Journal of Combinatorial Theory, Series A 119 (2012) 765–787

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Finite Eulerian posets which are binomial, Sheffer or triangular

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A R T I C L E   I N F O

Article history:
Received 14 July 2010
Available online 8 December 2011

Keywords:
Eulerian poset
Binomial poset
Sheffer poset
Triangular poset

A B S T R A C T

In this paper we study finite Eulerian posets which are binomial, Sheffer or triangular. These important classes of posets are related to the theory of generating functions and to geometry. The results of this paper are organized as follows:

- We completely determine the structure of Eulerian binomial posets and, as a conclusion, we are able to classify factorial functions of Eulerian binomial posets.
- We give an almost complete classification of factorial functions of Eulerian Sheffer posets by dividing the original question into several cases.
- In most cases above, we completely determine the structure of Eulerian Sheffer posets, a result stronger than just classifying factorial functions of these Eulerian Sheffer posets.

We also study Eulerian triangular posets. This paper answers questions posed by R. Ehrenborg and M. Readdy. This research is also motivated by the work of R. Stanley about recognizing the *boolean lattice* by looking at smaller intervals.

1. Introduction

There are many theories which unify various aspects of enumerative combinatorics and generating functions. One such successful theory introduced by Doubilet, Rota and Stanley [3] is that of binomial posets. Classically, binomial posets are infinite posets with the property that every two intervals of the same length have the same number of maximal chains. Doubilet, Rota and Stanley show this chain regularity condition gives rise to universal families of generating functions. Ehrenborg and Readdy [5]

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0097-3165/$ – see front matter
doi:10.1016/j.jcta.2011.11.002

and Reiner [9] independently generalized the notion of binomial posets to a larger class of posets called Sheffer posets or upper binomial posets.

Ehrenborg and Readdy [4] gave a complete classification of the factorial functions of infinite Eulerian binomial posets and infinite Eulerian Sheffer posets. Recall that infinite posets are those posets which contain an infinite chain. They posed the open question of characterizing the finite case. This paper deals with these questions.

A *triangular poset* is a graded poset such that the number of maximal chains in each interval $[x, y]$ depends only on $\rho(x)$ and $\rho(y)$, where $\rho(x)$ and $\rho(y)$ are the ranks of the elements x and y, respectively. Sheffer posets are a special class of triangular posets. A *Sheffer poset* is a graded poset such that the number of maximal chains $D(n)$ in an n-interval $[\hat{0}, y]$ depends only on $\rho(y) = n$, the rank of the element y, and the number $B(n)$ of maximal chains in an n-interval $[x, y]$, where $x \neq \hat{0}$, depends only on $\rho(x, y) = \rho(y) - \rho(x)$. The two functions $B(n)$ and $D(n)$ are called the *binomial factorial function* and *Sheffer factorial function*, respectively. *Binomial posets* are a special class of Sheffer posets. A binomial poset is a graded poset such that the number of maximal chains $B(n)$ in an n-interval $[x, y]$ depends only on $\rho(x, y) = \rho(y) - \rho(x)$.

Binomial posets were previously considered in [1,3,8,10,12]. They form a class of flag symmetric posets which were studied by Stanley [12]. Stanley [10] showed that the theory of binomial posets can be used to unify and extend various results dealing with the enumeration of permutations of sets and multisets with various conditions. Backelin [1] classified confluent binomial posets with the binomial factorial function $B(1) = 1$, $B(2) = 1$ and $B(i) = 2^{i-2}$ for $i \geqslant 2$. In particular, he showed that there is an uncountable number of isomorphism classes of them. Hetyei [8] studied the effect of the augmented Tchebyshev operator on binomial posets.

Ehrenborg and Readdy [5] used Sheffer posets and a generalization of R-labeling to study augmented r-signed permutations. Reiner [9] used them to derive generating functions counting signed permutations by two statistics.

A graded poset P is *Eulerian* if every non-singleton interval of P satisfies the *Euler–Poincaré* relation. (See Definition 2.1.) Eulerian posets form an important class of posets as there are many geometric examples such as the face lattices of convex polytopes, and more generally, the face posets of regular CW-spheres.

As we mentioned above, Ehrenborg and Readdy in [4] classify the factorial functions of infinite Eulerian binomial posets and infinite Eulerian Sheffer posets. Since we are concerned here with finite posets, we drop the requirement that binomial, Sheffer and triangular posets have an infinite chain. This paper studies the following natural questions, as suggested by Ehrenborg and Readdy in [4].

1. Which Eulerian posets are binomial?
2. Which Eulerian posets are Sheffer?

We also briefly consider Eulerian triangular posets.

Stanley has proved that one can recognize *boolean lattices* by looking at smaller intervals (see [7, Lemma 8]). Farley and Schmidt answer a similar question for *distributive lattices* in [6]. The project of studying Eulerian binomial posets and Eulerian Sheffer posets is also motivated by their work. In many cases we use the factorial function of smaller intervals to characterize the whole poset.

1.1. Our results

All posets considered in this paper are finite. Let us first describe two poset operations.

Let Q_i, $i = 1, \ldots, k$, be posets which contain a unique maximal element $\hat{1}$ and a unique minimal element $\hat{0}$. We define $\boxplus_{i=1,\ldots,k} Q_i$ to be the poset which is obtained by identifying all of the minimal elements of the posets Q_i as well as identifying all of their maximal elements. We define the *k-summation* of P, denoted by $\boxplus^k(P)$, to be $\boxplus_{i=1,\ldots,k} P$. This is also known as the *banana product*.

