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PROBABILITY LAWS RELATED TO THE JACOBI THETA AND RIEMANN ZETA FUNCTIONS, AND BROWNIAN EXCURSIONS

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Abstract. This paper reviews known results which connect Riemann's integral representations of his zeta function, involving Jacobi's theta function and its derivatives, to some particular probability laws governing sums of independent exponential variables. These laws are related to one-dimensional Brownian motion and to higher dimensional Bessel processes. We present some characterizations of these probability laws, and some approximations of Riemann's zeta function which are related to these laws.

Contents

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1. Introduction

In his fundamental paper [81], Riemann showed that the Riemann zeta function, initially defined by the series

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} \qquad (\Re s > 1) \tag{1.1}$$

admits a meromorphic continuation to the entire complex plane, with only a simple pole at 1, and that the function

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s) \qquad (\Re s > 1) \tag{1.2}$$

is the restriction to $(\Re s > 1)$ of a unique entire analytic function ξ, which satisfies the functional equation

$$\xi(s) = \xi(1-s) \tag{1.3}$$

for all complex s. These basic properties of ζ and ξ follow from a representation of 2ξ as the Mellin transform of a function involving derivatives of Jacobi's theta function. This function turns out to be the density of a probability distribution on the real line, which has deep and intriguing connections with the theory of Brownian motion.

This distribution first appears in the probabilistic literature in the 1950's in the work of Feller [31], Gnedenko [33], and Tákacs [90], who derived it as the asymptotic distribution as $n \to \infty$ of the range of a simple one-dimensional random walk conditioned to return to its origin after $2n$ steps, and found formula (1.5) below for $s = 1, 2, \cdots$. Combined with the approximation of random walks by Brownian motion, justified by Donsker's theorem [11], [26], [80], the random walk asymptotics imply that if

$$Y := \sqrt{\frac{2}{\pi}}\left(\max_{0\le u\le 1} b_u - \min_{0\le u\le 1} b_u\right) \tag{1.4}$$

where $(b_u, 0 \le u \le 1)$ is the standard Brownian bridge derived by conditioning a one-dimensional Brownian motion $(B_u, 0 \le u \le 1)$ on $B_0 = B_1 = 0$, then

$$E(Y^s) = 2\xi(s) \qquad (s \in \mathbb{C}) \tag{1.5}$$

where E is the expectation operator. Many other constructions of random variables with the same distribution as Y have since been discovered, involving functionals of the path of a Brownian motion or Brownian bridge in $\mathbb{R}^d$ for $d = 1, 2, 3$ or 4.

Our main purpose in this paper is to review this circle of ideas, with emphasis on the probabilistic interpretations such as (1.4)-(1.5) of various functions which play an important role in analytic number theory. For the most part this is a survey of known results, but the result of Section 5 may be new. As general background references we mention [40], [66], [83], [91] for analytic number theory related to the Riemann zeta function, and [11], [26], [80] for probability and stochastic processes.

Section 2 reviews the classical analysis underlying (1.5) and offers different analytic characterizations of the probability distribution of Y. We also point out how Li's criterion [59] for the Riemann Hypothesis can be expressed in terms of the sequence of cumulants of $\log(1/Y)$. Section 3 presents various formulae related to the distributions of the random variables S_h and C_h defined by

$$S_h := \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\Gamma_{h,n}}{n^2} \quad \text{and} \quad C_h := \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\Gamma_{h,n}}{(n-\frac{1}{2})^2}, \tag{1.6}$$

for independent random variables $\Gamma_{h,n}$ with the *gamma*(h) *density*

$$P(\Gamma_{h,n} \in dx)/dx = \Gamma(h)^{-1}x^{h-1}e^{-x} \qquad (h > 0, x > 0). \tag{1.7}$$

We use the notation S for sinh, C for cosh, as a mnemonic for the Laplace transforms

$$E\left[e^{-\lambda S_h}\right] = \left(\frac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}\right)^h \quad \text{and} \quad E[e^{-\lambda C_h}] = \left(\frac{1}{\cosh\sqrt{2\lambda}}\right)^h. \tag{1.8}$$

We were motivated to study the laws of S_h and C_h both by the connection between these laws and the classical functions of analytic number theory and by the repeated appearances of these laws in the study of Brownian motion, which we recall in Section 4. To illustrate, the law of Y featured in (1.4) and (1.5) is characterized by the equality in distribution

$$Y \overset{d}{=} \left(\frac{\pi}{2}S_2\right)^{\frac{1}{2}}. \tag{1.9}$$

As we discuss in Section 4, Brownian paths possess a number of distributional symmetries, which explain some of the remarkable coincidences in distribution implied by the repeated appearances of the laws of S_h and C_h for various h. See also [75] for further study of these laws. Section 5 shows how one of the probabilistic results of Section 3 leads us to an approximation of the zeta function, valid in the entire complex plane, which is similar to an approximation obtained by Sondow [88]. We conclude in Section 6 with some consideration of the Hurwitz zeta function and Dirichlet L-functions, and some references to other work relating the Riemann zeta function to probability theory.

2. Probabilistic interpretations of some classical analytic formulae

2.1. **Some classical analysis.** Let us start with Jacobi's theta function identity

$$\frac{1}{\sqrt{\pi t}}\sum_{n=-\infty}^{\infty} e^{-(n+x)^2/t} = \sum_{n=-\infty}^{\infty}\cos(2n\pi x)\, e^{-n^2\pi^2 t} \qquad (x \in \mathbb{R},\, t > 0) \tag{2.1}$$

which is a well known instance of the Poisson summation formula [6]. This identity equates two different expressions for $p_{t/2}(0,x)$, where $p_t(0,x)$ is the fundamental solution of the heat equation on a circle identified with $[0,1]$, with initial condition δ_0, the delta function at zero. In probabilistic terms, $p_t(0,x)$ is the probability density at $x \in [0,1]$ of the position of a Brownian motion on the circle started at 0 at time 0 and run for time t. The left hand expression is obtained by wrapping the Gaussian solution on the line, while the right hand expression is obtained by Fourier analysis. In particular, (2.1) for $x = 0$ can be written

$$\sqrt{t}\,\theta(t) = \theta(t^{-1}) \qquad (t > 0) \tag{2.2}$$

where θ is the Jacobi theta function

$$\theta(t) := \sum_{n=-\infty}^{\infty} \exp(-n^2 \pi t) \qquad (t > 0). \tag{2.3}$$

For the function ξ defined by (1.2), Riemann obtained the integral representation

$$\frac{4\xi(s)}{s(s-1)} = \int_0^\infty t^{\frac{s}{2}-1}(\theta(t) - 1)dt \qquad (\Re s > 1) \tag{2.4}$$

by switching the order of summation and integration and using

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx \qquad (\Re s > 0). \tag{2.5}$$

He then deduced his functional equation $\xi(s) = \xi(1-s)$ from (2.4) and Jacobi's functional equation (2.2). Following the notation of Edwards [28, §10.3], let

$$G(y) := \theta(y^2) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 y^2), \tag{2.6}$$

so Jacobi's functional equation (2.2) acquires the simpler form

$$yG(y) = G(y^{-1}) \qquad (y > 0). \tag{2.7}$$

The function

$$H(y) := \frac{d}{dy}\left[y^2 \frac{d}{dy} G(y)\right] = 2yG'(y) + y^2 G''(y), \tag{2.8}$$

that is

$$H(y) = 4y^2 \sum_{n=1}^{\infty} (2\pi^2 n^4 y^2 - 3\pi n^2) e^{-\pi n^2 y^2}, \tag{2.9}$$

satisfies the same functional equation as G:

$$yH(y) = H(y^{-1}) \qquad (y > 0). \tag{2.10}$$

As indicated by Riemann, this allows (2.4) to be transformed by integration by parts for $\Re s > 1$ to yield

$$2\xi(s) = \int_0^\infty y^{s-1} H(y) dy. \tag{2.11}$$

It follows immediately by analytic continuation that (2.11) serves to define an entire function $\xi(s)$ which satisfies Riemann's functional equation $\xi(s) = \xi(1-s)$ for all complex s. Conversely, the functional equation (2.10) for H is recovered from Riemann's functional equation for ξ by uniqueness of Mellin transforms. The representation of ξ as a Mellin transform was used by Hardy to prove that an infinity of zeros of ζ lie on the critical line ($\Re s = \frac{1}{2}$). It is also essential in the work of Pólya [78] and Newman [62], [63] on the Riemann hypothesis.

2.2. **Probabilistic interpretation of** $2\xi(s)$**.** As observed by Chung [18] and Newman [62], $H(y) > 0$ for all $y > 0$ (obviously for $y \geq 1$, hence too for $y < 1$ by (2.10)). By (1.2), (1.3) and $\zeta(s) \sim (s-1)^{-1}$ as $s \to 1$,

$$2\xi(0) = 2\xi(1) = 1,$$

so formula (2.11) for $s = 0$ and $s = 1$ implies

$$\int_0^\infty y^{-1} H(y) dy = \int_0^\infty H(y) dy = 1.$$

That is to say, the function $y^{-1}H(y)$ is the density function of a probability distribution on $(0, \infty)$ with mean 1. Note that the functional equation (2.10) for H can be expressed as follows in terms of a random variable Y with this distribution: for every non-negative measurable function g

$$E[g(1/Y)] = E[Y g(Y)]. \tag{2.12}$$

The distribution of $1/Y$ is therefore identical to the *size-biased* distribution derived from Y. See Smith-Diaconis [87] for further interpretations of this relation. The next proposition, which follows from the preceding discussion and formulae tabulated in Section 3, gathers different characterizations of a random variable Y with this density. Here Y is assumed to be defined on some probability space $(\Omega, \mathcal{F}, P)$, with expectation operator E.

