NORTH-HOLLAND

On Algebras of Toeplitz Plus Hankel Matrices

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ABSTRACT

We characterize a wide class of maximal algebras of Toeplitz plus Hankel matrices by exploiting properties of displacement operators.

1. INTRODUCTION

Various authors used algebras of Toeplitz plus Hankel matrices in application areas such as spectral problems for Toeplitz matrices [2, 3], preconditioning techniques for the conjugate gradient method [8, 13], and displacement operator based decompositions of matrices [5, 6, 9]. They introduced one or more algebras in a rather ad hoc fashion. In this paper we study algebras of Toeplitz plus Hankel matrices in a systematic way. In particular we give a precise characterization of a wide class of these algebras by using the displacement operators as our main tool and by developing some results obtained in [1] (see also [4]).

A systematic study of algebras of Toeplitz matrices has been performed in [14] with a more direct approach. See also [11].

The outline of the paper is as follows. In Section 2 we recall some relations known as orthogonality conditions for displacement operators. In Section 3 we give a characterization of Toeplitz plus Hankel structure by means of a specific displacement operator. In Section 4 we state the main

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result of the paper concerning certain maximal algebras of Toeplitz plus Hankel matrices. In Section 5 we present some examples.

Throughout the paper we denote by $\mathbb{C}$ the complex field and by $\mathbb{C}^n$ and $\mathbb{C}^{n\times n}$ the spaces of the n-vectors and of the $n \times n$ matrices with entries in $\mathbb{C}$ respectively. We denote by e_i, $i = 1, \dots, n$, the vectors of the canonical basis of $\mathbb{C}^n$, and we set $e = \sum_{i=1}^n e_i$. We denote by $J \in \mathbb{C}^{n\times n}$ the reversion matrix, i.e. $J = (\delta_{i,n-j+1})$, $i, j = 1, \dots, n$, where $\delta_{p,q} = 1$ if $p = q$ and $\delta_{p,q} = 0$ otherwise. Finally, by $\mathscr{T}$ and $\mathscr{H}$ we denote the linear spaces of Toeplitz and Hankel matrices respectively [$A = (a_{i,j})$ is a Toeplitz matrix if $a_{i,j} = \alpha_{j-i}$; $B = (b_{i,j})$ is a Hankel matrix if $b_{i,j} = \beta_{j+i}$].

2. DISPLACEMENT OPERATORS AND ORTHOGONALITY CONDITIONS

In 1979 Kailath et al. [12] introduced and studied various linear operators $\mathscr{K}$: $\mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ of the general form $\mathscr{K}(A) = A - NAM$, with N and M nilpotent matrices. They were called *displacement operators*, and this terminology has also been used for other linear operators studied thereafter in the spirit of the work of Kailath. The main motivation for the interest in the displacement operators has been the extension of effective techniques for Toeplitz matrix computations to more general classes of dense structured matrices. A recent reference to this wide subject is [4].

Here we consider displacement operators of the form

$$\mathscr{L}_X(A) = AX - XA, \tag{2.1}$$

with $X \in \mathbb{C}^{n\times n}$. Note that the operators $\mathscr{K}$ are nonsingular, while the operators $\mathscr{L}_X$ are singular. More precisely, given X, the kernel of the operator $\mathscr{L}_X$ is the set $Z(X) = \{A \in \mathbb{C}^{n\times n} | AX = XA\}$, which is a matrix algebra known as the *centralizer* of X. The theory of centralizers is discussed in [7].

First of all we need to characterize the range of $\mathscr{L}_X$. The same problem has been addressed by Gader in [10]: he studied a singular [though not of the form (2.1)] displacement operator $\mathscr{C}$, and he found necessary and sufficient conditions on a matrix B in order that $B = \mathscr{C}(A)$ for some A. He called such conditions *orthogonality conditions*, since they provide a convenient restatement of the orthogonality relation between the range of a linear operator and the kernel of its adjoint.

The ideas of Gader have been extended to all the operators of the form (2.1) in [9], and have been further discussed in [5]. An interesting algebraic development of Gader's ideas can also be found in [15].

Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be in $\mathbb{C}^{n\times n}$ and let "$\circ$" denote the entrywise matrix product, i.e., $A \circ B = (a_{i,j}b_{i,j})$. Let tr denote the trace operator, defined as $\operatorname{tr} A = \sum_{i=1}^{n} a_{i,i}$. This yields

$$e^T(A \circ B)e = \sum_{i,j=1}^{n} a_{i,j}b_{i,j} = \operatorname{tr}(AB^T).$$

LEMMA 2.1. *Let $X \in \mathbb{C}^{n\times n}$. For every $A \in \mathbb{C}^{n\times n}$ and every $B \in Z(X)$ we have*

$$e^T[\mathscr{L}_X(A) \circ B^T]e = \operatorname{tr}[\mathscr{L}_X(A)B] = 0.$$

Conversely, let $M \in \mathbb{C}^{n\times n}$. If for every $B \in Z(X)$ we have

$$e^T(M \circ B^T)e = \operatorname{tr}(MB) = 0,$$

then M belongs to the range of $\mathscr{L}_X$.

Proof. The proof can be obtained by direct inspection or by noting that the lemma simply restates the orthogonality relation between the range of the operator $\mathscr{L}_X$ and the kernel of its adjoint with respect to the scalar product $\langle A, B\rangle = \operatorname{tr}(AB^H)$. ∎

By using Lemma 2.1 it is possible to prove the following result.

