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The Group of Generalized Stirling Numbers

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In this paper we provide an algebraic approach to the generalized Stirling numbers (GSN). By defining a group $\mathscr{S}$ that contains the GSN, we obtain a unified interpretation for important combinatorial functions like the binomials, Stirling numbers, Gaussian polynomials. In particular we show that many GSN are products of others. We provide an explanation for the fact that many GSN appear as pairs and the inverse relations fulfilled by them. By introducing arbitrary boundary conditions, we show a Chu–Vandermonde type convolution formula for GSN. Using the group $\mathscr{S}$ we demonstrate a solution to the problem of finding the connection constants between two sequences of polynomials with persistent roots.

1. INTRODUCTION

In the past few years the study of *generalized Stirling numbers* (GSN) has gained increasing interest. Since the term GSN has appeared in [4], various authors analyzed this subject by using analytic, algebraic, and combinatorial approaches. As an important example Verde-Star's calculus of *divided differences* has provided a powerful tool for the unified treatment of GSN [15]. Other authors like de Médicis and Leroux [11] or Wagner [18] have focused on combinatorial techniques and interpretations for particular instances of GSN. Finally there are authors like D'Antona, Damiani, Hsu, Loeb, Naldi, and Shiue [5–7] who have looked at GSN basically as *connection constants* arising from the transformation between sequences of *polynomials with persistent roots*. Recently Bickel *et al.* [2] have shown that the GSN are well suited for the description of discrete time pure birth processes.

An article that has attained only little attention so far is due to Théorêt [14]. We have not been aware of this work and all results

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presented here have been developed independently. Nevertheless our Theorem 6 could be obtained directly from [14, Theorem 1, p. 199].

The GSN contain a lot of important combinatorial functions like the *binomials, Stirling numbers of the first and second kind, Lah numbers, and Gaussian polynomials*. For a survey see [7, 11]. For all but the most trivial GSN, explicit formulas are difficult to find. Hence one is interested in manipulating known GSN in order to build new ones. Another reason for searching for a unifying theory for GSN is that they appear in numerous combinatorial identities. For a better understanding of these identities it is essential to find an interpretation in a closed theoretical framework.

The purpose of this paper is to provide such a framework. In Section 2 we introduce the group $\mathcal{S}$ of all infinite lower triangular matrices A over a field $\mathbb{K}$ of characteristic 0 for which $A(n,n) = 1\ \forall n \in \mathbb{N}_0$. For our investigations we focus on the recursion

$$\begin{aligned} &A(0,\ell) = \delta(0,\ell) \qquad \text{and} \\ &A(n,\ell) = a_{n-1,\ell}\, A(n-1,\ell) + A(n-1,\ell-1) \end{aligned} \tag{1}$$

that is characteristic for the GSN. We then use (1) to define a mapping Φ from the set $\mathcal{M}$ of all coefficient functions $a : \mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{K}$ to $\mathcal{S}$. This approach is different from the one recently taken by several authors inasmuch as they have rather relayed on generating function methods [5, 7]. Our approach enables us to formulate and prove Theorem 6 which forms the main result of this paper. This result connects GSN of the first and second kind, which can be obtained by restricting Φ to appropriate subsets $\mathcal{N}$ and $\mathcal{L}$ of $\mathcal{M}$. $\mathcal{N}$ and $\mathcal{L}$ are formed by all $a \in \mathcal{M}$ that are constant in ℓ and n, respectively. As a consequence of Theorem 6 we derive some known inverse relations for GSN. In Section 3 we enlarge the applicability of the theory by introducing the notion of general *boundary conditions*. We then investigate the solutions of (1) under the assumption that $A(n,0) = g(n)$ for some arbitrary function $g : \mathbb{N}_0 \longrightarrow \mathbb{K}$ with $g(0) = 1$. In particular we show that an analogous result to Theorem 6 holds in this case too. We apply the theory in order to show a *Chu–Vandermonde type* convolution formula for GSN. This formula generalizes the corresponding formulas in [11, 15].

In Section 4 we investigate the problem of interpolating between *sequences of polynomials with persistent roots* or in short just *persistent sequences*. This problem has been treated extensively in [5, 6]. Applying Theorem 6 we are able to refine their analysis of the structure of the *connection constants*. We show that they can always be written as a product of elements of $\mathcal{S}$. Further we examine a group that has been introduced in [15]. We give explicit expressions for its elements and show that it is actually a subgroup of $\mathcal{S}$. Finally we make some remarks, concerning another class of subgroups of $\mathcal{S}$, and show some connections with *sequences of polynomials of binomial type*.

2. THE GROUP OF GENERALIZED STIRLING NUMBERS

Let $\mathbb{K}$ denote any field with characteristic 0. Usually $\mathbb{K}$ stands for the real or complex numbers. We consider the mappings

$$\begin{aligned} A : \mathbb{N}_0 \times \mathbb{N}_0 &\longrightarrow \mathbb{K} \\ (n, \ell) &\longmapsto A(n, \ell), \end{aligned}$$

where additionally A satisfies

$$A(n, n) = 1 \qquad \forall n \geq 0 \tag{2}$$

and

$$A(n, \ell) = 0 \qquad \forall \ell > n. \tag{3}$$

We denote the set of all such mappings by $\mathscr{S}$. In other words $\mathscr{S}$ is the set of all infinite lower triangular matrices over $\mathbb{K}$ with 1 on the diagonal. Because of (3) we can define a composition on $\mathscr{S}$ $\cdot : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ by

$$A \cdot B(n, \ell) := \sum_{k=0}^{\infty} A(n, k)\, B(k, \ell) = \sum_{k=\ell}^{n} A(n, k)\, B(k, \ell). \tag{4}$$

$(\mathscr{S}, \cdot)$ can be regarded as a closed subset of the incidence algebra for the trivial PO set $\mathbb{N}$.