Proposition 2.1 ([18], [81])**.** *For a non-negative random variable Y, each of the following conditions,* (i)-(iv)*, is equivalent to Y having density $y^{-1}H(y)$ for $y > 0$:*
(i)

$$E(Y^s) = 2\xi(s) \qquad (s \in \mathbb{C}); \tag{2.13}$$

(ii) *for $y > 0$*

$$P(Y \leq y) = G(y) + yG'(y) = -y^{-2}G'(y^{-1}), \tag{2.14}$$

that is

$$P(Y \leq y) = \sum_{n=-\infty}^{\infty} (1 - 2\pi n^2 y^2) e^{-\pi n^2 y^2} = 4\pi y^{-3} \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 / y^2}; \tag{2.15}$$

(iii) *with S_2 defined by* (1.6)

$$Y \stackrel{d}{=} \sqrt{\tfrac{\pi}{2} S_2}; \tag{2.16}$$

(iv) *for $\lambda > 0$*

$$E\left[e^{-\lambda Y^2}\right] = \left(\frac{\sqrt{\pi\lambda}}{\sinh \sqrt{\pi\lambda}} \right)^2. \tag{2.17}$$

As discussed further in Sections 1 and 4.1, the first probabilistic interpretations of Y defined by (2.14) were in terms of the asymptotic behaviour of random walks. Rényi and Szekeres [79] encountered the distribution of $\sqrt{\pi/2}\,Y$ as the limit distribution of $H_n/\sqrt{4n}$ as $n \to \infty$, where H_n is the maximum height above the root of a random tree with uniform distribution on the set of n^{n-1} rooted trees labeled by a set of n elements. They noted the equivalence of (2.15) with (2.13) for $s > 1$. By the previous discussion, this equivalence and the extension of (2.13) to all $s \in \mathbb{C}$ should be attributed to Riemann [81]. Aldous [1] explained why the asymptotic distribution of $H_n/\sqrt{4n}$ is identical to the distribution of the maximum of a standard Brownian excursion, as defined below (4.10). This distribution had been previously identified by Chung [18] using (2.15) and (2.17). See (3.4) below regarding (2.16). The connection between results for Brownian excursions and the functional equation for Riemann's ξ function was made in Biane-Yor [10].

2.3. **Li's criterion for the Riemann Hypothesis.** Li [59] showed that the Riemann hypothesis

$$\xi(\rho) = 0 \;\Rightarrow\; \Re\rho = \tfrac{1}{2} \tag{2.18}$$

is equivalent to the sequence of inequalities

$$\lambda_n > 0 \quad \text{for } n = 1, 2, \dots \tag{2.19}$$

where

$$\lambda_n := \frac{1}{(n-1)!}\frac{d^n}{ds^n}\left(s^{n-1}\log\xi(s)\right)\big|_{s=1} = \sum_\rho \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right]. \tag{2.20}$$

Here $\sum_\rho$ is the limit as $T \to \infty$ of a sum over the zeros ρ of ξ with $|\rho| \le T$, with repetition of zeros by multiplicity, and the second equality in (2.20) follows from the classical formula [22]

$$2\xi(s) = \prod_\rho \left(1 - \frac{s}{\rho}\right)$$

where the product is over all zeros ρ of ξ with ρ and $1-\rho$ paired together. Bombieri and Lagarias [13] remarked that Li's criterion (2.19) is closely connected to A. Weil's explicit formulae.

By an elementary binomial expansion

$$\lambda_n = n\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{k_{n-i}}{(n-i)!} \tag{2.21}$$

where

$$k_n := \frac{d^n}{ds^n}\left(\log\xi(s)\right)\big|_{s=1}. \tag{2.22}$$

We observe that k_n is the nth *cumulant* of the random variable $L := \log(1/Y)$ for Y satisfying the conditions of Proposition 2.1. This follows from the classical definition of cumulants [35], using the consequence of $E(Y^{1-s}) = \xi(1-s) = \xi(s)$ and (2.22) that

$$\log E[e^{(s-1)L}] = \log E[Y^{1-s}] = \log[2\xi(s)] = \sum_{n=1}^{\infty} k_n \frac{(s-1)^n}{n!}. \tag{2.23}$$

The cumulants k_n are related to the *moments*

$$\mu_n := E(L^n) = \int_0^\infty (-\log y)^n y^{-1} H(y)dy$$

by the general relation

$$\mu_n = \sum_{i=0}^{n-1}\binom{n-1}{i}\mu_i k_{n-i}. \tag{2.24}$$

We note that k_1 and k_2 are positive, because

$$k_1 = \mu_1 = -E[\log(Y)] > -\log(E[Y]) = 0$$

by Jensen's inequality, and $k_2 = \mu_2 - \mu_1^2$ is the variance of L. Hence λ_1 and λ_2 are positive by (2.21). As indicated in [13, p. 286], the λ_n can be explicitly expressed in terms of $\zeta(j)$ for $j = 2, 3, \ldots$ and the *Stieltjes constants* defined by the Laurent expansion of $\zeta(1+s)$. Approximate values of the first few λ_n and k_n are shown in the following table, which was computed using *Mathematica* for power series manipulation and numerical evaluation of the Stieltjes constants and $\zeta(j)$. The values of λ_n agree with those obtained independently by L. Pharamond, a student of J. Oesterlé, using *Maple* instead of *Mathematica*.

n	λ_n	k_n
1	0.0230957	0.0230957
2	0.0923457	0.0461543
3	0.207639	−0.000222316
4	0.36879	−0.0000441763
5	0.575543	0.0000171622
6	0.827566	0.0000337724

We learned from J. Oesterlé (private communication) that $\lambda_n \geq 0$ if every zero ρ of ξ with $|\Im\rho| < \sqrt{n}$ has $\Re\rho = \frac{1}{2}$. This is known to be true [94] for $n \leq 2.975\ldots \times 10^{17}$. See also Odlyzko [65] for a recent review of ongoing computations of zeros of ξ.

3. Two infinitely divisible families

This section presents an array of results regarding the probability laws on $(0,\infty)$ of the random variables S_h and C_h defined by (1.6), with special emphasis on results for $h = 1$ and $h = 2$, which are summarized by Table 1. Each column of the table presents features of the law of one of the four sums $\Sigma = S_1, S_2, C_1$ or C_2. Those in the S_2 column can be read from Proposition 2.1 and (2.12), while the formulae in other columns provide analogous results for S_1, C_1 and C_2 instead of S_2. While the Mellin transforms of S_1, S_2 and C_2 all involve the entire function ξ associated with the Riemann zeta function, the Mellin transform of C_1 involves instead the Dirichlet L-function associated with the quadratic character modulo 4, that is the entire function defined by

$$L_{\chi_4}(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s} \qquad (\Re s > 0). \tag{3.1}$$

We now discuss the entries of Table 1 row by row.

3.1. **Series representations and Laplace transforms.**

The first row represents each Σ in terms of independent random variables ε_n and $\hat{\varepsilon}_n$, $n = 1, 2, \ldots$ with the *gamma*(1) or *standard exponential distribution* $P(\varepsilon_n \geq x) = e^{-x}$, $x \geq 0$. So in (1.6) we have taken $\Gamma_{1,n} = \varepsilon_n$ and $\Gamma_{2,n} = \varepsilon_n + \hat{\varepsilon}_n$.

Recall that the distribution of the gamma(h) variables $\Gamma_{h,n}$ is characterized by the Laplace transform

$$E[\exp(-\lambda\Gamma_{h,n})] = (1+\lambda)^{-h}. \tag{3.2}$$

As pointed out in [19], [97], Euler's formulae

$$\sinh z = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2\pi^2}\right) \text{ and } \cosh z = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(n-\frac{1}{2})^2\pi^2}\right) \tag{3.3}$$

Table 1

Σ	$S_1 := \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\varepsilon_n}{n^2}$	$S_2 := \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\varepsilon_n+\hat{\varepsilon}_n}{n^2}$
$E[e^{-\lambda\Sigma}]$	$\frac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}$	$\left(\frac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}\right)^2$
Lévy density	$\rho_{S_1}(x) = \frac{1}{x}\sum_{n=1}^{\infty}e^{-\pi^2n^2x/2}$	$\rho_{S_2}(x) = 2\rho_{S_1}(x)$
$f(x) := \frac{d}{dx}P(\Sigma \le x)$	$\frac{d}{dx}\sum_{n=-\infty}^{\infty}(-1)^n e^{-n^2\pi^2x/2}$	$\frac{d}{dx}\sum_{n=-\infty}^{\infty}(1-n^2\pi^2x)e^{-n^2\pi^2x/2}$
reciprocal relations	$f_{S_1}(x) = \left(\frac{2}{\pi x^3}\right)^{1/2} f_{C_2}\left(\frac{4}{\pi^2 x}\right)$	$f_{S_2}(x) = \left(\frac{2}{\pi x}\right)^{5/2} f_{S_2}\left(\frac{4}{\pi^2 x}\right)$
$E\left[g\left(\frac{4}{\pi^2\Sigma}\right)\right]$	$\sqrt{\frac{\pi}{2}}E\left[(C_2)^{-1/2}g(C_2)\right]$	$\sqrt{\frac{\pi}{2}}E\left[(S_2)^{1/2}g(S_2)\right]$
$E[\Sigma^s]$	$\left(\frac{2^{1-2s}-1}{1-2s}\right)\left(\frac{2}{\pi}\right)^s 2\xi(2s)$	$\left(\frac{2}{\pi}\right)^s 2\xi(2s)$
$E[\Sigma^n]$	$\frac{n!}{(2n)!}(2^{3n}-2^{n+1})(-1)^{n+1}B_{2n}$	$\frac{n!}{(2n)!}(2n-1)2^{3n}(-1)^{n+1}B_{2n}$

yield the expressions (1.8) for the Laplace transforms of S_h and C_h, as displayed in the second row of Table 1 for $h = 1$ and 2. For instance, for $S_h := \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\Gamma_{h,n}}{n^2}$ we can compute for $\Re\lambda > 0$

(3.4)
$$E\left[e^{-\lambda S_h}\right] = E\left[\prod_{n=1}^{\infty}\exp\left(-\frac{2\lambda\Gamma_{h,n}}{n^2\pi^2}\right)\right] = \prod_{n=1}^{\infty}\left(1+\frac{2\lambda}{n^2\pi^2}\right)^{-h} = \left(\frac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}\right)^h.$$

3.2. **Lévy densities.** A probability distribution F on the line is called *infinitely divisible* if for each n there exist independent random variables $T_{n,1},\dots,T_{n,n}$ with the same distribution such that $\sum_{i=1}^n T_{n,i}$ has distribution F. According to the *Lévy-Khintchine representation*, a distribution F concentrated on $[0,\infty)$ is infinitely divisible if and only if its Laplace transform $\varphi(\lambda) := \int_0^\infty e^{-\lambda t}F(dt)$ admits the

