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On an Extension of Bilateral Generating Functions of Modified Jacobi Polynomials from the Existence of Partial-Quasi Bilinear Generating Function

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Abstract: In this note the authors have extended a novel result on bilateral generating functions involving modified Jacobi polynomials from the existence of partial-quasi bilinear generating function of the polynomial under consideration by utilizing group theoretic method. Some special cases of interest are also discussed.

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1. Introduction

In [1], the partial-quasi bilateral generating function is defined as follows:

(1.1) $$G(x,u,w) = \sum_{n=0}^{\infty} a_n \; p_{m+n}^{(\alpha)}(x)\, q_r^{(m+n)}(u)\; w^n ,$$

where a_n - the coefficients are quite arbitrary and $p_{m+n}^{(\alpha)}(x)$, $q_m^{(m+n)}(u)$ are two special functions of orders $m+n, r$ and of parameters $\alpha, m+n$ respectively. If $q_{m+n}^{(n)}(u) \equiv p_r^{(m+n)}(u)$, the generating relation is known as partial-quasi bilinear.

The aim at writing this note is to show that the existence of partial-quasi bilinear generating function involving modified Jacobi polynomials implies the existence of a more general generating relation by using group-theoretic method.

In [2], Chongdar and Chatterjea proved the following theorem on bilateral generating function involving $P_n^{(\alpha\ \beta-n)}(x)$, a modification of Jacobi polynomial by group-theoretic method .

Theorem 1: If there exists a unilateral generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n\ P_n^{(\alpha,\ \beta-n)}(x)\ w^n \tag{1.2}$$

then

$$(1+w)^{\beta}\left[1-\frac{w}{2}(1+x)\right]^{-1-\alpha-\beta} G\left(\frac{x-\frac{w}{2}(1+x)}{1-\frac{w}{2}(1+x)},\ \frac{vw}{(1-w)}\right) \tag{1.3}$$

$$= \sum_{n=0}^{\infty} w^n g_n(v) P_n^{(\alpha,\ \beta-n)}(x)$$

where

$$g_n(v) = \sum_{p=0}^{n} a_p \frac{(p+1)_{n-p}}{(n-p)!} v^p \ . \tag{1.4}$$

Subsequently, Chongdar obtained the following extension of Theorem-1 while investigating a problem on generating function involving modified Jacobi polynomials.

Theorem-2: If there exists a unilateral generating relation of the form:

$$G(x,w) = \sum_{n=0}^{\infty} a_n\ P_{n+m}^{(\alpha,\ \beta-n)}(x)\ w^n \tag{1.5}$$

then

$$(1.6)\qquad (1-w)^{\beta}\left[1-\frac{w}{2}(1+x)\right]^{-1-\alpha-\beta-m} G\left(\frac{x-\frac{w}{2}(1+x)}{1-\frac{w}{2}(1+x)},\ \frac{wt}{(1-w)}\right)$$

$$=\sum_{n=0}^{\infty} w^n g_n(t) P_{n+m}^{(\alpha,\,\beta-n)}(x)\,,$$

where

$$(1.7)\qquad g_n(t)=\sum_{p=0}^{n} a_p \binom{m+n}{m+p} t^p .$$

In this paper, we have obtained a nice extension of theorem-2 stated above from the existence of a partial quasi bilinear generating relation by using group-theoretic method.

In fact, we have obtained the following theorem as the main result of our investigation.

Theorem 3: If there exists a partial quasi-bilinear generating relation involving modified Jacobi polynomials of the following form:

$$(1.8)\qquad G(x,u,w)=\sum_{n=0}^{\infty} a_n\ P_{n+r}^{(\alpha,\,\beta-n)}\ (x)\ P_m^{(\alpha,\,n+r)}\ (u)\ \ w^n$$

then

$$(1.9)\qquad (1+w)^{-1-m-r-\alpha}\ (1+2w)^{\beta}\ [1+w(1+x)]^{-1-\alpha-\beta-m} \times G\left(\frac{x+w(1+x)}{1+w(1+x)},\ \frac{u+w}{1+w},\frac{wv}{(1+w)\,(1+2w)}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{p+q+n}}{p!\ q!}(-2)^p (n+r+1)_p \left(P_{n+p+r}^{(\alpha,\,\beta-n-p)}\ (x)\ \right) \times(-1)^q\ (1+n+\alpha+m+r)_q \left(P_m^{(\alpha,\,n+r+q)}\ (u)\ \right) v^n\ .$$