THEOREM 2.1. *Let X, $A \in \mathbb{C}^{n\times n}$. If $B \in Z(X)$ and $\mathscr{L}_X(A) = \sum_{m=1}^{\alpha} x_m y_m^T$ then*

$$\sum_{m=1}^{\alpha} x_m^T B^T y_m = 0.$$

Proof.

$$\begin{aligned}\sum_{m=1}^{\alpha} x_m^T B^T y_m &= \sum_{m=1}^{\alpha} e^T[(x_m y_m^T) \circ B^T]e \\ &= e^T\left[\left(\sum_{m=1}^{\alpha} x_m y_m^T\right) \circ B^T\right]e = e^T[\mathscr{L}_X(A) \circ B^T]e = 0.\end{aligned}$$ ∎

3. CROSS-SUM CONDITION, TOEPLITZ PLUS HANKEL MATRICES, AND DISPLACEMENT OPERATORS

Let $A = (a_{i,j})$ with $i, j = 1, \dots, n$. Suppose that the entries of A satisfy the following *cross-sum* condition:

$$a_{i-1,j} + a_{i+1,j} = a_{i,j-1} + a_{i,j+1}, \qquad i, j = 2, \dots, n-1. \qquad (3.1)$$

What kind of matrix is A? Obviously, every Toeplitz matrix satisfies the condition (3.1) as well as every Hankel matrix. Since the condition (3.1) is linear, every Toeplitz plus Hankel matrix must satisfy it. But what about the converse? If a matrix satisfies the condition (3.1) must it be Toeplitz plus Hankel? The answer is yes: the condition (3.1) is a "local" characterization of Toeplitz plus Hankel structure. Actually, this fact seems to be very little known, even if it is implicit in the results obtained by Bini in [1] (see also [4]). Here, in the line of the work of Bini but with a slightly different technique, we use the concept of displacement operator as our main tool for proving the previous characterization. In Section 4 the same concept will be used for studying the algebras of Toeplitz plus Hankel matrices.

First of all, let us restate the condition (3.1) in a more convenient form. Let us consider the matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

and the linear operator $\mathscr{L}_T$; see (2.1). In addition, let us define the linear space

$$\mathscr{B} = \left\{ e_1 u_1^T + e_n u_2^T + u_3 e_1^T + u_4 e_n^T | u_i \in \mathbb{C}^n \right\}.$$

The elements of $\mathscr{B}$ will be called *frame* matrices. If we denote by S the $(n-2) \times n$ matrix defined as

$$S^T = \begin{pmatrix} e_2 & e_3 & \cdots & e_{n-1} \end{pmatrix},$$

then the property of being a frame matrix can be expressed by using S, namely, $B \in \mathscr{B}$ iff $SBS^T = 0$.

Now, a simple direct check shows that $A = (a_{i,j})$ satisfies the condition (3.1) iff $\mathscr{L}_T(A) \in \mathscr{B}$. Putting it another way, A satisfies the condition (3.1) iff $A \in \mathscr{X}$, where $\mathscr{X}$ is the inverse image under $\mathscr{L}_T$ of $\mathscr{B}$, i.e.,

$$\mathscr{X} = \left\{ A \in \mathbb{C}^{n \times n} | \mathscr{L}_T(A) \in \mathscr{B} \right\}.$$

Thus, we have to prove that $\mathscr{X}$ is the space of Toeplitz plus Hankel matrices. To achieve this result we collect in the following proposition some information on the kernel of the operator $\mathscr{L}_T$, i.e., the algebra $\tau = Z(T)$. For more details the reader is referred to [3, 9].

PROPOSITION 3.1.

(1) *We have* $\dim \tau = n$.

(2) *We have* $\tau = \{\sum_{i=0}^{n-1} a_i T^i | a_i \in \mathbb{C}\}$. *As a consequence, every matrix in* τ *is symmetric and persymmetric and thus centrosymmetric.*

(3) *Let* $T_k = p_{k-1}(T)$, *where* $p_k(\lambda)$, *with* $k = 1, \dots, n$, *is the characteristic polynomial of the* $k \times k$ *top left submatrix of* T, *and* $p_0(\lambda) = 1$. *Then* $e_1^T T_k = e_k^T$. *Moreover, the set* $\{T_1, \dots, T_n\}$ *is a basis of the algebra* τ.

Actually, under suitable hypotheses, the results listed in Proposition 3.1 hold for a wide class of matrix algebras known as Hessenberg algebras, see [9].

Now, we are ready to prove the main result of this section.

THEOREM 3.1. *Let* $\mathscr{T}$ *and* $\mathscr{H}$ *be the linear spaces of Toeplitz and Hankel matrices respectively. Then*

$$\mathscr{X} = \mathscr{T} + \mathscr{H}.$$

Proof. First of all, observe that $\dim \mathscr{T} = \dim \mathscr{H} = 2n - 1$. Moreover, the space $\mathscr{T} \cap \mathscr{H}$ has dimension 2, since it is spanned by the two checkered matrices

$$\begin{pmatrix} 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus we have $\dim(\mathscr{T} + \mathscr{H}) = 4n - 4$.

Since, as already observed, both Toeplitz and Hankel matrices satisfy the condition (3.1), we have $\mathscr{T} \subset \mathscr{X}$ and $\mathscr{H} \subset \mathscr{X}$, so that $\mathscr{T} + \mathscr{H} \subseteq \mathscr{X}$.

Now, we will show that $\dim \mathscr{X} = 4n - 4$.