Let P be a poset with $\hat{0}$. The *dual suspension* of P, denoted by $\Sigma^*(P)$, is the poset P with two new elements a_1 and a_2 and with the following order relation: $\hat{0} <_{\Sigma^*(P)} a_i <_{\Sigma^*(P)} y$, for all $y > \hat{0}$ in P and $i = 1, 2$, the order relations of P hold in $\Sigma^*(P)$, and $\Sigma^*(P)$ has no other relations.

Let Q be a poset of odd rank. It is easily seen that if Q is an Eulerian Sheffer poset then so is $\boxplus^k(Q)$. Moreover, if P is an Eulerian binomial poset, then $\Sigma^*(P)$ is an Eulerian Sheffer poset.

We describe the structure of Eulerian binomial posets. For an Eulerian binomial poset P of rank n, the structure depends on n, as follows:

1. $n = 3$. Then $P \cong \boxplus_{i=1,\dots,r} P_{q_i}$ for some $q_1, \dots, q_r$ such that $q_i \geqslant 2$, where P_q denote the face lattice of a q-gon.
2. n is even. Then P is either isomorphic to B_n, the boolean lattice of rank n, or T_n, the butterfly poset of rank n (defined in Definition 2.7).
3. n is odd and $n \geqslant 5$. Then P is either isomorphic to $\boxplus^\alpha(B_n)$ or $\boxplus^\alpha(T_n)$ for some positive integer α.

For an Eulerian Sheffer poset P of rank n, we describe its structure and factorial functions.

1. $n = 3$. Then $P \cong \boxplus_{i=1,\dots,r} P_{q_i}$ for some $q_1, \dots, q_r$ such that $q_i \geqslant 2$.
2. $n = 4$. The complete classification of factorial functions of the poset P follows from Lemma 4.4.
3. n is odd and $n \geqslant 5$. Then one of the following is true:
   (a) $B(3) = D(3) = 6$. Then $P \cong \boxplus^\alpha(B_n)$ for some α.
   (b) $B(3) = 6$, $D(3) = 8$. This case is open.
   (c) $n = 5$, $B(3) = 6$, $D(3) = 10$. This case remains open.
   (d) $B(3) = 6$, $D(3) = 4$. Then $P \cong \boxplus^\alpha(\Sigma^*(B_{n-1}))$ for some positive integer α.
   (e) $B(3) = 4$. The classification follows from Theorems 3.11 and 3.13 in [4].
4. n is even and $n \geqslant 6$. Then one of the following is true:
   (a) $B(3) = D(3) = 6$. Then $P \cong B_n$.
   (b) $B(3) = 6$, $D(3) = 8$. The poset P has the same factorial functions as the cubical lattice of rank n, that is, $D(k) = 2^{k-1}(k-1)!$ and $B(k) = k!$.
   (c) $B(3) = 6$, $D(3) = 4$. Then $P \cong \Sigma^*(\boxplus^\alpha(B_{n-1}))$ for some positive integer α.
   (d) $B(k) = 2^{k-1}$, for $1 \leqslant k \leqslant 2m$, and $B(2m+1) = \alpha \cdot 2^{2m}$ for some positive integer $\alpha > 1$. In this case P is isomorphic to $\Sigma^*(\boxplus^\alpha(T_{2m+1}))$.
   (e) $B(k) = 2^{k-1}$, $1 \leqslant k \leqslant 2m+1$. The classification follows from Theorems 3.11 and 3.13 in [4].

The paper is structured as follows. In Section 2 we cover some basic definitions. In Section 3 we completely classify the structure of Eulerian binomial posets. See Lemma 3.6, Theorems 3.11 and 3.12. These results, coupled with Ehrenborg and Readdy's classification in the infinite case, complete the classification of Eulerian binomial posets. In Section 4, we give an almost complete classification of the factorial functions of Eulerian Sheffer posets. In fact, in most of the above cases we completely identify the structure of the finite Eulerian Sheffer posets, a result which is stronger than merely classifying the factorial functions. In Section 5 we review triangular posets. We classify Eulerian triangular posets such that the factorial functions of all of their 3-intervals are equal to 6. Finally, in Section 6 we provide some conclusions and remarks.

2. Definitions and background

We encourage readers to consult Chapter 3 of [11] for basic poset terminology. All the posets which are considered in this paper are finite.

We begin by recalling that a graded interval satisfies the *Euler–Poincaré relation* if it has the same number of elements of even rank as of odd rank.

Definition 2.1.

1. A graded poset is *Eulerian* if every non-singleton interval satisfies the Euler–Poincaré relation. Equivalently, a poset P is Eulerian if its Möbius function satisfies $\mu(x, y) = (-1)^{\rho(x,y)}$ for all $x \leqslant y$ in P, where ρ denotes the rank function of P.
2. Consider a graded poset P with rank function ρ. If $\rho(x, y) = n$, then we call $[x, y]$ an *n-interval*.

Definition 2.2. A finite graded poset P with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$ is called a (*finite*) *binomial poset* if it satisfies the following condition:

> For all $n \in \mathbb{N}$, $n \leqslant \operatorname{rank}(P)$, any two n-intervals have the same number $B(n)$ of maximal chains. We call $B(n)$ the *factorial function* or *binomial factorial function* of the poset P.

Next, we define the *atom function* $A(n)$ to be the number of coatoms in a binomial interval of length n. Therefore, $A(n) = \frac{B(n)}{B(n-1)}$ and $B(n) = A(n) \cdots A(1)$.