PROPOSITION 1. *$(\mathscr{S}, \cdot)$ is a group.*

Proof. We denote by $\mathscr{S}_n$ the set of $n \times n$ matrices that fulfill (2) and (3). $\mathscr{S}_n$ is a group by matrix multiplication. Now let $\phi_n : \mathscr{S}_n \to \mathscr{S}_{n-1}$ be given by omitting the nth row and column. Obviously ϕ_n is a surjective homomorphism for all n and we set

$$\widehat{\mathscr{S}} = \varprojlim (\mathscr{S}_n, \phi_n)$$

the inverse limit of $(\mathscr{S}_n, \phi_n)$. One easily verifies that $\widehat{\mathscr{S}} \cong \mathscr{S}$ and so that $\mathscr{S}$ is a group. ∎

Remark 2. The unit element in $\mathscr{S}$ is given by

$$\delta(n, \ell) := \begin{cases} 1 & \text{if } n = \ell \\ 0 & \text{if } n \neq \ell. \end{cases}$$

Next we consider mappings

$$a : \mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{K}$$
$$(n, \ell) \longmapsto a_{n,\ell}$$

and we denote the set of all such mappings by $\mathscr{M}$. $\mathscr{M}$ naturally carries a linear space structure. We write $a_{n,\ell}$ instead of $a(n, \ell)$ in order to distinguish the elements of $\mathscr{M}$ from the elements of $\mathscr{S}$. $\mathscr{M}$ contains two subsets $\mathscr{N}$ and $\mathscr{L}$ which are of special interest for us. We set

$$\mathscr{N} := \left\{ a \in \mathscr{M} \mid a(n, \ell_1) = a(n, \ell_2) \quad \forall\, \ell_1, \ell_2, n \in \mathbb{N}_0 \right\} \tag{5}$$

and accordingly

$$\mathscr{L} := \left\{ a \in \mathscr{M} \mid a(n_1, \ell) = a(n_2, \ell) \quad \forall\, n_1, n_2, \ell \in \mathbb{N}_0 \right\}. \tag{6}$$

This means the members of $\mathscr{N}$ and $\mathscr{L}$ are the elements of $\mathscr{M}$ that are constant in the first or second argument. Therefore we will often identify them with mappings $a : \mathbb{N}_0 \longrightarrow K$ and, instead of $a_{n,\ell}$, just write a_n or a_ℓ for $a \in \mathscr{N}, \mathscr{L}$, respectively.

In order to relate $\mathscr{M}$ and $\mathscr{S}$ we come to the central definition of this paper. We consider the recursion

$$A(0, \ell) = \delta(0, \ell) \qquad \forall \ell \geq 0 \tag{7}$$

$$A(n, \ell) = a_{n-1,\ell}\, A(n-1, \ell) + A(n-1, \ell-1) \qquad \forall n > 0, \ell \geq 0 \tag{8}$$

for any $a \in \mathscr{M}$. We set $A(n, \ell) = 0$ for $\ell < 0$ and get $A(n, 0) = a_{n-1,0}\, A(n-1, 0)$. Alternatively we could agree for (8) to hold only for $\ell > 0$ and that for $\ell = 0$ we have the boundary condition (BC)

$$A(n, 0) = \prod_{k=0}^{n-1} a_{k,0} \qquad \forall n > 0. \tag{9}$$

We call (9) the *natural boundary condition* (NBC) in contrast to an arbitrary BC

$$A(n, 0) = g(n), \quad \forall n > 0$$

that we will consider later. From (7) and (8) we infer immediately that $A(n, \ell) = 0$ if $\ell > n$ and $A(n, n) = 1 \;\; \forall n \geq 0$. So we can define a map

$$\Phi : \mathscr{M} \longrightarrow \mathscr{S} \tag{10}$$

by associating to any $a \in \mathscr{M}$ the solution $A \in \mathscr{S}$ of (7) and (8). One easily verifies that this map is well-defined. In the last 25 years, various authors studied the solutions of (7) and (8) for several $a \in \mathscr{M}$. Some took a more algebraic approach (e.g., [4, 5, 7, 8, 10, 15]) others used combinatorial methods (e.g., [11, 18]). Mostly the terms *generalized Stirling numbers* (GSN) of the first and second kind have been used for the members of $\Phi(\mathscr{N})$ and $\Phi(\mathscr{L})$. In the following we call the elements of $\Phi(\mathscr{M})$ just generalized Stirling numbers. The goal of this section is to investigate the map Φ and to clarify its interaction with the group structure of $\mathscr{S}$. We start with a negative result. Namely $\Phi : \mathscr{M} \longrightarrow \mathscr{S}$ is not injective.

LEMMA 3. *Let $a \in \mathscr{M}$. Then we have*

$$a_{n,n} = 0 \Longleftrightarrow \Phi(a) = \delta.$$

Proof. First assume there is an n with $a_{n,n} \neq 0$. Then

$$A(n+1, n) = a_{n,n} A(n, n) + A(n, n-1)$$

and hence either $A(n, n-1) \neq 0$ or $A(n+1, n) \neq 0$ in contradiction to $A = \delta$. On the other hand, if $a_{n,n} = 0$ for all $n \geq 0$ then

$$A(0,0) = 1$$

$$A(1,0) = a_{0,0} A(0,0) = 0, \qquad A(1,1) = A(0,0) = 1$$

and it follows easily by induction that $A = \delta$. ∎

The above lemma obviously implies that Φ is not injective. Now let $A \in \mathscr{S}$ with $A(n, 0) = 0$ for some n but with $A(n+1, 0) \neq 0$. Then it follows from (9) that there is no $a \in \mathscr{M}$ such that $A = \Phi(a)$. Thus Φ is not surjective either.

If Φ is analyzed on the whole domain $\mathscr{M}$, then the situation is very complicated. In order to obtain some meaningful results it seems necessary to restrict Φ to some subset $\mathscr{U} \subset \mathscr{M}$. Usually $\mathscr{U}$ will be $\mathscr{N}$, $\mathscr{L}$ or $\mathscr{N}+\mathscr{L}$.

The following result is due to Comtet [4].

PROPOSITION 4. (a) *Let $a \in \mathscr{N}$. Then*

$$\Phi(a)(n, \ell) = e_{n-\ell}(a_0, a_1, \ldots, a_{n-1}),$$

where e_k is the kth elementary symmetric polynomial.

(b) *Let $b \in \mathscr{L}$. Then*

$$\Phi(b)(n, \ell) = h_{n-\ell}(b_0, b_1, \ldots, b_\ell),$$

where h_k is the kth complete symmetric polynomial.

In the sequel we use small Greek letters $\alpha, \beta, \gamma, \ldots$ for scalars in $\mathbb{K}$. We denote by $\Phi_{\mathcal{N}} := \Phi|_{\mathcal{N}} : \mathcal{N} \longrightarrow \mathcal{S}$ the restriction of Φ to $\mathcal{N}$ and $\Phi_{\mathcal{L}}$ analogously. Further for any $A \in \mathcal{S}$, $\lambda * A$ means the element of $\mathcal{S}$ for which

$$\lambda * A(n, \ell) = \lambda^{n-\ell} A(n, \ell).$$

One easily proves by induction that

$$\Phi(\lambda a) = \lambda * \Phi(a). \tag{11}$$

Next we define a family of operators $T_k : \mathcal{S} \longrightarrow \mathcal{S}$ by its action on $A \in \mathcal{S}$

$$T_k(A)(n, \ell) := \begin{cases} \delta(n, \ell) & \text{if } n < k \text{ or } \ell < k \\ A(n-k, \ell-k) & \text{if } n \geq k \text{ and } \ell \geq k. \end{cases} \tag{12}$$

In the following we write $(\,.\,)$ for $\Phi((1, 1, 1, \ldots))$, i.e., $(\,.\,)(n, \ell) = \binom{n}{\ell}$ the binomials. We now show some basic properties of Φ.