Table 1 continued	
$C_1 := \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\varepsilon_n}{(n-\frac{1}{2})^2}$	$C_2 := \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\varepsilon_n+\hat{\varepsilon}_n}{(n-\frac{1}{2})^2}$
$\frac{1}{\cosh\sqrt{2\lambda}}$	$\left(\frac{1}{\cosh\sqrt{2\lambda}}\right)^2$
$\rho_{C_1}(x) = \frac{1}{x}\sum_{n=1}^{\infty}e^{-\pi^2(n-1/2)^2x/2}$	$\rho_{C_2}(x) = 2\rho_{C_1}(x)$
$\pi\sum_{n=0}^{\infty}(-1)^n(n+\frac{1}{2})e^{-(n+\frac{1}{2})^2\pi^2x/2}$	$\frac{1}{2}\sum_{n=-\infty}^{\infty}\left((n+\frac{1}{2})^2\pi^2x-1\right)e^{-(n+\frac{1}{2})^2\pi^2x/2}$
$f_{C_1}(x) = \left(\frac{2}{\pi x}\right)^{3/2} f_{C_1}\left(\frac{4}{\pi^2x}\right)$	$f_{C_2}(x) = \frac{2}{\pi}\left(\frac{2}{\pi x}\right)^{3/2} f_{S_1}\left(\frac{4}{\pi^2x}\right)$
$\sqrt{\frac{2}{\pi}}E\left[(C_1)^{-1/2}g(C_1)\right]$	$\left(\frac{2}{\pi}\right)^{3/2}E\left[(S_1)^{-1/2}g(S_1)\right]$
$\Gamma(s+1)2^{s+1}\left(\frac{2}{\pi}\right)^{2s+1}L_{\chi_4}(2s+1)$	$\frac{(2^{2(s+1)}-1)}{s+1}\left(\frac{2}{\pi}\right)^{s+1}\xi(2(s+1))$
$\frac{n!}{(2n)!}2^n(-1)^nE_{2n}$	$\frac{(2^{2n+2}-1)2^{3n+1}\,n!}{(n+1)(2n)!}(-1)^nB_{2n+2}$

representation

$$\varphi(\lambda) = \exp\left(-c\lambda - \int_0^\infty(1-e^{-\lambda x})\nu(dx)\right) \qquad (\lambda \geq 0) \tag{3.5}$$

for some $c \geq 0$ and some positive measure ν on $(0,\infty)$, called the *Lévy measure* of F, with c and ν uniquely determined by F. This representation has a well known interpretation in terms of Poisson processes [82, I.28].

If $\nu(dx) = \rho(x)dx$, then $\rho(x)$ is called the *Lévy density* of F. It is elementary that for Γ_h with gamma(h) distribution and $a > 0$ the distribution of Γ_h/a is infinitely divisible with Lévy density $hx^{-1}e^{-ax}$. It follows easily that for $a_n > 0$ with $\sum_n a_n < \infty$ and independent gamma(h) variables $\Gamma_{h,n}$ the distribution of $\sum_n \Gamma_{h,n}/a_n$ is infinitely divisible with Lévy density $hx^{-1}\sum_n e^{-a_nx}$. Thus for each $h > 0$ the laws of S_h and C_h are infinitely divisible, with the Lévy densities indicated in Table 1 for $h = 1$ and $h = 2$.

We note that Riemann's formula (2.4) for $s = 2p$ can be interpreted as an expression for the pth moment of the Lévy density of S_1: for $\Re p > \frac{1}{2}$

$$\int_0^\infty t^p \rho_{S_1}(t)dt = \left(\frac{2}{\pi}\right)^p \frac{1}{2}\int_0^\infty y^{p-1}(\theta(y)-1)dy = \frac{2^p}{\pi^{2p}}\Gamma(p)\zeta(2p). \tag{3.6}$$

3.3. **Probability densities and reciprocal relations.** By application of the negative binomial expansion

$$\frac{1}{(1-x)^h} = \frac{1}{\Gamma(h)}\sum_{n=0}^\infty \frac{\Gamma(n+h)}{\Gamma(n+1)}x^n \qquad (h>0, |x|<1) \tag{3.7}$$

there is the expansion

$$\left(\frac{t}{\sinh t}\right)^h = \frac{2^h t^h e^{-th}}{(1-e^{-2t})^h} = \frac{2^h t^h}{\Gamma(h)}\sum_{n=0}^\infty \frac{\Gamma(n+h)}{\Gamma(n+1)}e^{-(2n+h)t} \tag{3.8}$$

which corrects two typographical errors in [10, (3.v)], and

$$\left(\frac{1}{\cosh t}\right)^h = \frac{2^h e^{-th}}{(1+e^{-2t})^h} = \frac{2^h}{\Gamma(h)}\sum_{n=0}^\infty (-1)^n \frac{\Gamma(n+h)}{\Gamma(n+1)}e^{-(2n+h)t}. \tag{3.9}$$

The Laplace transform of C_h displayed in (1.8) can be inverted by applying the expansion (3.9) and inverting term by term using Lévy's formula [56]

$$\int_0^\infty \frac{a}{\sqrt{2\pi t^3}}e^{-a^2/(2t)}e^{-\lambda t}dt = e^{-a\sqrt{2\lambda}}. \tag{3.10}$$

Thus there is the following expression for the density $f_{C_h}(t) := P(C_h \in dt)/dt$: for arbitrary real $h > 0$:

$$f_{C_h}(t) = \frac{2^h}{\Gamma(h)}\sum_{n=0}^\infty (-1)^n \frac{\Gamma(n+h)}{\Gamma(n+1)}\frac{(2n+h)}{\sqrt{2\pi t^3}}\exp\left(-\frac{(2n+h)^2}{2t}\right). \tag{3.11}$$

A more complicated formula for $f_{S_h}(t)$ was obtained from (3.8) by the same method in [10, (3.x)]. The formulae for the densities of S_h and C_h displayed in Row 4 of Table 1 for $h = 1$ and $h = 2$ can be obtained using the reciprocal relations of Row 5. The self-reciprocal relation involving S_2 is a variant of (2.12), while that involving C_1, which was observed by Ciesielski-Taylor [20], is an instance of the more general reciprocal relation recalled as (6.3) in Section 6.1. Lastly, the reciprocal relation involving the densities f_{S_1} and f_{C_2} amounts to the identity

$$P(S_1 \le x) = \sum_{n=-\infty}^\infty (-1)^n e^{-n^2\pi^2 x/2} = \sqrt{\frac{2}{\pi x}}\sum_{n=-\infty}^\infty e^{-2(n+\frac{1}{2})^2/x} \tag{3.12}$$

where the second equality is read from (2.1) with $x = 1/2$ and t replaced by $x/2$.

Row 6 of Table 1 displays various formulae for $E\left[g\left(\frac{4}{\pi^2\Sigma}\right)\right]$. These formulae, valid for an arbitrary non-negative Borel function g, are integrated forms of the reciprocal relations, similar to (2.12).

3.4. **Moments and Mellin transforms.** It is easily shown that the distributions of S_h and C_h have moments of all orders (see e.g. Lemma 5.2). The formulae for the Mellin transforms $E(\Sigma^s)$ can all be obtained by term-by-term integration of the densities for suitable s, followed by analytic continuation. Another approach to these formulae is indicated later in Section 5.2.

According to the self-reciprocal relation for C_1, for all $s \in \mathbb{C}$

$$E\left[\left(\tfrac{\pi}{2}C_1\right)^s\right] = E\left[\left(\tfrac{\pi}{2}C_1\right)^{-\frac{1}{2}-s}\right]. \tag{3.13}$$

Using the formula for $E((C_1)^s)$ in terms of L_{χ_4} defined by (3.1), given in Table 1, we see that if we define

$$\Lambda_{\chi_4}(t) := E\left[\left(\tfrac{\pi}{2}\,C_1\right)^{\frac{t-1}{2}}\right] = \Gamma\left(\frac{t+1}{2}\right)\left(\frac{4}{\pi}\right)^{\frac{t+1}{2}} L_{\chi_4}(t), \tag{3.14}$$

then (3.13) amounts to the functional equation

$$\Lambda_{\chi_4}(t) = \Lambda_{\chi_4}(1-t) \qquad (t \in \mathbb{C}). \tag{3.15}$$

This is an instance of the general functional equation for a Dirichlet L-function, which is recalled as (6.4) in Section 6.

The formulae for positive integer moments $E(\Sigma^n)$, given in the last row of the table, are particularizations of the preceding row, using the classical evaluation of $\zeta(2n)$ in terms of the Bernoulli numbers B_{2n}. The result for C_1 involves the *Euler numbers* E_{2n}, defined by the expansion

$$\frac{1}{\cosh(z)} = \frac{2}{e^z + e^{-z}} = \sum_{n=0}^{\infty} E_{2n}\frac{z^{2n}}{(2n)!}. \tag{3.16}$$

3.5. **A multiplicative relation.** Table 1 reveals the following remarkably simple relation:

$$E(S_1^s) = \left(\frac{2^{1-2s}-1}{1-2s}\right) E(S_2^s) \tag{3.17}$$

where the first factor on the right side is evaluated by continuity for $s = 1/2$. By elementary integration, this factor can be interpreted as follows:

$$\left(\frac{2^{1-2s}-1}{1-2s}\right) = E(W^{-2s}) \tag{3.18}$$

for a random variable W with uniform distribution on $[1,2]$. Thus (3.17) amounts to the following identity in distribution:

$$S_1 \stackrel{d}{=} W^{-2} S_2 \tag{3.19}$$

where W is assumed independent of S_2. An equivalent of (3.19) was interpreted in terms of Brownian motion in [72, (4)].

Note that (3.19) could be rewritten as $S_1 \stackrel{d}{=} W^{-2}(S_1 + \hat{S}_1)$ for S_1, $\hat{S}_1$ and W independent random variables, with $\hat{S}_1$ having the same distribution as S_1, and W uniform on $[1,2]$. By consideration of positive integer moments, this property uniquely characterizes the distribution of S_1 among all distributions with mean $1/3$ and finite moments of all orders.

3.6. **Characterizations of the distributions of S_2 and C_2.** As just indicated, the identity (3.19) allows a simple probabilistic characterization of the distribution of S_1. The following proposition offers similar characterizations of the distributions of S_2 and C_2.

Proposition 3.1. *Let X be a non-negative random variable, and let X^* denote a random variable such that*

$$P(X^* \in dx) = xP(X \in dx)/E(X).$$

(i) *X is distributed as S_2 if and only if $E(X) = 2/3$ and*

$$X^* \stackrel{d}{=} X + HX^* \tag{3.20}$$

where X, H and X^* are independent, with

$$P(H \in dh) = (h^{-1/2} - 1)dh \qquad (0 < h < 1).$$

(ii) *X is distributed as C_2 if and only if $E(X) = 2$ and*

$$X^* \stackrel{d}{=} X + U^2\hat{X} \tag{3.21}$$

where X, U and $\hat{X}$ are independent, with $\hat{X}$ distributed as X and U uniform on $[0, 1]$.

For the proof of this proposition, note that (3.20) (or (3.21)) implies that the Laplace transform of X satisfies an integro-differential equation whose only solution is given by the appropriate function. The "only if" part of (i) appears in [103, p. 26]. Details of the remaining parts are provided in [75].

The distribution of X^*, known as the *size-biased* or *length-biased* distribution of X, has a natural interpretation in renewal theory [95]. As shown in [95], the Lévy-Khintchine formula implies that the equation $X^* \stackrel{d}{=} X + Y$ is satisfied by independent non-negative random variables X and Y if and only if the law of X is infinitely divisible.