Remember that the algebra τ is the kernel of the operator $\mathscr{L}_T$. Hence, if we denote with $\mathscr{L}_T(\mathscr{X}) \subseteq \mathscr{B}$ the image of $\mathscr{X}$ under $\mathscr{L}_T$, we have

$$\dim \mathscr{X} = \dim(\tau \cap \mathscr{X}) + \dim \mathscr{L}_T(\mathscr{X}). \tag{3.2}$$

From the inclusion $\mathscr{T} + \mathscr{H} \subseteq \mathscr{X}$ we get $\dim \mathscr{X} \geqslant 4n - 4$, and obviously we have $\dim(\tau \cap \mathscr{X}) = \dim \tau = n$. Now, we want to obtain some information about $\dim \mathscr{L}_T(\mathscr{X})$.

If $B = e_1 u_1^T + e_n u_2^T + u_3 e_1^T + u_4 e_n^T \in \mathscr{L}_T(\mathscr{X})$, then, by virtue of Theorem 2.1 and Proposition 3.1 the following equalities must hold:

$$e_1^T T_k u_1 + e_n^T T_k u_2 + u_3^T T_k e_1 + u_4^T T_k e_n = 0, \qquad k = 1, \dots, n,$$

i.e., by using the symmetry and persymmetry of the matrices T_k (see Proposition 3.1),

$$e_k^T u_1 + e_{n+1-k}^T u_2 + u_3^T e_k + u_4^T e_{n+1-k} = 0, \qquad k = 1, \dots, n.$$

We can write the preceding equalities in a more compact form:

$$u_1 + J u_2 + u_3 + J u_4 = 0.$$

This means that the entries in, say, the first column of B are completely determined when the remaining $3n - 4$ nonzero entries of B are given. Thus $\dim \mathscr{L}_T(\mathscr{X}) \leqslant 3n - 4$. Hence the equality (3.2) can hold only if $\dim \mathscr{X} = 4n - 4$ and $\dim \mathscr{L}_T(\mathscr{X}) = 3n - 4$, proving the thesis and also the inclusion $\tau \subset \mathscr{T} + \mathscr{H}$. ∎

4. ALGEBRAS OF TOEPLITZ PLUS HANKEL MATRICES

Let A be in $\mathbb{C}^{n \times n}$, and consider $\mathscr{L}_T(A)$, the image of A under the operator $\mathscr{L}_T$ defined in Section 3. In the following we will make use of the $(n-2)$-vectors

$$\begin{aligned} a_N^T &= e_1^T A S^T, & \tilde{a}_N^T &= e_1^T \mathscr{L}_T(A) S^T, \\ a_S^T &= e_n^T A S^T, & \tilde{a}_S^T &= e_n^T \mathscr{L}_T(A) S^T, \\ a_W &= S A e_1, & \tilde{a}_W &= S \mathscr{L}_T(A) e_1, \\ a_E &= S A e_n, & \tilde{a}_E &= S \mathscr{L}_T(A) e_n, \end{aligned} \tag{4.1}$$

which lie on the borders of the matrices A and $\mathscr{L}_T(A)$ as shown here:

$$A = \left(\begin{array}{c|c|c} & a_N^T & \\ \hline a_W & & a_E \\ \hline & a_S^T & \end{array}\right), \qquad \mathscr{L}_T(A) = \left(\begin{array}{c|c|c} & \tilde{a}_N^T & \\ \hline \tilde{a}_W & & \tilde{a}_E \\ \hline & \tilde{a}_S^T & \end{array}\right).$$

By means of these vectors we construct the following matrices:

$$M = \begin{pmatrix} a_W & a_E & \tilde{a}_W & \tilde{a}_E \end{pmatrix}, \qquad N = \begin{pmatrix} \tilde{a}_N^T \\ \tilde{a}_S^T \\ a_N^T \\ a_S^T \end{pmatrix}.$$

In this section the images under $\mathscr{L}_T$ of products of matrices will be considered. For this reason, let us state the following:

LEMMA 4.1. *Let A and X be in $\mathbb{C}^{n\times n}$. The following equations hold:*

$$\mathscr{L}_X(AB) = \mathscr{L}_X(A)B + A\mathscr{L}_X(B); \tag{4.2}$$

$$\mathscr{L}_X(A_1A_2 \cdots A_m) = \sum_{k=1}^{m} A_1 \cdots A_{k-1}\mathscr{L}_X(A_k)A_{k+1} \cdots A_m. \tag{4.3}$$

Proof. (4.2): obvious. (4.3): by induction on m. ∎

THEOREM 4.1. *Let $A \in \mathscr{T}+\mathscr{H}$. Then $A^2 \in \mathscr{T}+\mathscr{H}$ iff $MN = 0$.*

Proof. As we already observed in Section 3, A^2 is a Toeplitz plus Hankel matrix iff $\mathscr{L}_T(A^2)$ is a frame matrix, or, in other terms, iff $S\mathscr{L}_T(A^2)S^T = 0$. Since, by Lemma 4.1,

$$\mathscr{L}_T(A^2) = \mathscr{L}_T(A)A + A\mathscr{L}_T(A),$$

we have that A^2 is a Toeplitz plus Hankel matrix iff

$$S[\mathscr{L}_T(A)A + A\mathscr{L}_T(A)]S^T = 0. \tag{4.4}$$

Taking into account the structures of S and of $\mathscr{L}_T(A)$, which is a frame matrix, we have also

$$S\mathscr{L}_T(A)AS^T = \begin{pmatrix} \tilde{a}_W & \tilde{a}_E \end{pmatrix} \begin{pmatrix} a_N^T \\ a_S^T \end{pmatrix},$$

and

$$SA\mathscr{L}_T(A)S^T = \begin{pmatrix} a_W & a_E \end{pmatrix} \begin{pmatrix} \tilde{a}_N^T \\ \tilde{a}_S^T \end{pmatrix}.$$

The thesis follows from these equalities and from Equation (4.4). ■

COROLLARY 4.1. *If both A and A^2 are in $\mathscr{T}+\mathscr{H}$, then* rank M + rank N $\leqslant 4$.