PROPOSITION 5. (a) *$\Phi_{\mathcal{N}}$ and $\Phi_{\mathcal{L}}$ are injective.*

(b) $\Phi(\mathcal{N} \cap \mathcal{L}) = \{\lambda * (\,.\,) \mid \lambda \in \mathbb{K}\}$

(c) $\Phi(\mathcal{N}) \cap \Phi(\mathcal{L}) = \bigcup_{k=0}^{\infty} T_k(\{\lambda * (\,.\,) \mid \lambda \in \mathbb{K}\})$

(d) $\forall k > 0$, $A \in T_k(\{\lambda * (\,.\,) \mid \lambda \in \mathbb{K}\})$ *there is exactly one* $a \in \mathcal{N}$, $b \in \mathcal{L}$ *with* $\Phi(a) = A = \Phi(b)$. *Moreover we have* $a = b^T$.

Proof. (a) Let $A \in \mathcal{S}$ and $a, b \in \mathcal{N}$ with $\Phi(a) = \Phi(b) = A$. Then we have

$$a_n = A(n+1, n) - A(n, n-1) = b_n \qquad \forall n > 0$$

and $a_0 = A(1, 0) = b_0$. So $a = b$ and $\Phi_{\mathcal{N}}$ is injective. For $\Phi_{\mathcal{L}}$ one proceeds analogously.

(b) Let $a \in \mathcal{N} \cap \mathcal{L}$. Then there are sequences $(c_k)_k$ and $(d_k)_k$ with

$$a_{n,\ell} = c_n \qquad \text{and} \qquad a_{n,\ell} = d_\ell \quad \forall n, \ell \geq 0.$$

In particular we have $a_{n,n} = c_n = d_n$ and so $c = d$. From this we infer $a_{n,\ell} = c_n = c_\ell$. Hence $a_{n,\ell} = \lambda$ for some $\lambda \in \mathbb{K}$ and with (11) the claim follows.

(c) Let $A \in \Phi(\mathcal{N}) \cap \Phi(\mathcal{L})$. Then there are $a \in \mathcal{N}$ and $b \in \mathcal{L}$ with $\Phi(a) = \Phi(b) = A$. There exist sequences $(c_k)_k$ and $(d_k)_k$ with

$$a_{n,\ell} = c_n \qquad \text{and} \qquad b_{n,\ell} = d_\ell \; \forall n, \ell \geq 0.$$

We have

$$A(n+1, \ell) = c_n A(n, \ell) + A(n, \ell-1) = d_\ell A(n, \ell) + A(n, \ell-1).$$

This implies $c_n\, A(n,n) = d_n\, A(n,n)$ and hence $c = d$. Now let us assume that $c_k > 0\ \forall k \geq 0$. Then we have

$$
\begin{aligned}
A(1,0) &= c_0 \\
A(2,0) &= {c_0}^2 = c_0\, c_1 \Longrightarrow c_1 = c_0 \\
A(3,0) &= {c_0}^3 = c_0\, c_1\, c_2 \Longrightarrow c_2 = c_0.
\end{aligned}
$$

Inductively we find $c_k = c_0 = \lambda\ \forall k \geq 0$ and some $\lambda \in \mathbb{K}$ and hence $A = \lambda * (\,.\,)$. Now let us suppose there is exactly one i with $c_i = 0$. Because of

$$
{c_0}^{i+1} = c_0\, c_1 \ldots c_i = 0
$$

we infer $i = 0$. In a similar way we see that if $|\{i \mid c_i = 0\}| = k$ then

$$
c = \Bigl(\underbrace{0, \ldots, 0}_{k}, c_{k+1}, c_{k+2}, \ldots\Bigr).
$$

Analogously to the case above one finds $c_i = \lambda\ \forall i > k$ and it is easily verified that with $a_{n,\ell} = c_n$ and $b = a^T$ we have

$$
\Phi(a) = \Phi(b) = T_k\,(\lambda * (\,.\,)) \tag{13}
$$

and hence $\Phi(\mathscr{N}) \cap \Phi(\mathscr{L}) = \bigcup_{k=0}^{\infty} T_k(\{\lambda * (\,.\,) \mid \lambda \in \mathbb{K}\})$. We note that in this case $a \neq b$ and that $\forall k > 0$ and $A \in T_k(\{\lambda * (\,.\,) \mid \lambda \in \mathbb{K}\})$ we can find exactly one $a \in \mathscr{N}$ and $b \in \mathscr{L}$ that fulfill (13). This proves the other inclusion and (d). ∎

Next we want to demonstrate the central result of this paper. In short it says that $\Phi(\mathscr{N} + \mathscr{L}) = \Phi(\mathscr{N}) \cdot \Phi(\mathscr{L})$. More explicitly we have

THEOREM 6. *Let $a \in \mathscr{N}$ and $b \in \mathscr{L}$. Then we have*

$$
\Phi(a+b) = \Phi(a) \cdot \Phi(b).
$$

Proof. Let $A = \Phi(a)$, $B = \Phi(b)$, and $C = A \cdot B$. Then we have $C(0,\ell) = \delta(0,\ell)$, i.e., C fulfills the initial condition. Next for $\ell = 0$ we have

$$
C(n,0) = \sum_{k=0}^{n} A(n,k)\, B(k,0) = \sum_{k=0}^{n} e_{n-k}(a_0, \ldots, a_{n-1})\, {b_0}^k.
$$

On the other hand

$$
\Phi(a+b)(n,0) = \prod_{k=0}^{n-1} (a_k + b_0) = \sum_{k=0}^{n} e_{n-k}(a_0, \ldots, a_{n-1})\, {b_0}^k.
$$

For $n, \ell > 0$ we have

$$
\begin{aligned}
C(n,\ell) &= \sum_{k=\ell}^{n} A(n,k)\,B(k,\ell) \\
&= \sum_{k=\ell}^{n} (a_{n-1}\,A(n-1,k) + A(n-1,k-1))\,B(k,\ell) \\
&= \sum_{k=\ell}^{n} a_{n-1}\,A(n-1,k)\,B(k,\ell) \\
&\quad + \sum_{k=\ell}^{n} A(n-1,k-1)\,(b_\ell\,B(k-1,\ell) + B(k-1,\ell-1)) \\
&= (a_{n-1} + b_\ell) \sum_{k=\ell}^{n-1} A(n-1,k)\,B(k,\ell) \\
&\quad + \sum_{k=\ell-1}^{n-1} A(n-1,k)\,B(k,\ell-1) \\
&= (a_{n-1} + b_\ell)\,C(n-1,\ell) + C(n-1,\ell-1).
\end{aligned}
$$

It follows that C fulfills (7) and (8) with $a_{n,\ell} = a_n + b_\ell$ and from the definition of Φ we infer $C = \Phi(a+b)$. ∎

The above theorem shows a relation between GSN of the first and second kind which is known for special cases, but has never been noted in full generality. Note that the theorem does not hold if $\Phi(a)$ and $\Phi(b)$ are interchanged. Indeed the recursion fulfilled by $\Phi(b)\cdot\Phi(a)$ can be arbitrarily complicated and the non-commutativity of the group $\mathscr{S}$ is clearly recognized. But the theorem and the following corollary suggest that $(\mathscr{S}, \cdot)$ is the right algebraic framework to study Φ.

