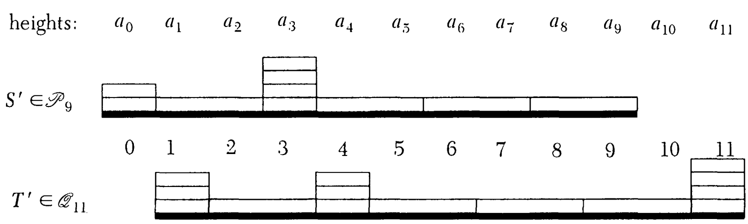

FIGURE 9
The result of swapping tails in Figure 8.

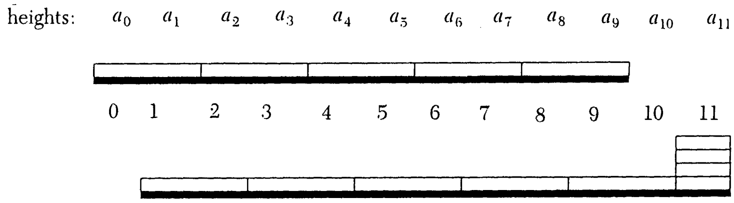

FIGURE 10
Problem pairings are fault-free.

elements of $\mathscr{P}_{n-2} \times \mathscr{Q}_n$, consisting of a stack of squares on the nth cell, and dominoes everywhere else (FIGURE 10). Likewise when n is even, there are no fault-free elements of $\mathscr{P}_{n-2} \times \mathscr{Q}_n$, but there are a_n fault-free elements of $\mathscr{P}_n \times \mathscr{Q}_{n-2}$, consisting of a stack of squares on the nth cell, and dominoes everywhere else. Thus we have established $|\mathscr{P}_n \times \mathscr{Q}_{n-2}| - |\mathscr{P}_{n-2} \times \mathscr{Q}_n| = (-1)^n a_n$, as desired.

Using the combinatorially clear fact that $q_n \to \infty$ as $n \to \infty$, the last two identities demonstrate that $(r_0, r_1), (r_2, r_3), (r_4, r_5), \ldots$ is a sequence of nested intervals whose lengths are going to zero. Hence, $\lim_{n \to \infty} r_n$ exists.

Extensions

Next we examine the quantity $K(i,j)$, for $i \leq j$, that counts the number of tilings of the sub-board with cells $i, i+1, \ldots, j$ with height conditions $a_i, a_{i+1}, \ldots, a_j$. For convenience we define $K(i, i-1) = 1$. We see that $K(i,j)$ is the numerator of the continued fraction $[a_i, a_{i+1}, \ldots, a_j]$ and the denominator of the continued fraction $[a_{i-1}, a_i, \ldots, a_j]$. Thus the $K(i,j)$ are identical to the classical *continuants* of Euler [4].

The following theorem, due to Euler, can also be proved by the same tail-swapping technique.

THEOREM 4. *For* $i < m < j < n$,

$$K(i,j)\ K(m,n) - K(i,n)\ K(m,j) = (-1)^{j-m} K(i,m-2) K(j+2,n).$$

This result follows by considering tilings of sub-boards S from cells i to j and T from m to n. Every faulty pair (S,T) corresponds to another faulty pair (S',T') obtained by swapping the tails after the last fault. The term on the right side of Theorems 4 counts the number of fault-free tilings that only occur when the overlapping regions (of S and T, or of S' and T', depending on the parity of $j-m$) consist entirely of dominoes in staggered formation. See FIGURES 11 and 12. Setting $i = 0$ and $m = 1$, Theorem 4 generalizes Theorems 2 and 3 by allowing us to compare arbitrary convergents r_j and r_n.

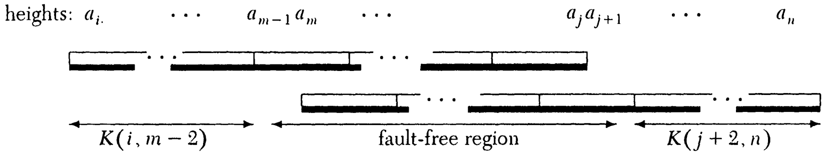

FIGURE 11
When $j-m$ is even, there are $K(i,m-2)K(j+2,n)$ fault-free tilings (S,T).

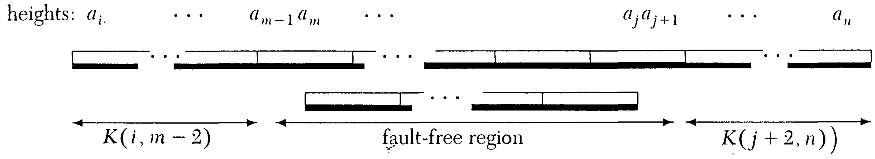

FIGURE 12

When $j - m$ is odd, there are $K(i, m-2)K(j+2, n)$ fault-free tilings (S', T').

Finally, we generalize in a different direction. Suppose we allow dominoes to be stacked as well as squares. Specifically, suppose we impose height conditions $b_1, b_2, \ldots$ so that we may stack as many as b_i dominoes on cells $i-1$ and i. We let $\hat{P}_n$ count the number of ways to tile the board with cells $0, 1, \ldots, n$ and height conditions $a_0, \ldots, a_n$ and $b_1, \ldots, b_n$ for the squares and dominoes respectively. We let $\hat{Q}_n$ count the same problem with cell 0 removed. As before, we see that $\hat{P}_n$ and $\hat{Q}_n$ satisfy

$$\hat{P}_n = a_n \hat{P}_{n-1} + b_n \hat{P}_{n-2}$$

$$\hat{Q}_n = a_n \hat{Q}_{n-1} + b_n \hat{Q}_{n-2}$$

for $n \geq 2$, with initial conditions $\hat{P}_0 = a_0$, $\hat{P}_1 = a_1 a_0 + b_1$, $\hat{Q}_0 = 1$, $\hat{Q}_1 = a_1$. But these are precisely the conditions that define the convergents of the expansion

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots + \cfrac{b_n}{a_n + \ddots}}}}$$

In other words, when the above continued fraction is truncated at the b_n/a_n term, it simplifies to the rational number $\hat{P}_n/\hat{Q}_n$. All of the preceding theorems have generalizations along these lines with similar combinatorial interpretations. We invite the reader to *continue* these investigations.

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