4. Brownian interpretations

It is a remarkable fact that the four distributions considered in Section 3 appear in many different problems concerning Brownian motion and related stochastic processes. These appearances are partially explained by the relation of these distributions to Jacobi's theta function, which provides a solution to the heat equation [6], [29], and is therefore related to Brownian motion [52, §5.4]. We start by introducing some basic notation for Brownian motion and Bessel processes, then present the main results in the form of another table.

4.1. **Introduction and notation.** Let $\beta := (\beta_t, t \geq 0)$ be a standard one-dimensional Brownian motion, that is a stochastic process with continuous sample paths and independent increments such that $\beta_0 = 0$, and for all $s, t > 0$ the random variable $\beta_{s+t} - \beta_s$ has a Gaussian distribution with mean $E(\beta_{s+t} - \beta_s) = 0$ and mean square $E[(\beta_{s+t} - \beta_s)^2] = t$, meaning that for all real x

$$P(\beta_{s+t} - \beta_s \leq x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} e^{-y^2/(2t)}\, dy.$$

Among continuous time stochastic processes, such as semimartingales, processes with independent increments, and Markov processes, Brownian motion is the paradigm of a stochastic process with continuous paths. In particular, among processes with stationary independent increments, the Brownian motions $(\sigma B_t + \mu t, t \geq 0)$ for $\sigma > 0, \mu \in \mathbb{R}$ are the only ones with almost surely continuous paths [82, I.28.12]. Brownian motion arises naturally as the limit in distribution as $n \to \infty$ of a rescaled random walk process $(W_n, n = 0, 1, \dots)$ where $W_n = X_1 + \cdots + X_n$ for independent random variables X_i with some common distribution with mean $E(X_i) = 0$ and variance $E(X_i^2) = 1$. To be more precise, let the value of W_r be extended to all real $r \geq 0$ by linear interpolation between integers. With this definition of $(W_r, r \geq 0)$ as a random continuous function, it is known that no matter what the distribution of the X_i with mean 0 and variance 1, as $n \to \infty$

$$\left(\frac{W_{nt}}{\sqrt{n}}, t \geq 0\right) \xrightarrow{d} (\beta_t, t \geq 0) \tag{4.1}$$

in the sense of weak convergence of probability distributions on the path space $C[0,\infty)$. In particular, convergence of finite dimensional distributions in (4.1) follows easily from the central limit theorem, which is the statement of convergence of one dimensional distributions in (4.1); that is for each fixed $t > 0$

$$\frac{W_{nt}}{\sqrt{n}} \xrightarrow{d} \beta_t \tag{4.2}$$

where $\xrightarrow{d}$ denotes weak convergence of probability distributions on the line. Recall that, for random variables $V_n, n = 1, 2, \dots$ and V such that V has a continuous distribution function $x \mapsto P(V \leq x)$, $V_n \xrightarrow{d} V$ means $P(V_n \leq x) \to P(V \leq x)$ for all real x. See [12], [80] for background.

Let $(b_t, 0 \leq t \leq 1)$ be a *standard Brownian bridge*, that is the centered Gaussian process with the conditional distribution of $(\beta_t, 0 \leq t \leq 1)$ given $\beta_1 = 0$. Some well known alternative descriptions of the distribution of b are [80, Ch. III, Ex (3.10)]

$$(b_t, 0 \leq t \leq 1) \stackrel{d}{=} (\beta_t - t\beta_1, 0 \leq t \leq 1) \stackrel{d}{=} ((1-t)\beta_{t/(1-t)}, 0 \leq t \leq 1) \tag{4.3}$$

where $\stackrel{d}{=}$ denotes equality of distributions on the path space $C[0,1]$, and the rightmost process is defined to be 0 for $t = 1$. According to a fundamental result in the theory of non-parametric statistics [25], [85], the Brownian bridge arises in another way from the asymptotic behaviour of the *empirical distribution*

$$F_n(x) := \frac{1}{n} \sum_{k=1}^{n} 1(X_k \leq x)$$

where the X_k are now supposed independent with common distribution $P(X_i \leq x) = F(x)$ for an arbitrary continuous distribution function F. As shown by Kolmogorov [53], the distribution of $\sup_x |F_n(x) - F(x)|$ is the same no matter what the choice of F, and for all real y

$$\lim_{n\to\infty} P(\sqrt{n} \sup_x |F_n(x) - F(x)| \leq y) = \sum_{n=-\infty}^{\infty} (-1)^n e^{-2n^2y^2}. \tag{4.4}$$

For F the uniform distribution on $[0,1]$, so $F(t) = t$ for $0 \leq t \leq 1$, it is known that

$$(\sqrt{n}(F_n(t) - t), 0 \leq t \leq 1) \xrightarrow{d} (b_t, 0 \leq t \leq 1). \tag{4.5}$$

As a well known consequence of (4.5), Kolmogorov's limiting distribution in (4.4) is identical to the distribution of $\max_{0 \leq t \leq 1} |b_t|$. On the other hand, as observed by Watson [97], Kolmogorov's limit distribution function in (4.4) is identical to that of

$\frac{\pi}{2}\sqrt{S_1}$. Thus we find the first appearance of the law of S_1 as the law of a functional of Brownian bridge.

To put this in terms of random walks, if (W_n) is a *simple random walk*, meaning $P(X_i = +1) = P(X_i = -1) = 1/2$, then

$$\left(\frac{W_{2nt}}{\sqrt{2n}}, 0 \le t \le 1 \,\middle|\, W_{2n} = 0 \right) \xrightarrow{d} (b_t, 0 \le t \le 1) \tag{4.6}$$

where on the left side the random walk is conditioned to return to zero at time $2n$, and on the right side the Brownian motion is conditioned to return to zero at time 1. Thus

$$\left(\frac{1}{\sqrt{2n}} \max_{0 \le k \le 2n} |W_k| \,\middle|\, W_{2n} = 0 \right) \xrightarrow{d} \max_{0 \le t \le 1} |b_t| \overset{d}{=} \frac{\pi}{2}\sqrt{S_1} \tag{4.7}$$

where the equality in distribution summarizes the conclusion of the previous paragraph. In the same vein, Gnedenko [33] derived another asymptotic distribution from random walks, which can be interpreted in terms of Brownian bridge as

$$\left(\frac{1}{\sqrt{2n}} \left[\max_{0 \le k \le 2n} W_k - \min_{0 \le k \le 2n} W_k \right] \,\middle|\, W_{2n} = 0 \right) \xrightarrow{d} \max_{0 \le t \le 1} b_t - \min_{0 \le t \le 1} b_t \overset{d}{=} \frac{\pi}{2}\sqrt{S_2}. \tag{4.8}$$

The equalities in distribution in both (4.7) and (4.8) can be deduced from the formula

$$P\left(\min_{0 \le u \le 1} b_u \ge -a, \max_{0 \le u \le 1} b_u \le b \right) = \sum_{k=-\infty}^{\infty} e^{-2k^2(a+b)^2} - \sum_{k=-\infty}^{\infty} e^{-2[b+k(a+b)]^2} \tag{4.9}$$

of Smirnov [86] and Doob [25]. Kennedy [45] found that these distributions appear again if the random walk is conditioned instead on the event $(R = 2n)$ or $(R > 2n)$, where

$$R := \inf\{n \ge 1 : W_n = 0\}$$

is the time of the first return to zero by the random walk. Thus

$$\left(\frac{1}{\sqrt{2n}} \max_{0 \le k \le 2n} |W_k| \,\middle|\, R = 2n \right) \xrightarrow{d} \max_{0 \le t \le 1} e_t \overset{d}{=} \frac{\pi}{2}\sqrt{S_2} \tag{4.10}$$

where $(e_t, 0 \le t \le 1)$ denotes a *standard Brownian excursion*, that is the process with continuous sample paths defined following [27], [42], [45] by the limit in distribution on $C[0,1]$

$$\left(\frac{|W_{2nt}|}{\sqrt{2n}}, 0 \le t \le 1 \,\middle|\, R = 2n \right) \xrightarrow{d} (e_t, 0 \le t \le 1). \tag{4.11}$$

A satisfying explanation of the identity in distribution between the limit variables featured in (4.8) and (4.10) is provided by the following identity of distributions on $C[0,1]$ due to Vervaat [96]:

$$(e_u, 0 \le u \le 1) \overset{d}{=} (b_{\rho+u(\mathrm{mod}\,1)} - b_\rho, 0 \le u \le 1) \tag{4.12}$$

where ρ is the almost surely unique time that the Brownian bridge b attains its minimum value. As shown by Tákacs [90] and Smith-Diaconis [87], either of the approximations (4.8) or (4.10) can be used to establish the differentiated form (2.14) of Jacobi's functional equation (2.7) by a discrete approximation argument involving quantities of probabilistic interest. See also Pólya [77] for a closely related proof of Jacobi's functional equation based on the local normal approximation to the binomial distribution.

In the same vein as (4.8) and (4.10) there is the result of [27], [45] that

$$\left(\frac{1}{\sqrt{2n}} \max_{0 \le k \le 2n} |W_k| \,\middle|\, R > 2n \right) \xrightarrow{d} \max_{0 \le t \le 1} m_t \stackrel{d}{=} \pi \sqrt{S_1} \tag{4.13}$$

where $(m_t, 0 \le t \le 1)$ denotes a *standard Brownian meander*, defined by the limit in distribution on $C[0,1]$

$$\left(\frac{|W_{2nt}|}{\sqrt{2n}}, 0 \le t \le 1 \,\middle|\, R > 2n \right) \xrightarrow{d} (m_t, 0 \le t \le 1). \tag{4.14}$$

The surprising consequence of (4.7) and (4.13), that $\max_{0 \le t \le 1} m_t \stackrel{d}{=} 2 \max_{0 \le t \le 1} |b_t|$, was explained in [10] by a transformation of bridge b into a process distributed like the meander m. For a review of various transformations relating Brownian bridge, excursion and the meander see [8].