COROLLARY 7. *Let $a \in \mathscr{N}$ and $b \in \mathscr{L}$. Then there exist unique $c \in \mathscr{L}$ and $d \in \mathscr{N}$ with*

$$\Phi(a)\cdot\Phi(c) = \Phi(c)\cdot\Phi(a) = \delta$$

and

$$\Phi(b)\cdot\Phi(d) = \Phi(d)\cdot\Phi(b) = \delta.$$

Moreover we have $c = -a^T$ and $d = -b^T$.

Proof. Let $a \in \mathscr{N}$. Then we have $a^T \in \mathscr{L}$ and

$$(a - a^T)_{n,n} = a_{n,n} - a^T_{n,n} = 0 \qquad \forall n \geq 0.$$

Hence Lemma 3 and Theorem 6 imply

$$\delta = \Phi(a - a^T) = \Phi(a) \cdot \Phi(-a^T) = \Phi(-a^T) \cdot \Phi(a).$$

The third equality follows from the fact that in any group the left inverse is also the right inverse. So the inverse element is uniquely determined and from the injectivity of $\Phi_{\mathscr{L}}$ we infer that $-a^T$ is also unique with this property. The statements for b and d follow in a similar way. ∎

The above corollary points out a relation between GSN of the first and second kind that has often been called the *orthogonality relation*. In the present context the term *inverse relation* seems to be more appropriate. Corollary 7 can be summarized in the following commutative diagram

$$\begin{array}{ccc} \mathscr{N} & \xrightarrow{\Phi} & \mathscr{S} \\ {\scriptstyle a \mapsto -a^T}\Big\downarrow & & \Big\downarrow{\scriptstyle A \mapsto A^{-1}} \\ \mathscr{L} & \xrightarrow{\Phi} & \mathscr{S} \end{array}$$

Theorem 6 suggests unifying the GSN as the members of $\Phi(\mathscr{N} + \mathscr{L})$ and considering the GSN of the first and second kind as special instances of the former. The GSN in this sense coincide with the connection constants in [5, 6]. In [5] the term *complementary symmetric functions* has been introduced. For some GSN like the *Lah numbers* it is known that they are self inverse [9]. The following corollary clarifies the situation and shows that this holds in a more general context.

COROLLARY 8. *Let* $a \in \mathscr{N}, b \in \mathscr{L}$. *Then we have*

$$\Phi(a + b) \cdot \Phi(-a^T - b^T) = \Phi(-a^T - b^T) \cdot \Phi(a + b) = \delta.$$

Proof.

$$\Phi(a + b) \cdot \Phi(-a^T - b^T) = \Phi(a) \cdot \underbrace{\Phi(b) \cdot \Phi(-b^T)}_{\delta} \cdot \Phi(-a^T) = \delta.$$

∎

In particular if $b = a^T$ we get

$$\Phi(a + a^T) \cdot (-1) * \Phi(a + a^T) = \delta. \tag{14}$$

EXAMPLE 9 (Binomials). Let $\mathscr{B} = \langle (\, . \,) \rangle \subset \mathscr{S}$ be the cyclic subgroup generated by $(\, . \,)$. We call $\mathscr{B}$ the *binomial group*. Then we have $(\mathscr{B}, \cdot) \cong (\mathbb{Z}, +)$. To see this we set

$$\varphi : \mathbb{Z} \longrightarrow \mathscr{N} \cap \mathscr{L}$$

$$\alpha \longmapsto a_{n,\ell} \equiv \alpha.$$

Then we have that

$$\begin{aligned}\Phi \circ \varphi : \mathbb{Z} &\longrightarrow \mathscr{B} \\ \alpha &\longmapsto \alpha * (\,.\,)\end{aligned}$$

is a bijection. From

$$\Phi(\varphi(\alpha + \beta)) = \Phi(\varphi(\alpha) + \varphi(\beta)) = \Phi(\varphi(\alpha)) \cdot \Phi(\varphi(\beta))$$

we infer that $\Phi \circ \varphi$ is an isomorphism. In particular we find the well known identity

$$\Phi \circ \varphi(1-1)(n, \ell) = \sum_{k=\ell}^{n} (-1)^{k-\ell} \binom{n}{k} \binom{k}{\ell} = \delta(n, \ell).$$

For $\alpha \in \mathbb{N}$ we have the powers in $\mathscr{S}$

$$(\,.\,)^{\alpha} = \underbrace{(\,.\,) \cdot (\,.\,) \cdot \cdots \cdot (\,.\,)}_{\alpha \times} = \Phi(\varphi(\underbrace{1 + \cdots + 1}_{\alpha \times})) = \Phi(\varphi(\alpha)).$$

This suggests defining powers of the binomials for arbitrary $\lambda \in \mathbb{K}$ by setting (and extending φ in an obviuos manner)

$$(\,.\,)^{\lambda} := \Phi(\varphi(\lambda)). \tag{15}$$

We have

$$(\,.\,)^{\lambda_1} \cdot (\,.\,)^{\lambda_2} = (\,.\,)^{\lambda_1 + \lambda_2}.$$

EXAMPLE 10 (Stirling Numbers, Lah Numbers). We set $a_{n,\ell} = n$ and $b_{n,\ell} = \ell$. Then we have $\Phi(a) = S$ the Stirling numbers of the first kind, $\Phi(b) = s$ the Stirling numbers of the second kind, and $\Phi(a + b) = L$ the sign-less Lah numbers. Theorem 6 yields

$$L = S \cdot s.$$

From Corollary 7 we infer the well known identity

$$\sum_{k=\ell}^{n} (-1)^{k-\ell} S(n, k)\, s(k, \ell) = \delta(n, \ell)$$

and from Corollary 8 we find

$$\sum_{k=\ell}^{n} (-1)^{k-\ell} L(n, k)\, L(k, \ell) = \delta(n, \ell).$$

EXAMPLE 11 (Non-central Stirling Numbers). This example is due to Koutras [8]. If we set $a_{n,\ell} = (\alpha - n)$ and $b_{n,\ell} = (\beta - \ell)$ we obtain the non-central Stirling numbers S_α, s_β of the first and second kind. We then have

$$S_\alpha(n,\ell) = \sum_{k=\ell}^{n} \alpha^{k-\ell}(-1)^{n-k} S(n,k)\binom{k}{\ell}$$

and

$$s_\beta(n,\ell) = \sum_{k=\ell}^{n} \beta^{n-k}(-1)^{k-\ell}\binom{n}{k} s(k,\ell).$$

For some applications of these numbers in probability theory see [8].