Consider a binomial poset P. The number of maximal chains passing through each element of rank k in any interval of rank n is $B(k)B(n-k)$, for $1 \leqslant k \leqslant n$. The total number of chains in this interval is $B(n)$. Hence, the number of elements of rank k in any interval of rank n is equal to

$$\frac{B(n)}{B(k)B(n-k)}. \tag{2.1}$$

Definition 2.3. A finite graded poset P with a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$ is called a (*finite*) *Sheffer poset* if it satisfies the following two conditions:

1. Any pair of n-intervals $[\hat{0}, y]$ and $[\hat{0}, v]$ have the same number $D(n)$ of maximal chains.
2. Any pair of n-intervals $[x, y]$ and $[u, v]$ such that $x \neq \hat{0}$ and $u \neq \hat{0}$ have the same number $B(n)$ of maximal chains.

Let us consider a Sheffer poset P. An interval $[\hat{0}, y]$, where $y \neq \hat{0}$, is called a *Sheffer interval* whereas an interval $[x, y]$ with $x \neq \hat{0}$ is called a *binomial interval*. The functions $B(n)$ and $D(n)$ are called the *binomial factorial function* and *Sheffer factorial function* of P, respectively. Next we define $A(n)$ and $C(n)$ to be the number of coatoms in a binomial interval of length n, respectively, a Sheffer interval of length n. The functions $A(n)$ and $C(n)$ are called the *atom function* and *coatom function* of P, respectively. It is not hard to see that $A(n) = \frac{B(n)}{B(n-1)}$ and $B(n) = A(n) \cdots A(1)$, as well as $C(n) = \frac{D(n)}{D(n-1)}$ and $D(n) = C(n)C(n-1) \cdots C(1)$.

The number of elements of rank k in a Sheffer interval of rank n is

$$\frac{D(n)}{D(k)B(n-k)}. \tag{2.2}$$

Moreover, for a binomial interval $[x, y]$ of rank n in a Sheffer poset, the number of elements of rank k is equal to

$$\frac{B(n)}{B(k)B(n-k)}. \tag{2.3}$$

The *dual suspension* of a poset P is defined in [4] as follows.

Definition 2.4. Let P be a poset with $\hat{0}$. We define the *dual suspension* of P, denoted $\Sigma^*(P)$, to be the poset P with two new elements a_1 and a_2. The elements a_1 and a_2 have the following order relations: $\hat{0} <_{\Sigma^*(P)} a_i <_{\Sigma^*(P)} y$, for all $y > \hat{0}$ in P and $i = 1, 2$. That is, the elements a_1 and a_2 are inserted between $\hat{0}$ and atoms of P. Clearly if P is Eulerian then so is $\Sigma^*(P)$. Moreover, if P is a binomial poset then $\Sigma^*(P)$ is a Sheffer poset with the factorial function $D_{\Sigma^*(P)}(n) = 2B(n-1)$, for $n \geqslant 2$.

Definition 2.5. Let P be a poset with $\hat{1}$. We define the *suspension* of P, denoted by $\Sigma(P)$, to be the poset P with two new elements a_1 and a_2 adjoined with the additional order relations that $y <_{\Sigma(P)} a_i <_{\Sigma(P)} \hat{1}$, for all $y < \hat{1}$ in P and $i = 1, 2$.

The *dual* of the poset P, denoted P^*, is defined as follows: P^* has the same set of elements as P and the following order relation: $x <_{P^*} y$ if and only if $y <_P x$.

Definition 2.6. The *boolean lattice* B_n of rank n is the poset of subsets of $[n] = \{1, \ldots, n\}$ ordered by inclusion.

Definition 2.7. The *butterfly poset* T_n of rank n consists of the elements of $\{\hat{0}\} \cup (D_{n-1} \times \{1, 2\}) \cup \{\hat{1}\}$, where $D_{n-1} \times \{1, 2\}$ is the direct product of the chain of length $n-1$, denoted by D_{n-1}, and the anti-chain of rank 2, with the order relation $(k, i) \prec (k+1, j)$ for all $i, j \in \{1, 2\}$. Also $\hat{0}$ and $\hat{1}$ are the unique minimal and maximal elements of this poset, respectively. Clearly, $T_n \cong \Sigma^*(T_{n-1})$.

A larger class of posets to consider is the class of *triangular posets*.

Definition 2.8. A finite poset P with $\hat{0}$ and $\hat{1}$ is called a (*finite*) *triangular poset* if it satisfies the following two conditions.

1. Every interval $[x, y]$ is graded; hence P has a rank function ρ.
2. Every two intervals $[x, y]$ and $[u, v]$ such that $\rho(x) = \rho(u) = m$ and $\rho(y) = \rho(v) = n$ have the same number $B(m, n)$ of maximal chains.

All posets considered in this paper are finite. By binomial, Sheffer and triangular posets, we mean finite binomial, finite Sheffer and finite triangular posets.

3. Finite Eulerian binomial posets

In this section, we classify the structure of finite Eulerian binomial posets. (These results are summarized in Section 1.1.) For undefined poset terminology and further information about binomial posets, see [11].

First we provide some examples of finite binomial posets. See [4] for infinite versions of Examples 3.1 and 3.3.

Example 3.1. The boolean lattice B_n of rank n is an Eulerian binomial poset with factorial function $B(k) = k!$ and atom function $A(k) = k$, $k \leqslant n$. Every interval of length k of this poset is isomorphic to B_k.

Example 3.2. Let D_n be the chain containing $n+1$ elements. This poset has factorial function $B(k) = 1$ and atom function $A(k) = 1$ for each $k \leqslant n$.

Example 3.3. The butterfly poset T_n of rank n is an Eulerian binomial poset with factorial function $B(k) = 2^{k-1}$ for $1 \leqslant k \leqslant n$ and atom function $A(k) = 2$, for $2 \leqslant k \leqslant n$, and $A(1) = 1$.