Proof. By Theorem 4.1 we have $MN = 0$; thus null $M \geqslant$ rank N. The thesis is proved by taking into account the equality rank M + null $M = 4$. ■

In the following a special kind of frame matrices is involved, that is, matrices V of the form

$$V = ae_1e_1^T + \beta e_ne_n^T + \delta e_1e_n^T + \gamma e_ne_1^T = \left(\begin{array}{c|ccc|c} \alpha & 0 & \cdots & 0 & \delta \\ \hline 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \cdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \hline \gamma & 0 & \cdots & 0 & \beta \end{array}\right)$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. We call these matrices *corner* matrices. The following lemmas hold.

LEMMA 4.2. *Let V be a corner matrix. Then, for any matrix A, $\mathscr{L}_T(A)$ is a frame matrix iff $\mathscr{L}_{T+V}(A)$ is a frame matrix.*

Proof. We have

$$\begin{aligned} S\mathscr{L}_{T+V}(A)S^T &= S\mathscr{L}_T(A)S^T + S\mathscr{L}_V(A)S^T \\ &= S\mathscr{L}_T(A)S^T + SAVS^T - SVAS^T \\ &= S\mathscr{L}_T(A)S^T, \end{aligned}$$

since $(SV)^T = VS^T = 0$. ■

LEMMA 4.3. *Let V and V' be two corner matrices in $\mathbb{C}^{n\times n}$ with $n \geqslant 4$. If $Z(T+V) = Z(T+V')$ then $V = V'$.*

Proof. We have $(T+V)(T+V') = (T+V')(T+V)$, which gives $TV' + VT + VV' = V'T + TV + V'V$. A direct inspection of both the sides of this equation shows that V and V' must be the same corner matrix. ■

The next theorem shows that the centralizers of matrices having the form $T+V$, where V is a corner matrix, are algebras of Toeplitz plus Hankel matrices.

THEOREM 4.2. *Let V be a corner matrix. Then $Z(T+V) \subseteq \mathscr{T} + \mathscr{H}$.*

Proof. If $A \in Z(T+V)$, then $\mathscr{L}_{T+V}(A) = 0$, which means that $\mathscr{L}_{T+V}(A)$ is a frame matrix, thus by Lemma 4.2 also $\mathscr{L}_T(A)$ is a frame matrix. In other words, $A \in \mathscr{T} + \mathscr{H}$. ■

It is not true that every algebra in $\mathscr{T} + \mathscr{H}$ is a centralizer $Z(T+V)$ for a corner matrix V. However this is the case under suitable hypotheses. First of all let us introduce the following Condition 1, involving some of the vectors defined in (4.1):

CONDITION 1. (a_N, a_S) and (a_W, a_E) are pairs of linearly independent vectors.

If a matrix $A \in \mathscr{T} + \mathscr{H}$ satisfies Condition 1, then we are able to give a criterion for A to belong to a centralizer $Z(T+V)$.

THEOREM 4.3. *Let A, A^2, and A^3 be in $\mathscr{T} + \mathscr{H}$. Moreover, assume that A satisfies Condition 1. Then there exists a corner matrix V such that $A \in Z(T+V)$.*

Proof. By virtue of Condition 1 and Corollary 4.1, we have rank $M =$ rank $N = 2$, and therefore two 2×2 complex matrices F and F' exist such that

$$\begin{pmatrix} \tilde{a}_W & \tilde{a}_E \end{pmatrix} = \begin{pmatrix} a_W & a_E \end{pmatrix} F'$$

and

$$\begin{pmatrix} \tilde{a}_N^T \\ \tilde{a}_S^T \end{pmatrix} = F \begin{pmatrix} a_N^T \\ a_S^T \end{pmatrix}.$$

By Theorem 4.1 we have

$$MN = \begin{pmatrix} a_W & a_E \end{pmatrix} \begin{pmatrix} I_{2\times 2} & F' \end{pmatrix} \begin{pmatrix} F \\ I_{2\times 2} \end{pmatrix} \begin{pmatrix} a_N^T \\ a_S^T \end{pmatrix} = 0,$$

and this implies, by Condition 1,

$$F + F' = 0. \tag{4.5}$$

Set

$$F = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix},$$

and let us consider the image of A under $\mathscr{L}_V$, where

$$V = \alpha e_1 e_1^T + \beta e_n e_n^T + \delta e_1 e_n^T + \gamma e_n e_1^T = \begin{pmatrix} e_1 & e_n \end{pmatrix} F \begin{pmatrix} e_1^T \\ e_n^T \end{pmatrix}. \tag{4.6}$$