EXAMPLE 12 (Möbius Inversion). Möbius and Zeta function play a central role in the theory of incidence algebras, e.g., see [13]. As mentioned before, $\mathcal{S}$ can be regarded as a multiplicatively closed subset of the incidence algebra over $\mathbb{N}$. It is easily verified that with $b \in \mathcal{L}$, $b_{n,\ell} = \delta_{0,\ell}$ $\forall n \geq 0$ we get

$$\Phi(b)(n,\ell) := \begin{cases} 1 & \text{if } n \geq \ell \\ 0 & \text{if } n < \ell. \end{cases}$$

So we have $\Phi(b) = \zeta$ the Zeta function. Let $a \in \mathcal{N}$, $a_{n,l} = -\delta_{n,0}$ $\forall \ell \geq 0$. Then with Lemma 3 we find $\Phi(a) \cdot \Phi(b) = \delta$ and thus $\Phi(a) = \mu$ the Möbius function.

3. BOUNDARY CONDITIONS AND CONVOLUTION FORMULAS

In this section we extend the algebraic background for the GSN and thus provide interpretations of additional identities for various combinatorial functions. In the previous section we have seen that these interpretations can often be found within the setting of the group $\mathcal{S}$. Unfortunately the mapping Φ was not surjective and therefore not all members of $\mathcal{S}$ could be treated in this way. By introducing the set of boundary conditions $\mathcal{G}$ we can extend Φ in a proper way to make it surjective. Consequently we get any $A \in \mathcal{S}$ as an image under Φ. We will then examine in which way $A \in \mathcal{S}$ depends on the boundary condition. The main result of this section will be Theorem 15 and as an important consequence of it, Corollary 16. Theorem 15 answers the question of dependence on the boundary conditions if A satisfies a recursion with coefficients in $\mathcal{N} + \mathcal{L}$. Corollary 16 provides a generalization of the convolution formulas found in [11, 15].

To begin with, we replace the natural boundary condition NBC

$$A(n,0) = \prod_{k=0}^{n-1} a_{k,0}$$

by an arbitrary function

$$g : \mathbb{N}_0 \longrightarrow \mathbb{K} \qquad \text{with } g(0) = 1.$$

This means we consider the recursion

$$A(n, \ell) = a_{n-1,\ell}\, A(n-1, \ell) + A(n-1, \ell-1) \qquad \forall n, \ell > 0 \tag{16}$$

together with the initial and boundary conditions

$$A(0, \ell) = \delta(0, \ell) \qquad \text{and} \qquad A(n, 0) = g(n) \quad \forall n, \ell \geq 0. \tag{17}$$

We denote by $\mathcal{G}$ the set of all admissible boundary conditions (BC)

$$\mathcal{G} := \big\{ g : \mathbb{N}_0 \to \mathbb{K} \mid g(0) = 1 \big\}.$$

Similar to the previous section we define a mapping

$$\Phi : \mathcal{G} \times \mathcal{M} \longrightarrow \mathcal{S}$$

by associating with a given BC and the coefficients $a \in \mathcal{M}$ the solution $A \in \mathcal{S}$ of (16) and (17). This Φ is well-defined and surjective. In the following we will write $\Phi(a)$ if we consider the NBC. For any $g_1, g_2 \in \mathcal{G}$ and $a, b \in \mathcal{M}$ we consider

$$g_3(n) = \Phi(g_1, a) \cdot \Phi(g_2, b)(n, 0). \tag{18}$$

We call g_3 the *product boundary condition* (PBC). The following result generalizes Theorem 6 for arbitrary BC.

THEOREM 13. *Let $g_1, g_2 \in \mathcal{G}$ and $a \in \mathcal{N}, b \in \mathcal{L}$. With g_3 the PBC we have*

$$\Phi(g_3, a + b) = \Phi(g_1, a) \cdot \Phi(g_2, b). \tag{19}$$

Proof. We set $A = \Phi(g_1, a), B = \Phi(g_2, b)$. As in the proof of Theorem 6 one verifies that $A \cdot B$ fulfills the recursion for $n, \ell > 0$. For $n = 0$ we have again $A \cdot B(0, \ell) = \delta(0, \ell)$ and for $\ell = 0$ we have

$$A \cdot B(n, 0) = \sum_{k=0}^{n} A(n, k)\, B(k, 0) = \sum_{k=0}^{n} A(n, k)\, g_2(k),$$

where the second equality follows from the definition of $B = \Phi(g_2, b)$. ∎

Next we investigate how $\Phi(g, a)$ depends on g. For this purpose we introduce two families of operators ν_k, λ_k $\forall k \geq 0$ on $\mathcal{M}$,

$$\begin{aligned} \nu_k : \mathcal{M} &\longrightarrow \mathcal{M} \\ a_{n,\ell} &\longmapsto a_{n+k,\ell} \end{aligned}$$

and

$$\begin{aligned} \lambda_k : \mathcal{M} &\longrightarrow \mathcal{M} \\ a_{n,\ell} &\longmapsto a_{n,\ell+k}. \end{aligned}$$

Then we have

$$\mathcal{N} = \left\{a \in \mathcal{M} \mid \lambda_k(a) = a \quad \forall k \geq 0\right\}$$

and

$$\mathcal{L} = \left\{a \in \mathcal{M} \mid \nu_k(a) = a \quad \forall k \geq 0\right\}.$$