4.2. **Bessel processes.** The work of Williams [98], [99], [100] shows how the study of excursions of one-dimensional Brownian motion leads inevitably to descriptions of these excursions involving higher dimensional Bessel processes. For $d = 1, 2, \dots$ let $R_d := (R_{d,t}, t \ge 0)$ be the *d-dimensional Bessel process BES(d)*, that is the non-negative process defined by the radial part of a d-dimensional Brownian motion:

$$R_{d,t}^2 := \sum_{i=1}^d B_{i,t}^2$$

where $(B_{i,t}, t \ge 0)$ for $i = 1, 2, \dots$ is a sequence of independent one-dimensional Brownian motions. Note that each of the processes $X = B$ and $X = R_d$ for any $d \ge 1$ has the *Brownian scaling property*:

$$(X_u, u \ge 0) \stackrel{d}{=} (\sqrt{c} X_{u/c}, u \ge 0) \tag{4.15}$$

for every $c > 0$, where $\stackrel{d}{=}$ denotes equality in distribution of processes. For a process $X = (X_t, t \ge 0)$ let $\overline{X}$ and $\underline{X}$ denote the past maximum and past minimum processes derived from X, that is

$$\overline{X}_t := \sup_{0 \le s \le t} X_s; \ \underline{X}_t := \inf_{0 \le s \le t} X_s.$$

Note that if X has the Brownian scaling property (4.15), then so too do $\overline{X}, \underline{X}$, and $\overline{X} - \underline{X}$. For a suitable process X, let

$$(L_t^x(X), t \ge 0, x \in \mathbb{R})$$

be the *process of local times of* X defined by the occupation density formula

$$\int_0^t f(X_s) ds = \int_{-\infty}^{\infty} f(x) L_t^x(X) dx \tag{4.16}$$

for all non-negative Borel functions f, and almost sure joint continuity in t and x. See [80, Ch. VI] for background and proof of the existence of such a local time process for $X = B$ and $X = R_d$ for any $d \ge 1$.

Let $r_d := (r_{d,u}, 0 \le u \le 1)$ denote the *d-dimensional Bessel bridge* defined by conditioning $R_{d,u}, 0 \le u \le 1$ on $R_{d,1} = 0$. Put another way, r_d^2 is the sum of squares of d independent copies of the standard Brownian bridge.

Table 2

0)	$\dfrac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}$	$\left(\dfrac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}\right)^2$	$\dfrac{1}{\cosh\sqrt{2\lambda}}$	$\left(\dfrac{1}{\cosh\sqrt{2\lambda}}\right)^2$
1)	$S_1 := \dfrac{2}{\pi^2}\sum_{n=1}^{\infty}\dfrac{\varepsilon_n}{n^2}$	$S_2 := \dfrac{2}{\pi^2}\sum_{n=1}^{\infty}\dfrac{\varepsilon_n+\hat{\varepsilon}_n}{n^2}$	$C_1 := \dfrac{2}{\pi^2}\sum_{n=1}^{\infty}\dfrac{\varepsilon_n}{(n-\frac{1}{2})^2}$	$C_2 := \dfrac{2}{\pi^2}\sum_{n=1}^{\infty}\dfrac{\varepsilon_n+\hat{\varepsilon}_n}{(n-\frac{1}{2})^2}$
2)	$\int_0^1 r_{2,u}^2\,du$	$\int_0^1 r_{4,u}^2\,du$	$\int_0^1 R_{2,u}^2\,du$	$\int_0^1 R_{4,u}^2\,du$
3)	$T_1(R_3)$	$T_1(R_3)+T_1(\hat{R}_3)$	$T_1(R_1)$	$4T_1(\overline{B}-\underline{B})$
4)	$\left(\overline{R}_{3,1}\right)^{-2}$		$\left(\overline{R}_{1,1}\right)^{-2}$	$4\left(\overline{B}_1-\underline{B}_1\right)^{-2}$
5)	$(\frac{2}{\pi}\overline{r}_{1,1})^2$	$(\frac{2}{\pi}\overline{r}_{3,1})^2$		
6)	$(\frac{1}{\pi}\overline{m}_1)^2$			
7)	$\left(\dfrac{1}{\pi}\int_0^1\dfrac{du}{m_u}\right)^2$	$\left(\dfrac{1}{\pi}\int_0^1\dfrac{du}{r_{3,u}}\right)^2$	$\left(\dfrac{1}{2}\int_0^1\dfrac{du}{R_{2,u}}\right)^{-2}$	$\left(\dfrac{1}{2}\int_0^1\dfrac{du}{R_{3,u}}\right)^{-2}$
8)	$\dfrac{\tau_1}{4(\overline{R}_{1,\tau_1})^2}$	$\dfrac{\tau_1}{(\overline{B}_{\tau_1}-\underline{B}_{\tau_1})^2}$	$\dfrac{4}{\tau_1^2}\int_0^{\tau_1}B_t^2\,dt$	

4.3. **A table of identities in distribution.** We now discuss the meaning of Table 2, which presents a number of known identities in distribution. The results are collected from the work of numerous authors, including Gikhman [32], Kiefer [49], Chung [18], and Biane-Yor [10]. See also [71], [74], [101]. In the following sections we review briefly the main arguments underlying the results presented in the table.

Each column of the table displays a list of random variables with the distribution determined by the Laplace transform in Row 0. Each variable in the second column is distributed as the sum of two independent copies of any variable in the first column, and each variable in the fourth column is distributed as the sum of two independent copies of any variable in the third column. The table is organized by rows of variables which are analogous in some informal sense. The next few paragraphs introduce row by row the notation used in the table, with pointers to explanations and attributions in following subsections. Blank entries in the table mean we do not know any construction of a variable with the appropriate distribution which respects the informal sense of analogy within rows, with the following exceptions. Entries for Rows 4 and 6 of the S_2 column could be filled as in Row 3 as the sums of two independent copies of variables in the S_1 column of the same row, but this would add nothing to the content of the table. The list of variables involved is by no means exhaustive: for instance, according to (4.8) the variable $(4/\pi^2)(\overline{b}_1 - \underline{b}_1)^2$ could be added to the second column. Many more constructions are possible involving Brownian bridge and excursion, some of which we mention in following sections. It is a consequence of its construction that each column of the table exhibits a family of random variables with the same distribution. Therefore it is a natural problem, coming from the philosophy of "bijective proofs" in enumerative combinatorics (see e.g. Stanley [89]), to try giving a direct argument for each distributional identity, not using the explicit computation of the distribution. Many such arguments can be given, relying on distributional symmetries of Brownian paths or some deeper results such as the Ray-Knight theorems. However, some identities remain for which we do not have any such argument at hand. As explained in Section 4.6, some of these identities are equivalent to the functional equation for the Jacobi theta (or the Riemann zeta) function.

Row 0. Here are displayed the Laplace transforms in λ of the four distributions under consideration.

Row 1. Here we recall from Section 3.1 the constructions of variables S_1, S_2, C_1 and C_2 with these Laplace transforms, from independent standard exponential variables ε_n and $\hat{\varepsilon}_n$, $n = 1, 2, \ldots$.

Row 2. Section 4.4 explains why the distributions of the random variables $\int_0^1 r_{d,u}^2 du$ and $\int_0^1 R_{d,u}^2 du$ for $d = 2$ and $d = 4$ are as indicated in this row.

Row 3. Most of the results of this row are discussed in Section 4.5. Here

$$T_a(X) := \inf\{t : X_t = a\}$$

is the hitting time of a by the process X, and $\hat{R}_d$ is an independent copy of the Bessel process R_d. Note that $R_1 := |B|$ is just Brownian motion with reflection at 0, and $T_1(\overline{B} - \underline{B})$ is the first time that the range of the Brownian B up to time t is an interval of length 1. The result that $4T_1(\overline{B} - \underline{B})$ has Laplace transform $1/\cosh^2 \sqrt{2\lambda}$ is due to Imhof [38]. See also Vallois [92], [93], Pitman [67] and Pitman-Yor [72] for various refinements of this formula.

Rows 4 and 5. These rows, which involve the distribution of the maximum of various processes over $[0, 1]$, are discussed in Section 4.6.

Row 6. Here $\overline{m}_1$ is the maximum of the standard Brownian meander $(m_u, 0 \leq u \leq 1)$. This entry is read from (4.13).

Row 7. The first two entries are obtained from their relation to the first two entries in Row 5, that is the equalities in distribution

$$\int_0^1 \frac{du}{m_u} \overset{d}{=} 2\overline{r}_{1,1} \text{ and } \int_0^1 \frac{du}{r_{3,u}} \overset{d}{=} 2\overline{r}_{3,1}.$$

These identities follow from descriptions of the local time processes $(L_1^x(r_d), x \geq 0)$ for $d = 1$ and $d = 3$, which involve m for $d = 1$ and r_3 for $d = 3$, as presented in Biane-Yor [10, Th. 5.3]. See also [68, Cor. 16] for another derivation of these results. The last two entries may be obtained through their relation to the last two entries of Row 2. More generally, there is the identity

$$\frac{1}{2}\int_0^1 \frac{ds}{R_{d,s}} \stackrel{d}{=} \left(\int_0^1 R^2_{2d-2,s}\, ds\right)^{-1/2} \qquad (d > 1)$$

which can be found in Biane-Yor [10] and Revuz-Yor [80, Ch. XI, Corollary 1.12 and p. 448].

Row 8. Here $\tau_1 := \inf\{t : L^0_t(B) = 1\}$ where $R_1 = |B|$ and $(L^x_t(B), t \geq 0, x \in \mathbb{R})$ is the local time process of B defined by (4.16). The distribution of $\tau_1/\overline{R}^2_{1,\tau_1}$ was identified with that of $4T_1(R_3)$ by Knight [52], while the distribution of $\tau_1/(\overline{B}_{\tau_1} - \underline{B}_{\tau_1})^2$ was identified with that of $T_1(R_3) + T_1(\hat{R}_3)$ by Pitman-Yor [72]. The result in the third column can be read from Hu-Shi-Yor [37, p. 188].

4.4. **Squared Bessel processes (Row 2).** For $d = 1, 2, \ldots$ the squared Bessel process R^2_d is by definition the sum of d independent copies of $R^2_1 = B^2$, the square of a one-dimensional Brownian motion B, and a similar remark applies to the squared Bessel bridge r^2_d. Following Lévy [57], [58], let us expand the Brownian motion $(B_t, 0 \leq t \leq 1)$ or the Brownian bridge $(b_t, 0 \leq t \leq 1)$ in a Fourier series. For example, the standard Brownian bridge b can be represented as

$$b_u = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{\pi} \frac{Z_n}{n} \sin(\pi n u) \qquad (0 \leq u \leq 1)$$

where the Z_n for $n = 1, 2, \ldots$ are independent standard normal random variables, so $E(Z_n) = 0$ and $E(Z^2_n) = 1$ for all n. Parseval's theorem then gives

$$\int_0^1 b^2_u du = \sum_{n=0}^{\infty} \frac{Z^2_n}{\pi^2 n^2}$$

so the random variable $\int_0^1 b^2_u du$ appears as a quadratic form in the normal variables Z_n. It is elementary and well known that $Z^2_n \stackrel{d}{=} 2\gamma_{1/2}$ for $\gamma_{1/2}$ with gamma$(\frac{1}{2})$ distribution as in (1.7) and (3.2) for $h = \frac{1}{2}$. Thus

$$E[\exp(-\lambda Z^2_n)] = (1 + 2\lambda)^{-1/2}$$

and

$$E\left[\exp\left(-\lambda \int_0^1 b^2_u du\right)\right] = \prod_{n=1}^{\infty}\left(1 + \frac{2\lambda}{\pi^2 n^2}\right)^{-1/2} = \left(\frac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}\right)^{1/2}$$

by another application of Euler's formula (3.3). Taking two and four independent copies respectively gives the first two entries of Row 2. The other entries of this row are obtained by similar considerations for unconditioned Bessel processes.