Since A is a Toeplitz plus Hankel matrix, by Lemma 4.2 both $\mathscr{L}_T(A)$ and $\mathscr{L}_{T+V}(A)$ are frame matrices. Moreover, we have

$$S\mathscr{L}_T(A) = \begin{pmatrix} \tilde{a}_W & \tilde{a}_E \end{pmatrix} \begin{pmatrix} e_1^T \\ e_n^T \end{pmatrix} = \begin{pmatrix} a_W & a_E \end{pmatrix} F' \begin{pmatrix} e_1^T \\ e_n^T \end{pmatrix}$$

and

$$S\mathscr{L}_V(A) = SAV - SVA = SAV = \begin{pmatrix} a_W & a_E \end{pmatrix} F \begin{pmatrix} e_1^T \\ e_n^T \end{pmatrix};$$

thus

$$\begin{aligned} S\mathscr{L}_{T+V}(A) &= S\mathscr{L}_T(A) + S\mathscr{L}_V(A) \\ &= \begin{pmatrix} a_W & a_E \end{pmatrix} F' + F \begin{pmatrix} e_1^T \\ e_n^T \end{pmatrix} = 0, \end{aligned} \tag{4.7}$$

as a consequence of Equation (4.5). We can see in the same way that

$$\mathscr{L}_{T+V}(A)S^T = 0. \tag{4.8}$$

Equations (4.7) and (4.8) imply that $\mathscr{L}_{T+V}(A)$ is a corner matrix, that is,

$$\mathscr{L}_{T+V}(A) = \begin{pmatrix} e_1 & e_n \end{pmatrix} C \begin{pmatrix} e_1^T \\ e_n^T \end{pmatrix}, \tag{4.9}$$

where $C \in \mathbb{C}^{2\times 2}$.

Now let us exploit the assumption that also A^3 is in $\mathscr{T}+\mathscr{H}$, which is equivalent to say

$$S\mathscr{L}_T(A^3)S^T = 0,$$

or, by Lemma 4.2,

$$S\mathscr{L}_{T+V}(A^3)S^T = 0. \tag{4.10}$$

We observe that by Equation (4.3),

$$\mathscr{L}_X(A^3) = \mathscr{L}_X(A)A^2 + A\mathscr{L}_X(A)A + A^2\mathscr{L}_X(A),$$

thus we obtain from Equations (4.10), (4.7), and (4.8)

$$\begin{aligned} &S\mathscr{L}_{T+V}(A)A^2S^T + SA\mathscr{L}_{T+V}(A)AS^T + SA^2\mathscr{L}_{T+V}(A)S^T \\ &\quad = SA\mathscr{L}_{T+V}(A)AS^T = 0. \end{aligned}$$

This equation, taking into account Equation (4.9) and the structure of S, can be rewritten as

$$\begin{aligned} &SA\mathscr{L}_{T+V}(A)AS^T \\ &\quad = \begin{pmatrix} a_W & a_E \end{pmatrix} \begin{pmatrix} e_1^T \\ e_n^T \end{pmatrix} C \begin{pmatrix} e_1 & e_n \end{pmatrix} C \begin{pmatrix} e_1^T \\ e_n^T \end{pmatrix} \begin{pmatrix} e_1 & e_n \end{pmatrix} \begin{pmatrix} a_N^T \\ a_S^T \end{pmatrix} \\ &\quad = \begin{pmatrix} a_W & a_E \end{pmatrix} C \begin{pmatrix} a_N^T \\ a_S^T \end{pmatrix} = 0, \end{aligned}$$

which, by Condition 1, implies $C = 0$, and therefore (4.9) becomes

$$\mathscr{L}_{T+V}(A) = 0,$$

which is the thesis. ∎

The fact that a Toeplitz plus Hankel matrix A satisfies Condition 1 has deep consequences for any algebra of Toeplitz plus Hankel matrices containing A.

THEOREM 4.4. *Let $\mathscr{A}$ be an algebra of Toeplitz plus Hankel matrices. If there exists a matrix $A \in \mathscr{A}$ satisfying Condition* 1, *then for any corner matrix V such that $A \in Z(T + V)$ we have $\mathscr{A} \subseteq Z(T + V)$.*

Proof. Let B be in $\mathscr{A}$, and set

$$\tilde{b}_N^T = e_1^T \mathscr{L}_{T+V}(B) S^T, \qquad \tilde{b}_S^T = e_n^T \mathscr{L}_{T+V}(B) S^T,$$

$$\tilde{b}_W = S \mathscr{L}_{T+V}(B) e_1, \qquad \tilde{b}_E = S \mathscr{L}_{T+V}(B) e_n.$$

We will prove that $B \in Z(T + V)$. We have that AB, being in $\mathscr{A}$ too, is a Toeplitz plus Hankel matrix, and therefore $\mathscr{L}_T(AB) = \mathscr{L}_T(A)B + A\mathscr{L}_T(B)$ is a frame matrix. By Lemma 4.2, this means $S\mathscr{L}_{T+V}(AB)S^T = S\mathscr{L}_{T+V}(A)BS^T + SA\mathscr{L}_{T+V}(B)S^T = 0$. This equation, since $\mathscr{L}_{T+V}(A) = 0$, gives

$$SA\mathscr{L}_{T+V}(B)S^T = 0. \tag{4.11}$$

$\mathscr{L}_{T+V}(B)$ is a frame matrix; thus Equation (4.11) can be rewritten in terms of the vectors lying on the borders of A and of $\mathscr{L}_{T+V}(B)$:

$$\begin{pmatrix} a_W & a_E \end{pmatrix} \begin{pmatrix} \tilde{b}_N^T \\ \tilde{b}_S^T \end{pmatrix} = 0,$$

which implies, by Condition 1, $\tilde{b}_N^T = \tilde{b}_S^T = 0$.