First we investigate $B = \Phi(g, b)$, $b \in \mathcal{L}$. We introduce the family of generating functions $(F_\ell)_{\ell \geq 0}$

$$F_\ell(x) := \sum_{k=\ell}^{\infty} B(k, \ell)\, x^k. \tag{20}$$

Then by applying (16) to F_ℓ we get

$$\begin{aligned} F_\ell(x) = \sum_{k=\ell}^{\infty} B(k, \ell)\, x^k &= \sum_{k=\ell}^{\infty} (b_\ell\, B(k-1, \ell) + B(k-1, \ell-1))\, x^k \\ &= b_\ell\, x \sum_{k=\ell}^{\infty} B(k, \ell)\, x^k + x \sum_{k=\ell-1}^{\infty} B(k, \ell-1))\, x^k \\ &= b_\ell\, x\, F_\ell(x) + x\, F_{\ell-1}(x) \end{aligned}$$

which implies

$$\begin{aligned} F_\ell(x) &= \frac{x}{1 - b_\ell\, x} F_{\ell-1}(x) = \prod_{k=1}^{\ell} \frac{x}{1 - b_k\, x} F_0(x) \\ &= x^\ell \left(\sum_{k=0}^{\infty} h_k(b_1, \ldots, b_\ell)\, x^k \right) \left(\sum_{k=0}^{\infty} g(k)\, x^k \right) \\ &= x^\ell \sum_{k=0}^{\infty} x^k \sum_{i=0}^{k} g(i)\, h_{k-i}(b_1, \ldots, b_\ell). \end{aligned}$$

By comparing the coefficients with (20) we get for $g \in \mathscr{G}$, $b \in \mathscr{L}$

$$\begin{aligned}\Phi(g,b)(n,\ell) &= \sum_{k=0}^{n-\ell} g(k)\, h_{(n-k)-\ell}(b_1,\ldots,b_\ell) \\ &= \sum_{k=0}^{n-\ell} g(k)\, \Phi(\lambda_1(b))(n-k-1,\ell-1). \qquad (21)\end{aligned}$$

Next we turn to the case $\Phi(g,a)$, $a \in \mathscr{N}$. In order to find an explicit expression we first prove the following lemma.

LEMMA 14. *Let $a \in \mathscr{N}$ and $A = \Phi(a)$. Then we have for $n \geq \ell \geq 0$, $i \geq 0$*

$$\sum_{k=i}^{n-\ell} A(n,k+\ell)\, A^{-1}(k,i) = \begin{cases} 0 & \textit{if } n < i \\ \delta_{0,\ell} & \textit{if } n = i \\ \Phi(\nu_{i+1}(a))(n-i-1,\ell-1) & \textit{if } n > i. \end{cases}$$

Proof. We set

$$C_i(n,\ell) := \sum_{k=i}^{n-\ell} A(n,k+\ell)\, A^{-1}(k,i).$$

Then for $\ell > 0$ we have

$$\begin{aligned}C_i(n,\ell) &= \sum_{k=i}^{n-\ell-1} a_{n-1}\, A(n-1,k+\ell)\, A^{-1}(k,i) \\ &\quad + \sum_{k=i}^{n-\ell} A(n-1,k+(\ell-1))\, A^{-1}(k,i) \\ &= a_{n-1}\, C_i(n-1,\ell) + C_i(n-1,\ell-1)\end{aligned}$$

and

$$C_i(n,0) = \sum_{k=i}^{n} A(n,k)\, A^{-1}(k,i) = \delta(n,i).$$

By abuse of notation we find

$$C_i = \Phi(\delta(\,.\,,i),a).$$

Now it is easily verified that

$$\Phi(\delta(\,.\,,i),a)(n,\ell) = \begin{cases} 0 & \text{if } n < i \\ \delta_{0,\ell} & \text{if } n = i \\ \Phi(\nu_{i+1}(a))(n-i-1,\ell-1) & \text{if } n > i \end{cases}$$

which completes the proof of the lemma. ∎

From Theorem 13 it follows that for $g_1 \in \mathscr{G}$ and $a \in \mathscr{N}$ we have

$$\Phi(g, a) = \Phi(a) \cdot \Phi(g_1, 0),$$

where g is given by the PBC

$$g(n) = \sum_{k=0}^{n} A(n, k)\, g_1(k). \tag{22}$$

Now for any given $g \in \mathscr{G}$, we choose a $g_1 \in \mathscr{G}$

$$g_1(n) = \sum_{k=0}^{n} A^{-1}(n, k)\, g(k)$$

and with this g_1, Eq. (22) holds. Using (21) one finds that

$$\Phi(g_1, 0)(n, \ell) = g_1(n - \ell)$$

and hence

$$\begin{aligned}
\Phi(g, a)(n, \ell) &= \Phi(a) \cdot \Phi(g_1, 0)(n, \ell) \\
&= \sum_{k=\ell}^{n} A(n, k)\, g_1(k - \ell) \\
&= \sum_{k=\ell}^{n} A(n, k) \sum_{i=0}^{k-\ell} A^{-1}(k - \ell, i)\, g(i) \\
&= \sum_{k=0}^{n-\ell} A(n, k + \ell) \sum_{i=0}^{k} A^{-1}(k, i)\, g(i) \\
&= \sum_{i=0}^{n-\ell} g(i) \sum_{k=i}^{n-\ell} A(n, k + \ell)\, A^{-1}(k, i) \\
&= \sum_{i=0}^{n-\ell} g(i)\, \Phi(\nu_{i+1}(a))(n - i - 1, \ell - 1).
\end{aligned}$$

The last equality follows from Lemma 14 and we have found the explicit expression

$$\begin{aligned}
\Phi(g, a)(n, \ell) &= \sum_{k=0}^{n-\ell} g(k)\, e_{(n-k)-\ell}(a_k, \ldots, a_{n-1}) \\
&= \sum_{k=0}^{n-\ell} g(k)\, \Phi(\nu_{k+1}(a))(n - k - 1, \ell - 1).
\end{aligned} \tag{23}$$

Now that we have seen how $\Phi(g, a)$, $\Phi(g, b)$ depend on g for $a \in \mathcal{N}$, $b \in \mathcal{L}$, the same question arises for $\Phi(g, a+b)$. First we want to introduce some abbreviations in order to make the formulas more readable. By setting

$$\widehat{A}_i = T_1(\Phi(\nu_{i+1}(a)))$$

and

$$\widehat{B} = T_1(\Phi(\lambda_1(b)))$$

we can restate (21) and (23) obtaining

$$\Phi(g, a)(n, \ell) = \sum_{i=0}^{n-\ell} g(i)\,\widehat{A}_i(n-i, \ell) \tag{24}$$

and

$$\Phi(g, b)(n, \ell) = \sum_{i=0}^{n-\ell} g(i)\,\widehat{B}(n-i, \ell). \tag{25}$$

We are now ready to state the analogous result to (24) and (25) for $\Phi(g,\ a+b)$. Theorem 13 suggests that in this case some kind of product $\widehat{A}_k \cdot \widehat{B}$ is involved such that for either $\widehat{A}_k = \delta$ or $\widehat{B} = \delta$ we get (25) and (24), respectively. Indeed we have the following theorem.