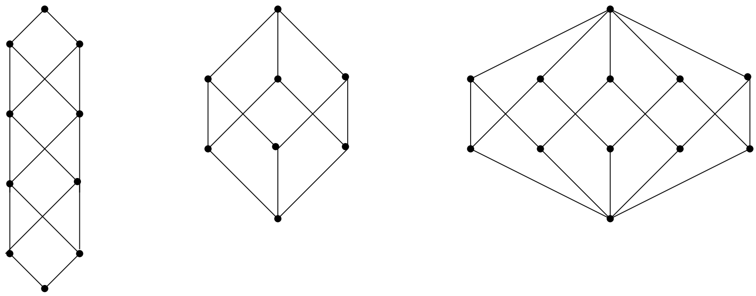
Example 3.4. Let $\mathbb{F}_q$ be the q-element field where q is a prime power and let $V_n = V_n(q)$ be an n-dimensional vector space over $\mathbb{F}_q$. Let $L_n = L_n(q)$ denote the poset of all subspaces of V_n, ordered by inclusion. L_n is a graded lattice of rank n. It is easy to see that every interval of size $1 \leqslant k \leqslant n$ is isomorphic to L_k. Hence $L_n(q)$ is a binomial poset. This poset is not Eulerian for $q \geqslant 2$.

It is not hard to see that in any n-interval of an Eulerian binomial poset P with factorial function $B(k)$ for $1 \leqslant k \leqslant n$, the Euler–Poincaré relation is stated as follows:

$$\sum_{k=0}^{n} (-1)^k \cdot \frac{B(n)}{B(k)B(n-k)} = 0. \tag{3.1}$$

The following is [4, Lemma 2.6].

Lemma 3.5. *Let P be a graded poset of odd rank such that every proper interval of P is Eulerian. Then P is an Eulerian poset.*



(1) (2) (3)

Fig. 1. (1): T_5, (2): B_3 and (3): P_5, the face lattice of a 5-gon.

Lemma 3.6. *Let P be an Eulerian binomial poset of rank 3. Then the poset P and its factorial function $B(n)$ satisfy the following conditions*:

(i) $B(2)=2$ *and* $B(3)=2q$, *where q is a positive integer such that* $q \geqslant 2$.
(ii) *There is a list of integers* $q_1,\ldots,q_r$, $q_i \geqslant 2$, *such that* $P \cong \boxplus_{i=1,\ldots,r} P_{q_i}$, *where P_{q_i} is the face lattice of the q_i-gon.*

This result is [4, Example 2.5]. It is also a consequence of Lemma 4.3. (See Fig. 1.)

R. Ehrenborg and M. Readdy proved the following two propositions. See [4, Lemma 2.17 and Prop. 2.15].

Proposition 3.7. *Let P be a binomial poset of rank n with factorial function $B(k)=2^{k-1}$ for $1 \leqslant k \leqslant n$. Then the poset P is isomorphic to the butterfly poset T_n.*

Proposition 3.8. *Let P be a binomial poset of rank n with factorial function $B(k)=k!$ for $1 \leqslant k \leqslant n$. Then the poset P is isomorphic to the boolean lattice B_n of rank n.*

The following is [4, Lemma 2.12].

Lemma 3.9. *Let P' and P be two Eulerian binomial posets of rank $2m+2$, $m \geqslant 2$, having atom functions $A'(n)$ and $A(n)$, respectively, which agree for $n \leqslant 2m$. Then the following equality holds*:

$$\frac{1}{A(2m+1)}\left(1-\frac{1}{A(2m+2)}\right)=\frac{1}{A'(2m+1)}\left(1-\frac{1}{A'(2m+2)}\right). \tag{3.2}$$

Lemma 3.10. *Every Eulerian binomial poset P of rank 4 is isomorphic to either T_4 or B_4.*

Proof. Applying Lemma 3.6 gives $B(3)=2k$, where $k \geqslant 2$. Eq. (2.3) implies that the number of elements of rank 1 is the same as the number of elements of rank 3 in P. We denote this number by n. Hence

$$n=\frac{B(4)}{B(3)B(1)}=\frac{B(4)}{B(3)}. \tag{3.3}$$

We can also enumerate the number r of elements of rank 2 as follows:

$$r=\frac{B(4)}{B(2)B(2)}. \tag{3.4}$$

The Euler–Poincaré relation on intervals of length four is $2 + r = 2n$. By enumerating the maximal chains, we conclude $B(4) = rB(2)B(2) = nB(3)$ and since always $B(2) = 2$, we have $2r = kn$. The Euler–Poincaré relation implies that $\frac{kn}{2} + 2 = 2n$, and so $k < 4$. We have the following cases.

(i) $k = 1$. Then $n = \frac{4}{3}$. This case is not possible.
(ii) $k = 2$. Then $n = 2$ and $r = 2$. We conclude that $B(k) = 2^{k-1}$, for $1 \leqslant k \leqslant 4$. By Proposition 3.7, $P \cong T_4$.
(iii) $k = 3$. Then $n = 4$ and $r = 6$. Thus $B(k) = k!$, for $1 \leqslant k \leqslant 4$. By Proposition 3.8, $P \cong B_4$. □

In the following theorem we obtain the structure of Eulerian binomial posets of even rank.

Theorem 3.11. *Every Eulerian binomial poset of even rank $n = 2m \geqslant 4$ is isomorphic to either T_n or B_n (the butterfly poset of rank n or boolean lattice of rank n).*

Proof. We proceed by induction on m. The claim is true for $2m = 4$, by Lemma 3.10. Assume that the theorem holds for Eulerian binomial posets of rank $2m \geqslant 4$. We wish to show that it also holds for Eulerian binomial posets of rank $2m + 2$.