If we exploit the fact that $BA \in \mathscr{A}$, we find that $\tilde{b}_W = \tilde{b}_E = 0$. Thus we have that $\mathscr{L}_{T+V}(B)$ is a corner matrix:

$$\mathscr{L}_{T+V}(B) = \begin{pmatrix} e_1 & e_n \end{pmatrix} C \begin{pmatrix} e_1^T \\ e_n^T \end{pmatrix}, \tag{4.12}$$

where $C \in \mathbb{C}^{2\times 2}$.

Finally, in order to show that $C = 0$, we note that $ABA \in \mathscr{A}$ and therefore $\mathscr{L}_{T+V}(ABA)$ is a frame matrix, that is

$$S\mathscr{L}_{T+V}(ABA)S^T = 0. \tag{4.13}$$

Observe that, by Equation (4.3), we have

$$\begin{aligned}\mathscr{L}_{T+V}(ABA) &= \mathscr{L}_{T+V}(A)BA + A\mathscr{L}_{T+V}(B)A + AB\mathscr{L}_{T+V}(A)\\ &= A\mathscr{L}_{T+V}(B)A;\end{aligned}$$

thus Equations (4.12) and (4.13) give

$$SA\mathscr{L}_{T+V}(B)AS^T = (a_W \quad a_E)C\begin{pmatrix} a_N^T \\ a_S^T \end{pmatrix} = 0.$$

This equation, by Condition 1, implies $C = 0$, and by (4.12)

$$\mathscr{L}_{T+V}(B) = 0,$$

thus concluding the proof. ■

As a consequence of Theorem 4.4, a Toeplitz plus Hankel matrix satisfying Condition 1 can belong to a centralizer $Z(T + V)$ for a unique corner matrix V. This is shown by the following:

COROLLARY 4.2. *Let A satisfy Condition 1. If $A \in Z(T + V) \cap Z(T + V')$, then $V = V'$.*

Proof. As A obeys Condition 1, both $Z(T + V)$ and $Z(T + V')$ can be taken as the algebra $\mathscr{A}$ involved in Theorem 4.4, which implies $Z(T + V) = Z(T + V')$. By Lemma 4.3, this means $V = V'$. ■

An algebra is maximal in $\mathscr{T} + \mathscr{H}$ if it is not a proper subalgebra of an algebra in $\mathscr{T} + \mathscr{H}$. The next theorem gives a characterization of all maximal algebras in $\mathscr{T} + \mathscr{H}$ containing a matrix satisfying Condition 1.

THEOREM 4.5. *For any corner matrix V, $Z(T + V)$ is a maximal algebra of Toeplitz plus Hankel matrices.*

Conversely, let $\mathscr{A}$ be a maximal algebra of Toeplitz plus Hankel matrices. If there exists a matrix $A \in \mathscr{A}$ satisfying Condition 1, then $\mathscr{A} = Z(T + V)$ for a unique corner matrix V.

Proof. If V is a given corner matrix, then $Z(T+V) \subset \mathcal{T}+\mathcal{H}$, by Theorem 4.2. If $\mathcal{A}$ is an algebra such that

$$Z(T+V) \subseteq \mathcal{A} \subset \mathcal{T}+\mathcal{H},$$

then $\mathcal{A}$ agrees with the assumptions of Theorem 4.4, as $T+V$ satisfies Condition 1. Henceforth

$$\mathcal{A} \subseteq Z(T+V).$$

The above inclusions prove that $\mathcal{A} = Z(T+V)$, that is $Z(T+V)$ is maximal.

Let us turn to the converse statement. By virtue of Theorem 4.3, we have that a corner matrix V exists such that $A \in Z(T+V)$. By applying Theorem 4.4 to $\mathcal{A}$ we obtain the inclusion $\mathcal{A} \subseteq Z(T+V)$. The equality $\mathcal{A} = Z(T+V)$ follows from the maximality of $\mathcal{A}$. The uniqueness of V is a consequence of Corollary 4.2. ∎

5. EXAMPLES

Throughout this section we use the upper shift matrix $U = (\delta_{i+1,j})$, $i, j = 1, \ldots, n$, and the matrices T and S defined in Section 3. Unless otherwise stated, matrices are square of order n.

The results in Section 4 completely characterize the matrix algebras of the form $Z(T+V)$. Various of these algebras have been used for applicative purposes. For example, the algebras $Z(T+V)$ with $V = \delta e_1 e_n^T + \gamma e_n e_1^T$ where $(\delta, \gamma) \in \{(1,1), (-1,-1)\}$ have been used in [5]. The ones with $V = \alpha e_1 e_1^T + \beta e_n e_n^T$ where $(\alpha, \beta) \in \{(0,0), (1,1), (-1,-1), (1,-1), (-1,1)\}$ and where $(\alpha, \beta) = (0,1)$ have been used in [6] and in [16] respectively.

Actually, to our knowledge, the algebras $Z(T+V)$ and their subalgebras, together with the Toeplitz triangular matrices (defined later on), are the only algebras in $\mathcal{T}+\mathcal{H}$ that have received attention in the literature. Nonetheless, it would be interesting to obtain a complete map of the algebras in $\mathcal{T}+\mathcal{H}$. Perhaps the examples in this section will be of some interest.

In the second part of the Theorem 4.5 the following two hypotheses appear:

(Hp1) $\mathcal{A}$ is a maximal algebra of Toeplitz plus Hankel matrices;
(Hp2) there exists a matrix $A \in \mathcal{A}$ satisfying Condition 1.