THEOREM 15. *Let $g \in \mathcal{G}$, $a \in \mathcal{N}$, $b \in \mathcal{L}$. Then*

$$\Phi(g, a+b)(n, \ell) = \sum_{i=0}^{n-\ell} g(i)\,\widehat{A}_i \cdot \widehat{B}(n-i, \ell)$$

with $\widehat{A}_i$, $\widehat{B}$ as above.

Proof. Let $g \in \mathcal{G}$. Then we set

$$g_1(n) = \sum_{k=0} (\Phi(a))^{-1}(n, k)\,g(k)$$

and we have $\Phi(g, a+b) = \Phi(a) \cdot \Phi(g_1, b)$. From this we infer

$$\begin{aligned}
\Phi(g, a+b)(n, m) &= \sum_{\ell=m}^{n} \left(\sum_{k=\ell}^{n} A(n, k)\,g_1(k-\ell) \right) \widehat{B}(\ell, m) \\
&= \sum_{\ell=m}^{n} \left(\sum_{k=0}^{n-\ell} g(k)\,\widehat{A}_k(n-k, \ell) \right) \widehat{B}(\ell, m) \\
&= \sum_{k=0}^{n-m} g(k) \sum_{\ell=m}^{n-k} \widehat{A}_k(n-k, \ell)\,\widehat{B}(\ell, m) \\
&= \sum_{k=0}^{n-m} g(k)\,\widehat{A}_k \cdot \widehat{B}(n-k, m).
\end{aligned}$$

∎

Using Theorem 15 we can generalize the convolution formulas for the GSN of the first kind found in [11] and of the second kind found in [15].

Let $a \in \mathcal{M}$ and $\nu_k, \lambda_k : \mathcal{M} \longrightarrow \mathcal{M}$ as above. Then it holds for the composition

$$\nu_k \circ \lambda_i(a)_{n,\,\ell} = \lambda_i \circ \nu_k(a)_{n,\,\ell} = a_{n+k,\,\ell+i} \qquad \forall i, k \geq 0.$$

COROLLARY 16. *Let $a \in \mathcal{N}$, $b \in \mathcal{L}$. Then we have*

$$\Phi(a+b)(n,\ \ell+k) = \sum_{i=k}^{n-\ell} \Phi(a+b)(i,k) \times T_1\big(\Phi\big(\nu_{i+1} \circ \lambda_{k+1}(a+b)\big)\big)(n-i,\ell) \quad \forall k \geq 0.$$

Proof. Let

$$C = \Phi(a+b), \quad g(n) = C(n+k,k), \quad \text{and} \quad C_k = \Phi(g, \nu_k \circ \lambda_k(a+b)).$$

Then we have $C(n+k, \ell+k) = C_k(n,\ell)\ \forall k \geq 0$ and from Theorem 15 we infer

$$\begin{aligned} C_k(n,\ell) &= \sum_{i=0}^{n-\ell} C(i+k,k)\,(T_1(\Phi(\nu_{i+k+1}(a))) \cdot T_1(\Phi(\lambda_{k+1}(b))))\,(n-i,\ell) \\ &= \sum_{i=k}^{n-\ell+k} C(i,k)\,T_1(\Phi(\nu_{i+1} \circ \lambda_{k+1}(a+b)))(n-(i-k),\ell) \end{aligned}$$

and hence

$$\begin{aligned} C(n,\ell+k) &= C_k(n-k,\ell) \\ &= \sum_{i=k}^{n-\ell} C(i,k)\,T_1(\Phi(\nu_{i+1} \circ \lambda_{k+1}(a+b)))(n-i,\ell). \end{aligned}$$

∎

If we set

$$\widehat{B}_k = T_1(\Phi(\lambda_{k+1}(b)))$$

and observe that for any $A, B \in \mathcal{S}$ we have $T_1(A \cdot B) = T_1(A) \cdot T_1(B)$, then we can restate Corollary 16 in the form

$$A \cdot B(n, \ell+k) = \sum_{i=k}^{n-\ell} A \cdot B(i,k)\ \widehat{A}_i \cdot \widehat{B}_k(n-i,\ell). \tag{26}$$

If we set $b = 0$, then we have $B = \widehat{B}_k = \delta$ and

$$A(n, \ell+k) = \sum_{i=k}^{n-\ell} A(i,k)\ \widehat{A}_i(n-i,\ell), \tag{27}$$

namely the convolution formula due to de Medicis and Leroux [11]. On the other hand if we set $a = 0$, i.e., $A = \widehat{A}_i = \delta$, we get

$$B(n, \ell + k) = \sum_{i=k}^{n-\ell} B(i, k)\ \widehat{B}_k(n - i, \ell), \tag{28}$$

that is, the convolution formula due to Verde-Star [15]. As $(1, 1, 1, \dots) \in \mathcal{N} \cap \mathcal{L}$, both formulas generalize the well-known Chu–Vandermonde convolution formula

$$\binom{n}{\ell + k} = \sum_{i=k}^{n-\ell} \binom{i}{k} \binom{n - i - 1}{\ell - 1}.$$

4. SEQUENCES OF POLYNOMIALS WITH PERSISTENT ROOTS

In this section we demonstrate some relations between sequences of polynomials with persistent roots (*persistent sequences*) and the group $\mathcal{S}$. In particular we investigate the transformation of one persistent sequence to another. This problem has been extensively studied for *sequences of binomial type* by Rota and Mullin in [12]. In [5, 6] much progress in the same problem for *persistent sequences* has been made. With the theory developed in Section 2 we are able to refine some of their results and interpret them in terms of the group $\mathcal{S}$. We give an explicit representation for a group that has been introduced in [15] and show that it is indeed a subgroup of $\mathcal{S}$. Finally we make some remarks on connections between conjugate subgroups of the *binomial group* $\mathcal{B}$ and *sequences of binomial type*.

We consider persistent sequences of polynomials $p_{a,n} \in \mathbb{K}[x]$

$$p_{a,0}(x) \equiv 1 \text{ and } p_{a,n}(x) = (x + a_{n-1})\, p_{a,n-1}(x) \qquad \forall n > 0, \tag{29}$$

where $a_k \in \mathbb{K}\ \forall k \geq 0$. Equation (29) says that all roots of $p_{a,n-1}$ persist in $p_{a,n}$. From (29) we infer

$$p_{a,n}(x) = \prod_{k=0}^{n-1} (x + a_k) \qquad \forall n > 0.$$

Let $P^n(x)$ denote the set of all polynomials of degree n in x with coefficients in $\mathbb{K}$. $P^n(x)$ forms an $n + 1$ dimensional linear space over the field $\mathbb{K}$. It is obvious that, for any persistent sequence, $p_{a,0}, \dots, p_{a,n}$ forms a basis for $P^n(x)$. Hence any $q \in P^n(x)$ can be represented as