Watson [97] found that

$$\int_0^1 \left(b_t - \int_0^1 b_u du\right)^2 dt \stackrel{d}{=} \tfrac{1}{4} S_1. \tag{4.17}$$

Shi-Yor [84] give a proof of (4.17) with the help of a space-time transformation of the Brownian bridge. See also [102, pp. 18-19], [103, pp. 126-127] and papers cited there for more general results in this vein. In particular, we mention a variant of

(4.17) for B instead of b, which can be obtained as a consequence of a stochastic Fubini theorem [102, pp. 21-22]:

$$\int_0^1 \left(B_t - \int_0^1 B_u du\right)^2 dt \stackrel{d}{=} \int_0^1 b_u^2 du \stackrel{d}{=} S_{1/2}. \tag{4.18}$$

As remarked by Watson [97], it is a very surprising consequence of (4.17) and (4.7) that

$$\int_0^1 \left(b_t - \int_0^1 b_u du\right)^2 dt \stackrel{d}{=} \pi^{-2} \max_{0\le t\le 1} b_t^2. \tag{4.19}$$

As pointed out by Chung [18], the identities in distribution (4.8) and (4.10), where S_2 is the sum of two independent copies of S_1, imply that the distribution of

$$(\max_{0\le t\le 1} b_t - \min_{0\le t\le 1} b_t)^2$$

is that of the sum of two independent copies of $\max_{0\le t\le 1} b_t^2$. In a similar vein, the first column of Table 2 shows that the distribution of

$$\frac{4}{\pi^2}\max_{0\le t\le 1} b_t^2$$

is that of the sum of two independent copies of $\int_0^1 b_t^2 dt \stackrel{d}{=} S_{1/2}$. There is still no explanation of these coincidences in terms of any kind of transformation or decomposition of Brownian paths, or any combinatorial argument involving lattice paths, though such methods have proved effective in explaining and generalizing numerous other coincidences involving the distributions of S_h and C_h for various $h > 0$. Vervaat's explanation (4.12) of the identity in law between the range of the bridge and the maximum of the excursion provides one example of this. Similarly, (4.12) and (4.17) imply that

$$\int_0^1 \left(e_t - \int_0^1 e_u du\right)^2 dt \stackrel{d}{=} \tfrac{1}{4} S_1. \tag{4.20}$$

4.5. **First passage times (Row 3).** It is known [20], [39], [46] that by solving an appropriate Sturm-Liouville equation, for $\lambda > 0$

$$E[e^{-\lambda T_1(R_d)}] = \frac{(\sqrt{2\lambda})^\nu}{2^\nu \Gamma(\nu+1) I_\nu(\sqrt{2\lambda})} = \prod_{n=1}^\infty \left(1 + \frac{2\lambda}{j_{\nu,n}^2}\right)^{-1}$$

where $\nu := (d-2)/2$ with I_ν the usual modified Bessel function, related to J_ν by $(ix)^\nu / I_\nu(ix) = x^\nu / J_\nu(x)$, and $j_{\nu,1} < j_{\nu,2} < \cdots$ is the increasing sequence of positive zeros of J_ν. That is to say,

$$T_1(R_d) \stackrel{d}{=} \sum_{n=1}^\infty \frac{2\varepsilon_n}{j_{\nu,n}^2} \tag{4.21}$$

where the ε_n are independent standard exponential variables. See also Kent [47], [48], and literature cited there, for more about this *spectral decomposition* of $T_1(X)$, which can be formulated for a much more general one-dimensional diffusion X instead of $X = R_d$. The results of Row 3, that

$$T_1(R_1) \stackrel{d}{=} C_1 \text{ and } T_1(R_3) \stackrel{d}{=} S_1, \tag{4.22}$$

are the particular cases $d = 1$ and $d = 3$ of (4.21), corresponding to $\nu = \pm 1/2$, when I_ν and J_ν can be expressed in terms of hyperbolic and trigonometric functions. In particular, $j_{-1/2,n} = (n - \frac{1}{2})\pi$ and $j_{1/2,n} = n\pi$ are the nth positive zeros of the cosine and sine functions respectively. Comparison of Rows 2 and 3 reveals the identities

$$T_1(R_1) \stackrel{d}{=} \int_0^1 R_{2,u}^2 du \text{ and } T_1(R_3) \stackrel{d}{=} \int_0^1 r_{2,u}^2 du.$$

As pointed out by Williams [98], [99], [100], these remarkable coincidences in distribution are the simplest case $g(u) = 1$ of the identities in law

$$\int_0^{T_1(R_1)} g(1 - R_{1,t})dt \stackrel{d}{=} \int_0^1 R_{2,u}^2 g(u)du \tag{4.23}$$

and

$$\int_0^{T_1(R_3)} g(R_{3,t})dt \stackrel{d}{=} \int_0^1 r_{2,u}^2 g(u)du \tag{4.24}$$

where the two Laplace transforms involved are again determined by the solutions of a Sturm-Liouville equation [69], [80, Ch. XI]. Let $L_t^x(R_d), t \geq 0, x \in \mathbb{R}$ and $L_t^x(r_d), 0 \leq t \leq 1, x \in \mathbb{R}$ be the local time processes of R_d and r_d defined by the occupation density formula (4.16) with B replaced by R_d or r_d. Granted existence of local time processes for R_d and r_d, the identities (4.23) and (4.24) are expressions of the Ray-Knight theorems [80, Ch. XI, §2] that

$$(L_{T_1(R_1)}^{1-u}(R_1), 0 \leq u \leq 1) \stackrel{d}{=} (R_{2,u}^2, 0 \leq u \leq 1) \tag{4.25}$$

and

$$(L_{T_1(R_3)}^{u}(R_3), 0 \leq u \leq 1) \stackrel{d}{=} (r_{2,u}^2, 0 \leq u \leq 1). \tag{4.26}$$

The next section gives an interpretation of the variable $T_1(R_3) + T_1(\hat{R}_3)$ appearing in column 2 in terms of Brownian excursions.

4.6. **Maxima and the agreement formula (Rows 4 and 5).** The entries in Row 4 are equivalent to corresponding entries in Row 3 by application to $X = R_d$ and $X = \overline{B} - \underline{B}$ of the elementary identity

$$(\overline{X}_1)^{-2} \stackrel{d}{=} T_1(X) \tag{4.27}$$

which is valid for any process X with continuous paths which satisfies the Brownian scaling identity (4.15), because

$$P((\overline{X}_1)^{-2} > t) = P(\overline{X}_1 < t^{-\frac{1}{2}}) = P(\overline{X}_t < 1) = P(T_1(X) > t).$$

The first entry of Row 5, with $\overline{r}_{1,1} := \max_{0 \leq u \leq 1} |b_u|$, is read from (4.7). The second entry of Row 5, involving the maximum $\overline{r}_{3,1}$ of a three-dimensional Bessel bridge $(r_{3,u}, 0 \leq u \leq 1)$, is read from the work of Gikhman [32] and Kiefer [49], who found a formula for $P(\overline{r}_{d,1} \leq x)$ for arbitrary $d = 1, 2, \ldots$. See also [74]. This result involving $\overline{r}_{3,1}$ may be regarded as a consequence of the previous identification (4.10) of the law of $\overline{e}_1$, the maximum of a standard Brownian excursion, and the identity in law $\overline{e}_1 \stackrel{d}{=} \overline{r}_{3,1}$ implied by the remarkable result of Lévy-Williams [58], [98], that

$$(e_t, 0 \leq t \leq 1) \stackrel{d}{=} (r_{3,t}, 0 \leq t \leq 1). \tag{4.28}$$

Another consequence of the scaling properties of Bessel processes is provided by the following absolute continuity relation between the law of $(\overline{r}_{d,1})^{-2}$ and the law of

$$\Sigma_{2,d} := T_1(R_d) + T_1(\hat{R}_d)$$

for general $d > 0$. This result, obtained in [9], [10], [70], [71], we call the *agreement formula*: for every non-negative Borel function g

$$E\left[g((\overline{r}_{d,1})^{-2})\right] = c_d E\left[\Sigma_{2,d}^{\nu}\, g(\Sigma_{2,d})\right] \tag{4.29}$$

where $c_d := 2^{(d-2)/2}\Gamma(d/2)$. In [71] the agreement formula was presented as the specialization to Bessel processes of a general result for one-dimensional diffusions. As explained in [10], [71], [101], the agreement formula follows from the fact that a certain σ-finite measure on the space of continuous non-negative paths with finite lifetimes can be explicitly disintegrated in two different ways, according to the lifetime, or according to the value of the maximum.

Note from (4.21) that $\Sigma_{2,3} \stackrel{d}{=} S_2$ and $\Sigma_{2,1} \stackrel{d}{=} C_2$. For $d = 3$ formula (4.29) gives for all non-negative Borel functions g

$$E\left[g(\overline{r}_{3,1})\right] = \sqrt{\tfrac{2}{\pi}}\, E\left[\sqrt{S_2}\, g(1/\sqrt{S_2})\right]. \tag{4.30}$$

In view of (4.30), the symmetry property (2.12) of the common distribution of Y and $\sqrt{\frac{\pi}{2}S_2}$, which expresses the functional equations for ξ and θ, can be recast as the following identity of Chung [18], which appears in the second column of Table 2:

$$\left(\tfrac{2}{\pi}\,\overline{r}_{3,1}\right)^2 \stackrel{d}{=} S_2. \tag{4.31}$$

As another application of (4.29), we note that for $d = 1$ this formula shows that the reciprocal relation between the laws of S_1 and C_2 discussed in Section 3 is equivalent to the equality in distribution of (4.7), that is

$$\left(\tfrac{2}{\pi}\,\overline{r}_{1,1}\right)^2 \stackrel{d}{=} S_1. \tag{4.32}$$

We do not know of any path transformation leading to a non-computational proof of (4.31) or (4.32).