The first one is not really restrictive (clearly, once one has found maximal algebras, one can study their subalgebras). Concerning the second one, we note that (1) it is fairly simple to check and (2) it allows us to characterize precisely the algebras $Z(T + V)$. Moreover, conditions of linear dependence of the vectors $a_W, a_E, \tilde{a}_W, \tilde{a}_E$ and $a_N, a_S, \tilde{a}_N, \tilde{a}_S$ might be used in order to locate other maximal algebras in $\mathscr{T}+\mathscr{H}$ not satisfying hypothesis (Hp2). At this regard, let us consider a couple of examples.

EXAMPLE 5.1. Let $T \in \mathbb{C}^{(n-2)\times(n-2)}$ with $n \geqslant 2$ and let $\mathscr{V}$ be the linear space of all matrices $A \in \mathbb{C}^{n\times n}$ such that

$$A = \left(\begin{array}{c|ccc|c} a & 0 & \cdots & 0 & b \\ \hline 0 & & & & 0 \\ \vdots & & B & & \vdots \\ 0 & & & & 0 \\ \hline d & 0 & \cdots & 0 & c \end{array}\right),$$

where $B \in Z(T)$. It is easy to check that $\mathscr{V}$ is an algebra in $\mathscr{T}+\mathscr{H}$ of dimension $n + 2$. Clearly $\mathscr{V}$ does not satisfy hypothesis (Hp2).

Observe (and compare with Theorem 4.1) that the algebra $\mathscr{V}$ is the set of the matrices in $\mathscr{T}+\mathscr{H}$ such that $a_W = a_E = a_N = a_S = 0$. In the next example we consider the matrices in $\mathscr{T}+\mathscr{H}$ such that $a_W = a_E = \tilde{a}_W = \tilde{a}_E = 0$.

EXAMPLE 5.2. First of all, let us define the following matrices:

$$[T_1]_{i,j} = \begin{cases} 0 & \text{if} \quad i > j, \\ 1 & \text{if} \quad i \leqslant j \text{ and } i + j \text{ is even}, \\ 0 & \text{if} \quad i \leqslant j \text{ and } i + j \text{ is odd}, \end{cases}$$

$$[T_2]_{i,j} = \begin{cases} 0 & \text{if} \quad i > j, \\ 0 & \text{if} \quad i \leqslant j \text{ and } i + j \text{ is even}, \\ 1 & \text{if} \quad i \leqslant j \text{ and } i + j \text{ is odd}, \end{cases}$$

$i, j = 1, \dots, n$, and

$$A = \begin{cases} T_1 - JT_2 & \text{if } n \text{ is even,} \\ T_1 - JT_1 & \text{if } n \text{ is odd,} \end{cases} \qquad B = \begin{cases} T_2 - JT_1 & \text{if } n \text{ is even,} \\ T_2 - JT_2 & \text{if } n \text{ is odd.} \end{cases}$$

Obviously I, J, A, B are linearly independent matrices (it is sufficient to compare their first rows) and belong to $\mathscr{T} + \mathscr{H}$. Using the identities

$$\begin{aligned} AJ &= B + J && \text{if } n \text{ is even,} \\ AB &= 0 && \text{if } n \text{ is even,} \\ AJ &= A - I + J && \text{if } n \text{ is odd,} \\ BJ &= B && \text{if } n \text{ is odd,} \end{aligned}$$

whose simple proof is left to the reader, we can construct the following multiplication tables:

	I	J	A	B
I	I	J	A	B
J	J	I	$-B$	$-A$
A	A	$B+J$	A	0
B	B	$A-I$	B	0

n even,

	I	J	A	B
I	I	J	A	B
J	J	I	$-A$	$-B$
A	A	$A-I+J$	A	B
B	B	B	0	0

n odd.

From these tables we deduce that the linear space $\mathscr{A} = \text{span}\{I, J, A, B\}$ is closed under multiplication and hence it is an algebra of dimension 4 in $\mathscr{T} + \mathscr{H}$. Clearly this algebra does not satisfy hypothesis (Hp2).

REMARK 5.1. Let $\mathscr{A}$ be an algebra in $\mathscr{T} + \mathscr{H}$ that does not satisfy hypothesis (Hp2). Then $\mathscr{A}$ and/or $\mathscr{A}^T$ are made up of matrices A such that a_W and a_E are linearly dependent. In fact, if A and B were matrices in $\mathscr{A}$ such that a_W and a_E are linearly independent and b_N and b_S are linearly independent, the matrix $\alpha A + \beta B$ would satisfy Condition 1 for certain α and β. In the following let us assume that $\mathscr{A}$ is an algebra made up of matrices A such that a_W and a_E are linearly dependent. There are two possibilities:

(i) There exists a vector u such that $a_W = \lambda_A u$ and $a_E = \mu_A u$ for any $A \in \mathscr{A}$. This is the case for the algebras presented in Examples 5.1 and 5.2.

(ii) There exist two linear independent vectors u and v and two matrices $A, B \in \mathscr{A}$ such that $a_W = u$ and $b_W = v$ (or $a_E = u$ and $b_E = v$). This is the case for the algebra of the Toeplitz triangular matrices, presented later on.

Let us consider case (ii). If A and B are as before and $a_E = \lambda_A a_W = \lambda_A u$ and $b_E = \lambda_B b_W = \lambda_B v$ then $\lambda_A = \lambda_B = \lambda$. Moreover if $C \in \mathscr{A}$ and $c_E = 0$ then $c_W = 0$. With these observations at hand it is possible to prove that the vector $w = (-\lambda, 0, \dots, 0, 1)^T$ is a common eigenvector of all the matrices in $\mathscr{A}$. We leave the verification to the reader. ■

We close the paper showing that hypotheses (Hp1) and (Hp2) are independent. In fact, all of the following four cases can occur:

(c1) (Hp1) true, (Hp2) true;
(c2) (Hp1) false, (Hp2) true;
(c3) (Hp1) false, (Hp2) false;
(c4) (Hp1) true, (Hp2) false.