$$q(x) = \sum_{k=0}^{n} q_k\, p_{a,k}(x).$$

In particular for any two persistent sequences $(p_{a,k})_{k\geq 0}$ and $(p_{b,k})_{k\geq 0}$ we have

$$p_{a,n}(x) = \sum_{k=0}^{n} R(n,k)\, p_{b,k}(x). \tag{30}$$

It was shown in [5, 6] that $R(n,k)$ satisfy the recurrence

$$R(n,\ell) = (a_{n-1} - b_\ell)\, R(n-1,\ell) + R(n-1,\ell-1) \qquad \forall n,\ell > 0,$$

where

$$R(0,\ell) = \delta(0,\ell).$$

Hence with $a \in \mathscr{N}$ and $b \in \mathscr{L}$ we have

$$R = \Phi(a-b) = \Phi(a)\cdot\Phi(-b) = \Phi(a)\cdot(\Phi(b^T))^{-1}$$

by Theorem 6. In the sequel we identify $a \in \mathscr{N}$ and $b \in \mathscr{L}$ with the sequences $(a_k)_{k\geq 0}$, $(b_k)_{k\geq 0}$. We have thus proved

THEOREM 17. *Let $a, b \in \mathscr{N}$. If $p_{a,0}(x) = p_{b,0}(x) \equiv 1$ and*

$$p_{a,n}(x) = \prod_{k=0}^{n-1}(x + a_k), \qquad p_{b,n}(x) = \prod_{k=0}^{n-1}(x + b_k)$$

then

$$p_{a,n}(x) = \sum_{k=0}^{n} \Phi(a)\cdot(\Phi(b))^{-1}(n,k)\, p_{b,k}(x).$$

In other words Theorem 17 states that

$$p_{a,n}(x) = \sum_{k=0}^{n} R(n,k)\, p_{b,k}(x) \quad \Longrightarrow \quad \Phi(a) = R\cdot\Phi(b). \tag{31}$$

Special GSN have often been defined by property (30). In [7], for example, Eq. (30) forms the starting point for the investigation of *Stirling-type pairs*, although only for the particular case where the sequences of zeros $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ are of the form $a_k = \alpha k - r$, $b_k = \beta k - r$ $\alpha, \beta, r \in \mathbb{K}$.

As another application of Theorem 17 we describe a non-trivial subgroup of $\mathscr{S}$. This example has already been treated in [15] but can now be integrated in the general theory. Let $a \in \mathscr{N}$ and $(p_{a,k})_{k\geq 0}$ as above. The map $x \mapsto \lambda x + \tau$ operates on $P^n(x)$ by

$$p(x) \mapsto q(x) = p(\lambda x + \tau).$$

In particular we have

$$p_{a,n}(\lambda x + \tau) = \sum_{k=0}^{n} R_{\lambda,\tau}(n,k)\, p_{a,k}(x). \tag{32}$$

On the other hand

$$p_{a,n}(\lambda x+\tau)=\lambda^n\prod_{i=0}^{n-1}\left(x+\frac{\tau+a_i}{\lambda}\right). \tag{33}$$

Thus with $\hat{a}_n=(\tau+a_n)/\lambda$ we have

$$\Phi(\hat{a})=\frac{1}{\lambda}*\big(\Phi(a)\cdot\tau*(\,.\,)\big)$$

and with $\alpha\uparrow\Phi(a)(n,\ell):=\alpha^n\,\Phi(a)(n,\ell)$ and $\alpha\downarrow\Phi(a)(n,\ell):=\alpha^\ell\Phi(a)\times(n,\ell)$ we infer from (31)

$$R_{\lambda,\tau}=\lambda\downarrow\Phi(a)\cdot\tau*(\,.\,)\cdot(\Phi(a))^{-1}. \tag{34}$$

Now if we set $\mathscr{R}=\big\{R_{\lambda,\tau}\mid\lambda\in\mathbb{K}\backslash\{0\},\ \tau\in\mathbb{K}\big\}$ then $(\mathscr{R},\cdot)$ is a subgroup of $(\mathscr{S},\cdot)$. We have

$$R_{\lambda_1,\tau_1}\cdot R_{\lambda_2,\tau_2}=R_{\lambda_1\lambda_2,\,\tau_1+\lambda_1\tau_2}.$$

The straightforward proof of these facts is left to the interested reader.

Other examples of nontrivial subgroups of $\mathscr{S}$ are given by $\mathscr{B}$ in Example 9. Of special interest are the conjugate subgroups of $\mathscr{B}$

$$A\cdot\mathscr{B}\cdot A^{-1}\qquad\text{with}\quad A\in\mathscr{S}.$$

We show that they are connected to *sequences of binomial type*. First we observe that for $a\in\mathscr{N}$ and $(p_{a,k})_{k\geq0}$ as above and $A_x(n,\ell)=\binom{n}{\ell}\,p_{a,n-\ell}(x)$ we have

$$\begin{aligned}&A_x(0,\ell)=\delta(0,\ell)\qquad\text{and}\\&A_x(n,\ell)=(x+a_{n-1-\ell})\,A_x(n-1,\ell)+A_x(n-1,\ell-1).\end{aligned}$$

In particular if a_k is linear in k and $A=\Phi(a)$ then we infer

$$A_x=A\cdot(\,.\,)^x\cdot A^{-1}$$

and hence with Example 9

$$A_{x+y}=A_x\cdot A_y.$$

From this we obtain the following corollary.

COROLLARY 18. *If $a\in\mathscr{N}$ and $a_{\lambda(n+\ell)}=\lambda\,(a_n+a_\ell)$ then*

$$\binom{n}{\ell}\,p_{a,n-\ell}(x+y)=\sum_{k=\ell}^{n}\binom{n}{k}\,p_{a,n-k}(x)\binom{k}{\ell}\,p_{a,k-\ell}(y).$$

In particular from Corollary 18 we get for $\ell = 0$

$$p_{a,n}(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_{a,n-k}(x)\, p_{a,k}(y)$$

so that $(p_{a,k})_{k\geq 0}$ is a *sequence of binomial type* and a *persistent sequence* at the same time. For a complete list of such sequences refer to [3]. On the other hand for any $A \in \mathscr{S}$

$$A_x(n,\ell) = A \cdot (\,.\,)^x \cdot A^{-1}(n,\ell)$$

is a polynomial of degree $n-\ell$. Thus $A_x(n,\ell)$ can always be written as $A_x(n,\ell) = \binom{n}{\ell} p_{n,\ell}(x)$ some $p_{n,\ell} \in P^{n-\ell}(x)$. For these polynomials we get we get as above

$$\binom{n}{\ell} p_{n,\ell}(x+y) = \sum_{k=\ell}^{n} \binom{n}{k} p_{n,k}(x) \binom{k}{\ell} p_{k,\ell}(y).$$

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