Let P be an Eulerian binomial poset of rank $2m + 2$. The factorial and atom functions of this poset are denoted by $B(n)$ and $A(n)$, respectively. By Lemma 3.10, every interval of size 4 is either isomorphic to B_4 or T_4. So the factorial function $B(3)$ of the intervals of rank 3 can only take the values 4 or 6 and we have the following two cases where $B(3) = 4$ and $B(3) = 6$.

- $B(3) = 6$. We wish to show that P is isomorphic to B_{2m+2} by induction on m. By Lemma 3.10, the claim is true for $2m = 4$. By the induction hypothesis, the claim holds for $n = 2m$, and we wish to prove it for $n = 2m+2$. Let $P' = B_{2m+2}$, so P' has the atom function $A'(n) = n$ for $1 \leqslant n \leqslant 2m+2$. By the induction hypothesis, $A(j) = A'(j) = j$ for $j \leqslant 2m$. Lemma 3.9 implies that Eq. (3.2) holds. Since $2m = A(2m) \leqslant A(2m+2) < \infty$, we obtain the following inequalities:
$$2m + 1 - \frac{2}{2m} < A(2m+1) < 2m + 2. \tag{3.5}$$
Thus $A(2m+1) = 2m+1$. Eq. (3.2) implies that $A(2m+2) = 2m+2$. By Proposition 3.8, the poset P is isomorphic to B_{2m+2}, as desired.
- $B(3) = 4$. We claim that the poset P of rank $n = 2m + 2$ is isomorphic to T_n. By the induction hypothesis, our claim holds for even $n \leqslant 2m$, and we would like to prove it for $n = 2m + 2$. Consider the poset T_{2m+2}. This poset has the atom function $A(n) = 2$ for $1 \leqslant n \leqslant 2m + 2$. By the induction hypothesis the intervals of length $2m$ in P are isomorphic to T_{2m}, so $A(j) = 2$ for $1 \leqslant j \leqslant 2m$. We wish to show that $A(2m+1) = A(2m+2) = 2$, which implies that $P \cong T_{2m+2}$. Clearly $2 = A(2m) \leqslant A(2m+2) < \infty$. Eq. (3.2) implies that $2 \leqslant A(2m+1) < 4$. We show that the case $A(2m+1) = 3$ is forbidden by an idea similar to one that appears in the proof of Theorem 2.16 in [4]: Assume that $A(2m+1) = 3$. Let $[x, y]$ be a $(2m+1)$-interval in P. For $1 \leqslant k \leqslant 2m$ there are $B(2m+1)/B(k) \cdot B(2m+1-k) = 3 \cdot 2^{2m-1}/(2^{k-1} \cdot 2^{2m-k}) = 3$ elements of rank k in this interval.
Let c be a coatom. The interval $[x, c]$ has two atoms, say a_1 and a_2. Moreover, the interval $[x, c]$ has two elements of rank 2, say b_1 and b_2. Moreover we know that each b_j covers each a_i. Let a_3 and b_3 be the third atom, respectively the third rank 2 element, in the interval $[x, y]$. We know that b_3 covers two atoms in $[x, y]$. One of them must be a_1 or a_2, say a_1. However a_1 is covered by the three elements b_1, b_2 and b_3. This contradicts the fact that each atom is covered by exactly two elements. Hence this rules out the case $A(2m+1) = 3$. We have $A(2m+1) = A(2m+2) = 2$. Proposition 3.7 implies that P is isomorphic to T_{2m+2}. □

Theorem 3.12. *Let P be an Eulerian binomial poset of odd rank $n = 2m + 1 \geqslant 5$. Then the poset P satisfies one of the following conditions*:

(i) *There is a positive integer k such that P is the k-summation of the boolean lattice of rank n. In other words,* $P \cong \boxplus^k(B_n)$.
(ii) *There is a positive integer k such that P is the k-summation of the butterfly poset of rank n. In other words,* $P \cong \boxplus^k(T_n)$.

Proof. We prove the theorem for two different cases $B(3) = 4$ and $B(3) = 6$. Lemma 3.10 implies that every interval of length 4 is isomorphic either to B_4 or T_4. Thus the factorial function $B(3)$ can only take the values 4 or 6 and therefore we are in one of these two cases $B(3) = 4$ and $B(3) = 6$.

(i) $B(3) = 6$. In this case we claim that there is a positive integer k such that $P \cong \boxplus^k(B_n)$. In order to show that $P \cong \boxplus^k(B_n)$, we make the following construction. We remove $\hat{1}$ and $\hat{0}$ from P. The remaining poset is a disjoint union of connected components. We add a minimal element $\hat{0}$ and a maximal element $\hat{1}$ to each of these connected components. We show that the obtained posets are isomorphic to B_n. Consider one of the obtained connected components and add a minimal element $\hat{0}$ and a maximal element $\hat{1}$ to it. Denote the resulting poset by Q. We wish to show that $Q \cong B_n$. This implies that $P \cong \boxplus^k(B_n)$.
It is not hard to see that Q is an Eulerian binomial poset. The posets P and Q have the same factorial functions and atom functions up to rank $2m$. Hence $B_Q(k) = B_P(k)$ and $A_Q(k) = A_P(k)$, for $1 \leqslant k \leqslant 2m$. Therefore, Eq. (2.3) implies that the number of atoms and coatoms are the same in the poset Q. Denote this number by t. Let $x_1, \dots, x_t$ and $a_1, \dots, a_t$ be an ordering of the atoms and coatoms of Q, respectively. Also, let $c_1, \dots, c_l$ be the set of elements of rank $2m - 1$ in Q. We show that $t = 2m + 1$, and this implies that $Q \cong B_{2m+1}$.
For each element y of rank at least 2 in Q, let $S(y)$ be the set of atoms of Q that are below y. Set $A_i := S(a_i)$ for each element a_i of rank $2m$, $1 \leqslant i \leqslant t$, and also set $C_i := S(c_i)$ for each element c_i of rank $2m - 1$, $1 \leqslant i \leqslant l$. In order to show that $Q \cong B_n$, we prove the following.
(1) We show that $|A_i \cap A_j| = 2m - 1$ for $i \neq j$.
(2) We use part (1) to show that $t = 2m + 1$.