Case (c1): Let $\mathscr{A} = Z(T) = \tau \subset \mathscr{T} + \mathscr{H}$. Hypothesis (Hp1) is true, since the first part of Theorem 4.5 implies that $\mathscr{A}$ is a maximal algebra in $\mathscr{T} + \mathscr{H}$. Hypothesis (Hp2) is also true, since $T \in \mathscr{A}$ satisfies Condition 1. Observe that the algebra $Z(T^2)$ is such that $Z(T) \subset Z(T^2)$ and $\dim Z(T) < \dim Z(T^2)$, but this is not a contradiction, since $Z(T^2) \not\subset \mathscr{T} + \mathscr{H}$.

Case (c2): Let us set $C = U + e_n e_1^T$ and consider the algebra $\mathscr{A} = Z(C) \subset \mathscr{T}$, known as the algebra of circulant matrices (see [7]). The equality $C^T = C^{n-1}$ yields $C + C^T \in \mathscr{A}$. Since $C + C^T$ satisfies Condition 1, hypothesis (Hp2) is true. Nevertheless, hypothesis (Hp1) is false. In fact, it turns out that $Z(C) \subset Z(C + C^T)$ and $\dim Z(C) < \dim Z(C + C^T)$; see [5]. Moreover, $Z(C + C^T) = Z(T + V)$ with $V = e_n e_1^T + e_1 e_n^T$. Therefore, by Theorem 4.2, $Z(C + C^T) \subset \mathscr{T} + \mathscr{H}$.

Case (c3): Let $\mathscr{A} = \operatorname{span}\{I, ee^T\}$. Hypothesis (Hp2) is false since no matrix in $\mathscr{A}$ satisfies Condition 1. Hypothesis (Hp1) is false as well, since, for example, we have $\mathscr{A} \subset Z(C)$ and $\dim \mathscr{A} < \dim Z(C)$.

Case (c4): Finally, let us consider the algebra $\mathscr{A} = Z(U)$, known as the algebra of Toeplitz upper triangular matrices. Hypothesis (Hp2) is false, since $a_W = SAe_1 = a_S = SA^T e_n = 0$ for every matrix $A \in \mathscr{A}$. However, we will show that $\mathscr{A}$ is a maximal algebra in $\mathscr{T} + \mathscr{H}$, i.e., hypothesis (Hp1) is true.

Let $\mathscr{F} \subset \mathscr{T} + \mathscr{H}$ be an algebra such that

$$\mathscr{A} \subseteq \mathscr{F}. \tag{5.1}$$

For every $F \in \mathscr{F}$ we have $U^2 F, FU^2, U^2 FU^2 \in \mathscr{F}$ and

$$S\mathscr{L}_T(U^2 F) S^T = 0,$$

$$S\mathscr{L}_T(FU^2) S^T = 0,$$

$$S\mathscr{L}_T(U^2 FU^2) S^T = 0.$$

By applying Lemma 4.1, the previous relations can be rewritten as

$$S\mathscr{L}_T(U^2)FS^T + SU^2\mathscr{L}_T(F)S^T = 0,$$

$$S\mathscr{L}_T(F)U^2S^T + SF\mathscr{L}_T(U^2)S^T = 0,$$

$$S\mathscr{L}_T(U^2)FU^2S^T + SU^2\mathscr{L}_T(F)U^2S^T + SU^2F\mathscr{L}_T(U^2)S^T = 0.$$

Hence, taking into account that $\mathscr{L}_T(F)$ is a frame matrix, we obtain

$$F = \left(\begin{array}{c|ccc} & & & \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \\ \hline 0 & 0 & \cdots & 0 \end{array}\right),$$

$$\mathscr{L}_T(F) = \left(\begin{array}{c|ccc} & & & \\ \hline 0 & & & \\ \vdots & & 0 & \\ 0 & & & \\ \hline 0 & 0 & \cdots & 0 \end{array}\right).$$

Let us set $G = F - \tilde{F}$, where $\tilde{F}$ is the Toeplitz upper triangular matrix such that $e_1^T\tilde{F} = e_1^TF$. We have

$$G = \left(\begin{array}{c|ccc|c} 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & & & & \\ \vdots & & & & \\ 0 & & & & \\ \hline 0 & 0 & \cdots & 0 & \end{array}\right),$$

$$\mathscr{L}_T(G) = \left(\begin{array}{c|ccc|c} & & & & \\ \hline 0 & & & & \\ \vdots & & 0 & & \\ 0 & & & & \\ \hline 0 & 0 & \cdots & 0 & \end{array}\right).$$

Moreover, since $T = U + U^T$,

$$\mathscr{L}_T(G) = GU - UG + GU^T - U^TG. \tag{5.2}$$

If we use the relation (5.2) and take into account the structures of G and $\mathscr{L}_T(G)$, we can proceed by induction on i to show that $Ge_i = 0$, $i = 1, \dots, n$. Hence $G = 0$ and $F = \tilde{F}$, so that

$$\mathscr{F} \subseteq \mathscr{A}. \tag{5.2}$$

The relations (5.1) and (5.3) yield $\mathscr{F} = \mathscr{A}$, and this proves that $\mathscr{A}$ is maximal in $\mathscr{T} = \mathscr{H}$.

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