4.7. **Further entries.** The distributions of $T_1(R_d)$ and $T_1(R_d) + T_1(\hat{R}_d)$ for $d = 1, 3$ shared by the columns of Table 2also arise naturally from a number of other constructions involving Brownian motion and Bessel processes. Alili [3] found the remarkable result that

$$\frac{\mu^2}{\pi^2}\left[\left(\int_0^1 \coth(\mu r_{3,u})du\right)^2 - 1\right] \stackrel{d}{=} S_2 \text{ for all } \mu \neq 0. \tag{4.33}$$

As a check, the almost sure limit of the left side of (4.33) as $\mu \to 0$ is the variable $\pi^{-2}(\int_0^1 r_{3,u}^{-1}du)^2$ in the second column of Row 7. As shown by Alili-Donati-Yor [4], consideration of (4.33) as $\mu \to \infty$ shows that

$$\frac{4}{\pi^2}\int_0^\infty \frac{dt}{\exp(R_{3,t}) - 1} \stackrel{d}{=} S_1. \tag{4.34}$$

According to the identity of Ciesielski-Taylor [20]

$$\int_0^\infty 1(R_{d+2,t} \le 1)dt \stackrel{d}{=} T_1(R_d) \tag{4.35}$$

which for $d = 1$ and $d = 3$ provides further entries for the table via (4.22). See also [24], [51], [67], [73], [102, pp. 97-98, Ch. 7], [103, pp. 132-133] for still more functionals of Brownian motion whose Laplace transforms can be expressed in terms of hyperbolic functions.

5. Renormalization of the series $\sum n^{-s}$

5.1. **Statement of the result.** The expansion of S_1 as an infinite series (1.6) suggests that we use partial sums in order to approximate its Mellin transform. As we shall see, this yields an interesting approximation of the Riemann zeta function. Consider again the relationship (1.2) between ζ and ξ, which allows the definition of $\zeta(s)$ for $s \neq 1$ despite the lack of convergence of (1.1) for $\Re s \leq 1$. There are a number of known ways to remedy this lack of convergence, some of which are discussed in Section 5.3. One possibility is to look for an array of coefficients $(a_{n,N}, 1 \leq n \leq N)$ such that the functions

$$\kappa_N(s) := \sum_{n=1}^{N} \frac{a_{n,N}}{n^s} \tag{5.1}$$

converge as $N \to \infty$, for all values of s. For fixed N there are N degrees of freedom in the choice of the coefficients, so we can enforce the conditions $\kappa_N(s) = \zeta(s)$ at N choices of s, and it is natural to choose the points 0, where $\zeta(0) = -\frac{1}{2}$, and $-2, -4, -6, \ldots, -2(N-1)$ where ζ vanishes. It is easily checked that this makes

$$a_{n,N} = \frac{(-N)(1-N)(2-N)\ldots(n-1-N)}{(N+1)(N+2)\ldots(N+n)} = (-1)^n \frac{\binom{2N}{N-n}}{\binom{2N}{N}}. \tag{5.2}$$

Note that for each fixed n

$$a_{n,N} \to (-1)^n \text{ as } N \to \infty \tag{5.3}$$

and recall that

$$(2^{1-s} - 1)\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} \qquad (\Re s > 1) \tag{5.4}$$

extends to an entire function of s.

Theorem 5.1. *For $\varepsilon_i, 1 \leq i \leq N$ independent standard exponential variables, and $\Re s > -2N$*

$$E\left[\left(\sum_{n=1}^{N} \frac{\varepsilon_n}{n^2}\right)^{\frac{s}{2}}\right] = -s\Gamma(\tfrac{s}{2}) \sum_{n=1}^{N} \frac{a_{n,N}}{n^s} \tag{5.5}$$

where the $a_{n,N}$ are defined by (5.2)*, and*

$$\sum_{n=1}^{N} \frac{a_{n,N}}{n^s} \to (2^{1-s} - 1)\zeta(s) \text{ as } N \to \infty \tag{5.6}$$

uniformly on every compact subset of $\mathbb{C}$.

The proof of Theorem 5.1 is given in the next section. As an immediate consequence of Theorem 5.1, we deduce the formula in Table 1 for $E(S_1^s)$, first found by another method in [74, (86)].

5.2. **On sums of independent exponential random variables.** Let $(\varepsilon_n; n \geq 1)$ be a sequence of independent identically distributed random variables, with the standard exponential distribution $P(\varepsilon_n \geq x) = e^{-x}$, $x \geq 0$. Let $(a_n; n \geq 1)$ be a sequence of positive real numbers, such that $\sum_{n=1}^{\infty} a_n < \infty$; then the series $X = \sum_{n=1}^{\infty} a_n \varepsilon_n$ converges almost surely, and in every L^p space, for $1 \leq p < \infty$.

Lemma 5.2. *Let $X = \sum_{n=1}^{\infty} a_n \varepsilon_n$ as above, and let $X_N = \sum_{n=1}^{N} a_n \varepsilon_n$ be the partial sums; then for every real x one has $E[X^x] < \infty$, and $E[X_N^x] < \infty$ for $x > -N$. Furthermore one has*

$$E[X_N^s] \to E[X^s] \text{ as } N \to \infty$$

uniformly with respect to s on each compact subset of $\mathbb{C}$.

Proof. We have already seen that $E[X^x] < \infty$ if $x \geq 0$. Let us prove that $E[X_N^x] < \infty$ for $0 > x > -N$. Let $b_N = \min(a_n; n \leq N)$; then $X_N \geq b_N Y_N = b_N \sum_{n=1}^{N} \varepsilon_n$. But Y_N has a gamma distribution, with density $\frac{1}{\Gamma(N)} t^{N-1} e^{-t}$ at $t > 0$, so that $E[Y_N^x] < \infty$ for $0 > x > -N$ and thus $E[X_N^x] < \infty$. The assertion for X follows from $X \geq X_N$. It remains to check the uniform convergence. If $\Re s \in [-A, +A]$, then

$$\begin{aligned} |X_N^s - X^s| &= \left| \int_{X_N}^{X} s y^{s-1} dy \right| \\ &\leq |s| (X - X_N)(X_N^{-A-1} \vee X^{A-1}), \end{aligned}$$

and the required uniform convergence as $N \to \infty$ is now evident by application of the Cauchy-Schwarz inequality. ∎

We now compute the Mellin transform of the distribution of $\sum_{n=1}^{N} a_n \varepsilon_n$, assuming that the a_n are all distinct and strictly positive.

Lemma 5.3. *With the above notations, and $\Pi_{n,N} := \prod_{j \neq n, 1 \leq j \leq N} \left(1 - \frac{a_j}{a_n}\right)$, for $\Re s > -N$*

$$E\left[\left(\sum_{n=1}^{N} a_n \varepsilon_n\right)^s\right] = \Gamma(s+1) \sum_{n=1}^{N} \frac{a_n^s}{\Pi_{n,N}} \tag{5.7}$$

where the right side of (5.7) is defined by continuity for $s = -1, -2, \ldots, -N+1$.

Proof. The partial fraction expansion of the Laplace transform

$$E\left[\exp\left(-\lambda \sum_{n=1}^{N} a_n \varepsilon_n\right)\right] = \prod_{n=1}^{N} \frac{1}{(1 + \lambda a_n)} = \sum_{n=1}^{N} \frac{1}{\Pi_{n,N}} \frac{1}{(1 + \lambda a_n)} \tag{5.8}$$

implies that for every non-negative measurable function g such that $E[g(a_n \varepsilon_1)]$ is finite for every n

$$E\left[g\left(\sum_{n=1}^{N} a_n \varepsilon_n\right)\right] = \sum_{n=1}^{N} \frac{1}{\Pi_{n,N}} E[g(a_n \varepsilon_1)]. \tag{5.9}$$

For $g(x) = x^s$ this gives (5.7), first for real $s > -1$, then also for $\Re s > -N$ since the previous lemma shows that the left side is analytic in this domain, and the right side is evidently meromorphic in this domain. ∎

Note the implication of the above argument that the sum on the right side of (5.7) must vanish at $s = -1, -2, \dots, -N+1$.

Proof of Theorem 5.1. Apply Lemma 5.3 with $a_n = n^{-2}$ to obtain (5.5) with $a_{n,N}$ defined by

$$-2a_{n,N} = \frac{1}{\Pi_{n,N}} = \frac{\prod_{j\neq n}(j^2)}{\prod_{j\neq n}(j^2 - n^2)} \tag{5.10}$$

where both products are over j with $1 \le j \le N$ and $j \neq n$. The product in the numerator is $(N!/n)^2$, while writing $j^2 - n^2 = (j-n)(j+n)$ allows the product in the denominator to be simplified to $(-1)^{n-1}(N+n)!(N-n)!/(2n^2)$. Thus the expression for $a_{n,N}$ can be simplified to (5.2). The convergence (5.6) at each fixed s with $\Re s > 1$ follows immediately from (5.3) and (5.4) by dominated convergence. Uniform convergence on compact subsets of $\mathbb{C}$ then follows by Lemma 5.2 and analytic continuation. □

Approximation of the Dirichlet L_{χ_4} function. As a variation of the previous argument, Lemma 5.3 applied to $a_n = (n - \frac{1}{2})^{-2}$ yields the formula

$$E\left[\left(\sum_{n=1}^{N} \frac{\varepsilon_n}{(n-\frac{1}{2})^2}\right)^{\frac{s}{2}}\right] = \Gamma(1+\tfrac{s}{2})\frac{2^{s+2}}{\pi}\alpha_N L_{\chi_4}^{(N)}(s+1) \tag{5.11}$$

where

$$L_{\chi_4}^{(N)}(s) := \sum_{n=1}^{N} b_{n,N}\,\frac{(-1)^{n-1}}{(2n-1)^s} \text{ with } b_{n,N} := \frac{\binom{2N-1}{N-n}}{\binom{2N-1}{N}} \to 1 \text{ as } N \to \infty \tag{5.12}$$

for each fixed n, and

$$\alpha_N := \frac{\pi \prod_{j=1}^{N}(2j-1)^2}{4^N N!(N-1)!} \to 1 \text{ as } N \to \infty$$

by Stirling's formula. A reprise of the previous argument gives

$$L_{\chi_4}^{(N)}(s) \to L_{\chi_4}(s) \text{ as } N \to \infty \tag{5.13}$$

uniformly on every compact of $\mathbb{C}$, where $L_{\chi_4}(s)$ is defined by analytic continuation of (3.1) to all $s \in \mathbb{C}$. This argument also yields the formula in Table 1 for $E((C_1)^s)$ in terms of L_{χ_4}. To parallel the discussion above (5.2), we note that

$$L_{\chi_4}^{(N)}(1-2k) = 0 \text{ for } k = 1, 2, \dots N-1. \tag{5.14}$$

It is also possible to use formulae (1.5) and (1.9) to provide another approximation of ζ, but we leave this computation to the interested reader.