(1) We first show that $|A_i \cap A_j| = 2m - 1$ for $i \neq j$.
By considering the factorial functions, Theorem 3.11 implies that the intervals $[\hat{0}, a_i]$ and $[x_j, \hat{1}]$ have the same factorial functions as B_{2m} and so they are isomorphic to B_{2m} for $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant t$. We conclude that any interval $[\hat{0}, c_k]$ of rank $2m - 1$ is isomorphic to B_{2m-1}. As a consequence, we have $|A_i| = |S(a_i)| = 2m$, $1 \leqslant i \leqslant t$ and also $|C_k| = |S(c_k)| = 2m - 1$, $1 \leqslant k \leqslant l$.
If there exist i and j such that $A_i \cap A_j \neq \emptyset$, where $1 \leqslant i, j \leqslant t$, we claim that $2m - 1 \leqslant |A_i \cap A_j| \leqslant 2m$. Consider an atom $x_k \in A_i \cap A_j$, $1 \leqslant k \leqslant t$. Theorem 3.11 implies that $[x_k, \hat{1}] \cong B_{2m}$. Thus, by considering properties of boolean lattices, there is an element c_h of rank $2m - 2$ in this interval which is covered by a_i and a_j, $1 \leqslant h \leqslant l$. Notice that c_h is an element of rank $2m - 1$ in Q. Therefore, $|C_h| = 2m - 1 \leqslant |A_i \cap A_j| \leqslant |A_i| = |S(a_i)| = 2m$.
We claim that for all distinct pairs i and j, $1 \leqslant i, j \leqslant t$, we have $A_i \cap A_j \neq \emptyset$. In order to show this claim, associate the graph G_Q to the poset Q as follows: $A_1, \dots, A_t$ are vertices of this graph, and we connect vertices A_i and A_j if and only if $A_i \cap A_j \neq \emptyset$.
We will show that G_Q is a complete graph and so $|A_i \cap A_j| \neq 0$ for all $i \neq j$. Since $Q - \{\hat{0}, \hat{1}\}$ is connected, G_Q is also a connected graph. We show that if $\{A_i, A_j\}$ and $\{A_j, A_k\}$ are different edges of G_Q, $\{A_i, A_k\}$ is also an edge of G_Q. Since $\{A_i, A_j\}$ and $\{A_j, A_k\}$ are edges of G_Q, we have $|A_i \cap A_j| \geqslant 2m - 1$ as well as $|A_j \cap A_k| \geqslant 2m - 1$. On the other hand, since $|A_i| = |A_j| = |A_k| = 2m$, we conclude that $A_i \cap A_k \neq \emptyset$. Therefore $\{A_i, A_k\}$ is also an edge of G_Q. As a consequence, the connected graph G_Q is a complete graph. Thus $A_i \cap A_j \neq \emptyset$ and also $2m - 1 \leqslant |A_i \cap A_j| \leqslant 2m$ for $1 \leqslant i, j \leqslant t$ and $i \neq j$.
Now, we show that $|A_i \cap A_j| = 2m - 1$ for all $i \neq j$. We proceed by contradiction. Suppose this claim does not hold. Then there are different i and j such that $|A_i \cap A_j| = 2m$. We claim that in the case $|A_i \cap A_j| = 2m$, there are two elements of rank $2m - 1$ in Q such that they both are covered by coatoms a_i and a_j. To show this claim, consider an atom $x_f \in A_i \cap A_j$, so we have $[x_f, \hat{1}] \cong B_{2m}$. Hence, there is a unique element c_h of rank $2m - 2$ in this interval which is covered

by both a_i and a_j. By induction on m, Lemma 3.6, and the property that $|C_h| \leqslant |A_i \cap A_j| = 2m$, we conclude that $[\hat{0}, c_h]$ is isomorphic to B_{2m-1} and so $|C_h| = 2m-1$. Therefore there is an atom $x_d \in A_i \cap A_j \setminus C_h$. Since the interval $[x_d, \hat{1}]$ is isomorphic to B_{2m}, there is an element $c_k \neq c_h$ of rank $2m-1$ which is covered by coatoms a_i and a_j.

We know that $|C_h| = |S(c_h)| = |C_k| = |S(c_k)| = 2m-1$ and C_k and C_h are both subsets of $A_i \cap A_j$, so there should be an atom $x_s \in C_k \cap C_h$. Therefore the interval $[x_s, \hat{1}]$ has two elements c_k and c_h of rank $2m-2$ such that they both are covered by two elements a_i and a_j of rank $2m-1$ in the interval $[x_s, \hat{1}]$. We know $[x_s, \hat{1}] \cong B_{2m}$ and there are no two elements of rank $2m-2$ covered by two elements of rank $2m-1$ in B_{2m}. This contradicts our assumption, and so $|A_i \cap A_j| = 2m-1$ for pairs i and j of distinct elements.

In summary, we have:

(a) $|A_i| = 2m$ for $1 \leqslant i \leqslant t$,

(b) $|A_i \cap A_j| = 2m-1$ for all $1 \leqslant i < j \leqslant t$,

(c) $\bigcup_{i=1}^{t} A_i = \{x_1, \ldots, x_t\}$.

As a consequence, we have $t > 2m$.

(2) Now, we show that $t = 2m+1$.