2 Proof of the theorem

We first consider the following linear partial differential operators [3, 4],

$$R_1 = (1-x^2)\ \frac{\partial}{\partial x}\ .-xy^{-2}\ \frac{\partial}{\partial y}\ -[(1+\alpha+\beta+r)(1+x)-2\beta]\ y \quad .$$

and

$$R_2 = (1-u)\ \frac{\partial}{\partial u} - t^2\ \frac{\partial}{\partial t} - (1+\alpha+m+r)\ t$$

such that

(2.1)
$$R_1(P_{n+r}^{(\alpha,\beta-n)}(x)y^n) = -2(n+r+1)\ P_{n+r+1}^{(\alpha,\ \beta-n-1)}(x)\ y^{n+1}$$

and

(2.2)
$$R_2\left(P_m^{(\alpha,\ n+r)}(u)\ t^n\right) = -(1+n+\alpha+m+r)P_m^{(\alpha,\ n+r+1)}\ (u)\ t^{n+1}\ .$$

The extended form of the groups generated to R_1 and R_2 are given by

(2.3.)
$$e^{wR_1}\ f(x,y) = (1+2wy)^{\beta}\ [1+wy(1+x)]^{-1-\alpha-\beta-m} \times f\ \left(\frac{x+wy(1+x)}{1+wy(1+x)},\ \frac{y}{1+2wy}\right)$$

and

(2.4)
$$e^{wR_2}\ f(u,t) = (1+wt)^{-1-m-r-\alpha}\quad f\ \left(\frac{u+wt}{1+wt},\frac{t}{1+2wt}\right)\ .$$

We now consider the following generating relation:

(2.5)
$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\ P_{n+r}^{(\alpha,\ \beta-n)}\ (x)\ P_m^{(\alpha,\ n+r)}\ (u)\ w^n.$$

Replacing w by $wytv$ in (2.5), we get

(2.6)
$$G(x,u,wytv) = \sum_{n=0}^{\infty} a_n\ \ (wv)^n\ \left(P_{n+r}^{(\alpha,\ \beta-n)}\ (x)\ y^n\right)\left(P_m^{(\alpha,\ n+r)}\ (u)\ t^n\right)\ .$$

Now operating $e^{wR_1}e^{wR_2}$ on both sides of (2.6) we get,

(2.7)
$$e^{w\,R_1}e^{w\,R_2}\ G(x,u,wytv)$$
$$= e^{w\,R_1}e^{w\,R_2}\left[\sum_{n=0}^{\infty} a_n\ \ (wv)^n\ \left(P_{n+r}^{(\alpha,\ \beta-n)}\ (x)\ y^n\right)\left(P_m^{(\alpha,\ n+r)}\ (u)\ t^n\right)\right].$$

Left hand side of (2.7)

(2.8)
$$= e^{w\,R_1}e^{w\,R_2}\ G(x,u,wytv)$$

$$= e^{w\,R_1}\left((1+wt)^{-1-m-r-\alpha}\;\; G\left(x, \frac{u+wt}{1+wt}, \frac{wytv}{1+wt}\right)\right)$$

$$= (1+wt)^{-1-m-r-\alpha}\;(1+2wt)^{\beta}\;[1+wy(1+x)]^{-1-\alpha-\beta-m}$$

$$\times G\left(\frac{x+wy(1+x)}{1+wy(1+x)},\; \frac{u+wt}{1+wt}, \frac{wytv}{(1+wt)\,(1+2wt)}\right).$$

Right hand side of (2.7)

(2.9) $$e^{w\,R_1}\, e^{w\,R_2}\left[\sum_{n=0}^{\infty} a_n\;\;(wv)^n\;\left(P_{n+r}^{(\alpha,\,\beta-n)}\,(x)\;y^n\right)\left(P_m^{(\alpha,\,n+r)}\,(u)\,t^n\right)\right]$$

$$= \sum_{n=0}^{\infty}\;\sum_{p=0}^{\infty}\;\sum_{q=0}^{\infty} a_n\,(wv)^n\,\frac{w^{p+q}}{p!\;\;q!}\left(P_{n+p+r}^{(\alpha,\,\beta-n-p)}\,(x)\;y^{n+p}\right)\left(P_m^{(\alpha,\,n+r+q)}\,(u)\;t^{n+q}\right)$$