5.3. **Comparison with other summation methods.** Perhaps the simplest way to renormalize the series (1.1) is given by the classical formula

$$\zeta(s) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} n^{-s} - \frac{N^{1-s}-1}{1-s}\right) - \frac{1}{1-s} \qquad (\Re s > 0). \tag{5.15}$$

Related methods are provided by the approximate functional equation and by the Riemann-Siegel formula, which are powerful tools in deriving results on the behaviour of the zeta function in the critical strip ($0 \le \Re s \le 1$). See e.g. Edwards [28, Ch. 7] for a detailed discussion.

It is also known [36], [55] that the series $\sum_1^\infty (-1)^n/n^s$ is Abel summable for all values of $s \in \mathbb{C}$, meaning that as $z \to 1$ in the unit disk,

$$\sum_1^\infty \frac{(-z)^n}{n^s} \to (2^{1-s}-1)\zeta(s).$$

The Lerch zeta function

$$\Phi(x,a,s) = \sum_{n=0}^\infty \frac{e^{2i\pi nx}}{(n+a)^s} \qquad (x \in \mathbb{R},\ 0 < a \leq 1,\ \Re s > 1)$$

is known to have analytic continuation to $s \in \mathbb{C}$, with a pole at $s = 1$ for $a = 1$, $x \in \mathbb{Z}$. This allows us to sketch another proof of Theorem 5.1. The formula

$$\kappa_N(s) = 2^{2N} \binom{2N}{N}^{-1} \int_0^1 (\sin(\pi x))^{2N} \Phi(x,1,s)\, dx$$

is easily checked using (5.1)-(5.2) for $\Re s > 1$, and extended by analytic continuation to all values of $s \in \mathbb{C}$. Convergence of $\kappa_N(s)$ towards $(2^{1-s}-1)\zeta(s)$ then follows from continuity properties of the Lerch zeta function in the variable x and the fact that $2^{2N}\binom{2N}{N}^{-1}(\sin(\pi x))^{2N}\, dx \to \delta_{1/2}$ weakly as $N \to \infty$.

Finally, we note that J. Sondow [88] has shown that Euler's summation method yields the following series, uniformly convergent on every compact subset of $\mathbb{C}$:

$$(1-2^{1-s})\zeta(s) = \sum_{j=0}^\infty \frac{1 - \binom{j}{1}2^{-s} + \ldots + (-1)^j\binom{j}{j}(j+1)^{-s}}{2^{j+1}}. \tag{5.16}$$

Furthermore the sum of the first N terms of this series gives the exact values of ζ at $0, -1, -2, \ldots, -N+1$, so we can rewrite the partial sum in (5.16) as

$$\rho_N(s) = \sum_1^N \frac{c_{n,N}}{n^s}$$

where the $c_{n,N}$ are completely determined by $\rho_N(-j) = \zeta(-j)$ for $j = 0, 1, \ldots, N-1$. Compare with the discussion between (5.1) and (5.2) to see the close parallel between (5.6) and (5.16).

6. Final remarks

6.1. **Hurwitz's zeta function and Dirichlet's L-functions.** Row 3 of Table 2 involves hitting times of Bessel processes of dimension 1 and 3, started from 0. If the Bessel process does not start from zero, we still have an interesting formula for the Mellin transform of the hitting time, expressed now in terms of the Hurwitz zeta function. Specifically, one has

$$E[e^{-\lambda T_1^a(R_3)}] = \frac{\sinh(a\sqrt{2\lambda})}{a\sinh(\sqrt{2\lambda})};\ \ E[e^{-\lambda T_1^a(R_1)}] = \frac{\cosh(a\sqrt{2\lambda})}{\cosh(\sqrt{2\lambda})} \tag{6.1}$$

where T_1^a denotes the hitting time of 1, starting from $a \in]0,1[$, of the corresponding Bessel process. Expanding the denominator we get

$$\frac{\sinh(a\sqrt{2\lambda})}{a\sinh(\sqrt{2\lambda})} = \frac{1}{a}\sum_{n=0}^\infty e^{-(2n+1-a)\sqrt{2\lambda}} - e^{-(2n+1+a)\sqrt{2\lambda}}.$$

Inverting the Laplace transform yields the density of the distribution of $T_1^a(R_3)$

$$\frac{1}{a\sqrt{2\pi t^3}}\sum_{n=0}^{\infty}(2n+1-a)e^{-(2n+1-a)^2/(2t)}-(2n+1+a)e^{-(2n+1+a)^2/(2t)}.$$

Taking the Mellin transform we get

$$E[(T_1^a(R_3))^{s/2}]=\frac{\Gamma(\frac{s-1}{2})}{a2^{s/2}}\left[\zeta\left(s,\frac{1-a}{2}\right)-\zeta\left(s,\frac{1+a}{2}\right)\right]\qquad(\Re s>1)$$

where $\zeta(s,x)=\sum_{n=0}^{\infty}(n+x)^{-s}$ is the Hurwitz zeta function. This identity extends by analytic continuation to all $s\in\mathbb{C}$. A similar expression exists for $T_1^a(R_1)$.

One can use the product expansion for sinh in order to give an approximation of $\zeta(s,u)-\zeta(s,1-u); u\in]0,1[$. For it is easy to see that $\prod_{n=1}^{N}\left(1+\frac{2a^2\lambda}{n^2}\right)\left(1+\frac{2\lambda}{n^2}\right)^{-1}$ is the Laplace transform of a probability distribution on $[0,\infty[$, and that this probability distribution converges towards that of $T_1^a(R_3)$ in such a way that there is a result similar to Theorem 5.1.

The Hurwitz zeta function can be used to construct Dirichlet L-functions by linear combinations. However, direct probabilistic interpretations of general Dirichlet L-functions, in the spirit of what we did in Section 4, do not seem to exist. More precisely, let χ be a primitive character modulo N, and let

$$\theta_\chi(t)=\sum_{n=-\infty}^{+\infty}n^\epsilon\chi(n)e^{-\pi n^2 t}\tag{6.2}$$

where $\epsilon=0$ or 1 according to whether χ is even or odd, so $\chi(-1)=(-1)^\epsilon$. These functions satisfy the functional equation

$$\theta_\chi(t)=\frac{(-i)^\epsilon\tau(\chi)}{N^{1+\epsilon}t^{\epsilon+1/2}}\theta_{\bar\chi}\left(\frac{1}{N^2t}\right)\tag{6.3}$$

where $\tau(\chi)$ is a Gauss sum. Taking a Mellin transform, this yields the analytic continuation and functional equation for the associated Dirichlet L-function

$$L_\chi(s):=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},$$

namely

$$\Lambda(s,\chi)=(-i)^\epsilon\tau(\chi)N^{-s}\Lambda(1-s,\bar\chi)\tag{6.4}$$

where

$$\Lambda(s,\chi)=\pi^{\frac{-(s+\epsilon)}{2}}\Gamma\left(\frac{s+\epsilon}{2}\right)L_\chi(s).$$

See [21] or [15, §1.1] for the classical derivations of these results. For general real χ, there does not seem to be any simple probabilistic interpretation of $\theta_\chi(t)$. In particular, this function is not necessarily positive for all $t>0$. This can be seen as follows. We choose an odd character χ (the case of even characters is similar), and compute the Laplace transform of θ_χ using (6.2) and (6.3):

$$\int_0^\infty e^{-\lambda t}\theta_\chi(t)\,dt = \sum_{n=-\infty}^{+\infty}\int_0^\infty \frac{n\chi(n)}{N^{3/2}t^{3/2}}e^{-\lambda t}e^{-\pi n^2/(N^2 t)}\,dt$$
$$= \frac{1}{\sqrt{N}}\sum_{n=1}^{\infty}\chi(n)e^{-\frac{2n}{N}\sqrt{\pi\lambda}}.$$

Using the periodicity of χ, this equals

$$\frac{\sum_{n=1}^{N-1}\chi(n)e^{-\frac{2n}{N}\sqrt{\pi\lambda}}}{\sqrt{N}\,(1-e^{-2\sqrt{\pi\lambda}})} = \frac{\sum_{n=1}^{(N-1)/2}\chi(n)\sinh(\frac{N-2n}{N}\sqrt{\pi\lambda})}{\sqrt{N}\,\sinh(\sqrt{\pi\lambda})}. \tag{6.5}$$

For small values of N, and χ a real odd character modulo N, one can see by inspection that this indeed is the Laplace transform of a positive function; hence by uniqueness of the Laplace transform, $\theta_\chi(t) > 0$ for $t > 0$. However, Pólya [76] exhibited an infinite number of primes p such that for the quadratic character modulo p the polynomial $Q_p(x) := \sum_{n=1}^{p-1} x^n\chi(n)$ takes negative values somewhere on $[0,1]$. In particular, for $p = 43$ we find $Q_{43}(3/4) \approx -0.0075$. For such quadratic Dirichlet characters, the Laplace transform above also takes negative values, which implies that θ_χ does not stay positive on $]0,\infty[$. We note that if $\theta_\chi > 0$ on $]0,\infty[$, then obviously its Mellin transform has no zero on the real line, and hence the corresponding L-function has no Siegel zeros.

6.2. **Other probabilistic aspects of Riemann's zeta function.** It is outside the scope of this paper, and beyond the competence of its authors, to discuss at length the theory of the Riemann zeta function. But we mention in this final section some other works relating the zeta function to probability theory.

The work of Pólya has played a significant role in the proof of the Lee-Yang theorem in statistical mechanics: see the discussion in [78, pp. 424-426]. Other connections between Riemann zeta function and statistical mechanics appear in Bost and Connes [14] and in Knauf [50].

The Euler product for the Riemann zeta function is interpreted probabilistically in Golomb [34] and Nanopoulos [61] via the independence of various prime factors when choosing a positive integer according to the distribution with probability at n equal to $\zeta(s)^{-1}n^{-s}$ for some $s > 1$. See also Chung [17, p. 247] and [87].

It is an old idea of Denjoy [23] that the partial sums of the Möbius function should behave like a random walk (the law of iterated logarithm would imply Riemann hypothesis). See also Elliott [30].

There are fascinating connections between the distribution of spacings between zeros of the Riemann zeta function (and other L-functions) and the distribution of spacings between eigenvalues of random matrices. See for instance Odlyzko [64], Katz and Sarnak [43], [44], Berry and Keating [7], Michel [60].

A new connection between generalizations of Riemann's ξ function and infinitely divisible laws on the line has recently been made by Lagarias and Rains [54]. Finally, we mention some other recent papers which give probabilistic interpretations of some particular values of the Riemann zeta function: Alexander, Baclawski and Rota [2], Asmussen, Glynn and Pitman [5], Joshi and Chakraborty [41], Chang and Peres [16].

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