We are going to show that $t = 2m+1$. Without loss of generality, consider the three different sets $A_1 = S(a_1)$, $A_2 = S(a_2)$ and $A_3 = S(a_3)$ associated with the three coatoms a_1, a_2 and a_3. We know that $|A_1| = |A_2| = |A_3| = 2m$ and $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_1 \cap A_3| = 2m-1$. Without loss of generality, let us assume that $A_1 = \{x_1, x_2, \ldots, x_{2m-1}, y_1\}$ and $A_2 = \{x_1, x_2, \ldots, x_{2m-1}, y_2\}$ where $y_i \neq x_1, \ldots, x_{2m-1}$ for $i = 1, 2$. We have the following two different cases:

(a) A_3 does not contain y_1 and y_2.

(b) A_3 contains at least one of y_1 and y_2.

First we study the case, $A_3 = \{x_1, x_2, \ldots, x_{2m-1}, y_3\}$ where $y_3 \notin \{y_1, y_2, x_1, \ldots, x_{2m-1}\}$. Considering the $t-3$ other coatoms a_k, $4 \leqslant k \leqslant t$, there are different atoms y_k, $4 \leqslant k \leqslant t$, such that $y_k \notin \{y_1, y_2, y_3, x_1, \ldots, x_{2m-1}\}$ and $A_k = S(a_k) = \{x_1, x_2, \ldots, x_{2m-1}, y_k\}$. This implies that the number of atoms is $|\bigcup_{i=1}^{t} A_i| = t + 2m - 1$, which is a contradiction. So it must be the case that A_3 contains one of y_1 or y_2. In this case $|A_2 \cap A_3| = |A_1 \cap A_3| = 2m-1$ implies that $A_3 = \{x_1, x_2, \ldots, x_{2m-1}, y_1, y_2\} \setminus \{x_j\} \subset A_1 \cup A_2$ for some x_j. Since A_3 was chosen arbitrarily, it follows that for each A_k we have $A_k \subset A_1 \cup A_2$. Therefore,

$$\bigcup_{i=1}^{t} A_k = \{x_1, \ldots, x_{2m-1}, y_1, y_2\}, \tag{3.6}$$

thus the number of coatoms in the poset Q is $t = 2m+1$.

By Theorem 3.11, $B_Q(k) = k!$ for $1 \leqslant k \leqslant 2m$, therefore $B_Q(2m+1) = (2m+1)!$. By Proposition 3.8, Q is isomorphic to B_{2m+1} and so P is a union of copies of B_{2m+1} with their minimal elements and maximal elements identified. In other words, $P \cong \boxplus^k(B_{2m+1})$. It can be seen that P is binomial and Eulerian and the proof follows.

(ii) $B(3) = 4$. With the same argument as part (i), we remove $\hat{1}$ and $\hat{0}$ from P. The remaining poset is a disjoint union of connected components. We add a minimal element $\hat{0}$ and a maximal element $\hat{1}$ to each of these connected components. We show that the obtained posets are isomorphic to T_n. This implies that $P \cong \boxplus^k(T_n)$.

We construct the binomial poset Q by adding $\hat{1}$ and $\hat{0}$ to one of the connected components of $P - \{\hat{0}, \hat{1}\}$. We claim that Q is isomorphic to T_{2m+1}. Similar to part (i), let $a_1, \ldots, a_t$ and $x_1, \ldots, x_t$ denote coatoms and atoms of Q. We show that $t = 2$ which implies $Q \cong T_{2m+1}$.

Set $A_i = S(a_i)$. By Theorem 3.11, we have $|A_i| = 2$. It is easy to see that $\bigcup_{i=1}^{t} A_i = \{x_1, \ldots, x_t\}$. Define G_Q to be the graph with vertices $x_1, \ldots, x_t$ and edges $A_1, \ldots, A_t$. Since $Q \setminus \{\hat{0}, \hat{1}\}$ is connected, G_Q is also a connected graph. Since $[x_i, \hat{1}] \cong T_{2m}$, the degree of each vertex of G_Q is 2 and G_Q is the cycle of length t. Therefore if $t > 2$, we have $|A_i \cap A_j| = 1$ or 0, $1 \leqslant i < j \leqslant t$.

We claim that $t = 2$. Suppose this claim does not hold and $t > 2$. Consider an element c of rank 3 in Q. Lemma 3.6 and Theorem 3.11 imply that both intervals $[\hat{0}, c]$ and $[c, \hat{1}]$ are isomorphic to

butterfly posets. Hence there are two coatoms above c, say a_k and a_l, and similarly there are two atoms below c, say x_h and x_s. Therefore, we have $A_k = A_l = \{x_h, x_s\}$. This is not possible when $t > 2$. As a consequence, $t = 2$ and all the A_i's have two elements and $|\bigcup_1^t A_i| = |\{x_1, \ldots, x_t\}| = 2 = t$.
Similar to part (i), $B_Q(k) = 2^{k-1}$ for $1 \leqslant k \leqslant 2m+1$. By Proposition 3.7, we conclude that Q is isomorphic to T_{2m+1}. Therefore, there is an integer $k > 0$ such that $P \cong \boxplus^k(T_n)$. □

4. Finite Eulerian Sheffer posets

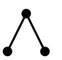
In this section, we give an almost complete classification of the factorial functions and the structure of Eulerian Sheffer posets. For basic definitions regarding Sheffer posets, see Section 2.