$$\times(-2)^p\,(n+r+1)_p\;\;(-1)^q\;\;(1+n+\alpha+m+r)_q\,.$$

Equating (2.8) and (2.9) and then putting $y = t = 1,$ we get

(2.10) $$(1+w)^{-1-m-r-\alpha}\;(1+2w)^{\beta}\;[1+w(1+x)]^{-1-\alpha-\beta-m}$$

$$\times G\left(\frac{x+w(1+x)}{1+w(1+x)},\; \frac{u+w}{1+w}, \frac{wv}{(1+w)\,(1+2w)}\right)$$

$$= \sum_{n=0}^{\infty}\;\sum_{p=0}^{\infty}\;\sum_{q=0}^{\infty} a_n\,\frac{w^{p+q+n}}{p!\;\;q!}\;(-2)^p\,(n+r+1)_p\;\left(P_{n+p+r}^{(\alpha,\,\beta-n-p)}\,(x)\;\right)$$

$$\times(-1)^q\;\;(1+n+\alpha+m+r)_q\left(P_m^{(\alpha,\,n+r+q)}\,(u)\;\right)v^n,$$

which completes the proof of the **Theorem 3**.

Corollary 1: Putting $r = 0$ in (2.10), we get

(2.11) $$(1+w)^{-1-m-\alpha}\;(1+2w)^{\beta}\;[1+w(1+x)]^{-1-\alpha-\beta-m}$$

$$\times G\left(\frac{x+w(1+x)}{1+w(1+x)},\; \frac{u+w}{1+w}, \frac{wv}{(1+w)\,(1+2w)}\right)$$

$$= \sum_{n=0}^{\infty}\;\sum_{p=0}^{\infty}\;\sum_{q=0}^{\infty} a_n\,(wv)^n\,\frac{w^{p+q}}{p!\;\;q!}\;\left(P_{n+p}^{(\alpha,\,\beta-n-p-\alpha)}\,(x)\;\right)$$

$$\times\left(P_m^{(\alpha,\,n+q)}\,(u)\;\right)\times(-2)^p\,(n+1)_p\;\;(-1)^q\;\;(1+n+\alpha+m+)_q\;,$$

a more general generating relation which can be obtained from the existence of quasi bilateral generating relation [5] involving Jacobi polynomial

Corollary 2: Putting $m = 0$, we get

$$(2.12)\qquad (1+2w)^{\beta}\,[1+w(1+x)]^{-1-\alpha-\beta}(1+w)^{-1-r-\alpha} \times G\left(\frac{x+w(1+x)}{1+w(1+x)},\ \frac{wv}{(1+w)\,(1+2w)}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n (-2w)^{n+p}\ P_{n+p+r}^{(\alpha,\,\beta-n-p)}(x)\,(-2)^{-n}\,\frac{(n+r+1)_p}{p!}(1+w)^{-1-n-r-\alpha}\ v^n$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n\ P_{n+p+r}^{(\alpha,\,\beta-n-p)}(x)\left(\frac{(n+r+1)_p}{p!}\left(\frac{-v}{2(1+w)}\right)^n\right)\times(-2w)^{n+p}(1+w)^{-1-r-\alpha}\ .$$

Thus we have

$$(2.13)\qquad (1+2w)^{\beta}\,[1+w(1+x)]^{-1-\alpha-\beta}\times G\left(\frac{x+w(1+x)}{1+wy(1+x)},\ \frac{wv}{(1+w)\,(1+2w)}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n\ P_{n+p+r}^{(\alpha,\,\beta-n-p)}(x)\left(\frac{(n+r+1)_p}{p!}\left(\frac{-v}{2(1+w)}\right)^n\right)\times(-2w)^{n+p}\ .$$

Putting $2w = -w,\ v = -2v$ so that $wv = (-\frac{w}{2})(-2v) = wv$ and replacing $-\frac{v}{2(1-w)}$ by v, we get

$$(1-w)^{\beta}\left[1-\frac{w}{2}(1+x)\right]^{-1-\alpha-\beta} G\left(\frac{x-\frac{w}{2}(1+x)}{1-\frac{w}{2}(1+x)},\ \frac{vw}{(1-w)}\right) = \sum_{n=0}^{\infty} w^n g_n(v)\, P_{n+r}^{(\alpha,\,\beta-n)}(x),$$

where

$$g_n(v) = \sum_{p=0}^{n} a_p \binom{n+r}{p+r} v^p\ ,$$

which is **theorem -2**.

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