First, we provide some examples of Eulerian Sheffer posets. We study Eulerian Sheffer posets of ranks $n = 3$ and 4 in Lemmas 4.3 and 4.4. By these two lemmas, we reduce the values of $B(3)$ to 4 or 6. In Section 4.1, Lemma 4.5 and Theorems 4.6, 4.11, 4.12 and 4.13 deal with Eulerian Sheffer posets with $B(3) = 6$. Finally in Section 4.2, Theorems 4.14, 4.15 and 4.16 deal with Eulerian Sheffer posets with $B(3) = 4$. The results of this section are summarized in Section 1.1.

It is clear that every binomial poset is also a Sheffer poset. Here are some other examples of Sheffer posets, some of which appear in [4] and [9].

Example 4.1. Let P be a binomial poset of rank n with the factorial function $B(k)$. By adjoining a new minimal element $\widehat{-1}$ to P, we obtain a Sheffer poset of rank $n+1$ with binomial factorial function $B(k)$ for $1 \leqslant k \leqslant n$ and Sheffer factorial function, $D(k) = B(k-1)$ for $1 \leqslant k \leqslant n+1$.

Example 4.2. Let T be the following three element poset:



Let T^n be the Cartesian product of n copies of the poset T. The poset $C_n = T^n \cup \{\hat{0}\}$ is the face lattice of an n-dimensional cube, also known as the *cubical lattice*. The cubical lattice is a Sheffer poset with $B(k) = k!$ for $1 \leqslant k \leqslant n$ and $D(k) = 2^{k-1}(k-1)!$ for $1 \leqslant k \leqslant n+1$.

Let P be an Eulerian Sheffer poset of rank n. The Euler–Poincaré relation for every m-Sheffer interval, $2 \leqslant m \leqslant n$, becomes

$$1 + \sum_{k=1}^{m} (-1)^k \cdot \frac{D(m)}{D(k)B(m-k)} = 0. \tag{4.1}$$

It is clear that B_2 is the only Eulerian Sheffer poset of length 2 in P.

In the next lemma, we characterize the structure of Eulerian Sheffer posets of rank 3. The characterization of the factorial functions is an immediate consequence.

Lemma 4.3. *Let P be an Eulerian Sheffer poset of rank* 3.

(i) *The poset P has the factorial function $D(2) = 2$ and $D(3) = 2q$, where q is a positive integer such that $q \geqslant 2$.*
(ii) *There is a list of integers $q_1, \ldots, q_r$, $q_i \geqslant 2$ such that $P \cong \boxplus_{i=1,\ldots,r} P_{q_i}$, where P_{q_i} denotes the face lattice of a q_i-gon.*

Proof. Consider an Eulerian Sheffer poset P of rank 3. Now $P - \{\hat{0}, \hat{1}\}$ consists of all elements of rank 1 and rank 2 of P. By the Euler–Poincaré relation, it is easy to see that $B(2) = 2$ and every interval of rank 2 is isomorphic to B_2. So in $P - \{\hat{0}, \hat{1}\}$, every element of rank 2 is connected to two elements of rank 1 and vice versa. Therefore, the Hasse diagram of $P - \{\hat{0}, \hat{1}\}$ decomposes as the disjoint union

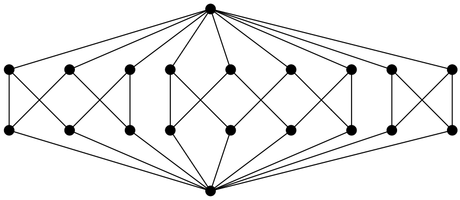


Fig. 2. $P_2 \boxplus P_3 \boxplus P_4$.

of cycles of even lengths $2q_1, \dots, 2q_r$ where $q_i \geqslant 2$ (see Fig. 2). We conclude that the poset P is obtained by identifying all minimal elements of the posets $P_{q_1}, \dots, P_{q_r}$ and identifying all of their maximal elements. Hence $P \cong \boxplus_{i=1,\dots,r} P_{q_i}$ and $D(3) = 2(q_1 + \cdots + q_r)$. Thus every Eulerian Sheffer poset of rank 3 has the factorial functions $D(3) = 2q$ where $q \geqslant 2$ and $B(2) = D(2) = 2$. $\square$

Lemma 4.4 deals with Eulerian Sheffer posets of rank 4.

Lemma 4.4. *Let the poset P be an Eulerian Sheffer poset of rank 4. Then one of the following conditions holds.*

(i) $B(3) = 2b$, $D(3) = 4$, $D(4) = 4b$, *where* $b \geqslant 2$.
(ii) $B(3) = 8$, $D(3) = 3!$, $D(4) = 2^3 \cdot 3!$.
(iii) $B(3) = 10$, $D(3) = 3!$, $D(4) = 5!$.
(iv) $B(3) = 4$, $D(3) = 3!$, $D(4) = 2 \cdot 3!$.
(v) $B(3) = 3!$, $D(3) = 3!$, $D(4) = 4!$.
(vi) $B(3) = 3!$, $D(3) = 4$, $D(4) = 2 \cdot 3!$.
(vii) $B(3) = 3!$, $D(3) = 10$, $D(4) = 5!$.
(viii) $B(3) = 3!$, $D(3) = 8$, $D(4) = 2^3 \cdot 3!$.
(ix) $B(3) = 4$, $D(3) = 2b$, $D(4) = 4b$, *where* $b \geqslant 2$.

Proof. Let P be an Eulerian Sheffer poset of rank 4. Note that for every Eulerian Sheffer poset $B(1) = D(1) = 1$ as well as $B(2) = D(2) = 2$. Let the variables a, b and c denote the number of elements of rank 1, 2 and 3 of P, respectively. By the Euler–Poincaré relation, we have $2 + b = a + c$. The number of maximal chains in P is given by $4b = B(3)a = D(3)c$. Lemma 4.3 implies that there are positive integers k_1, k_2 such that $D(3) = 2k_2$ and $B(3) = 2k_1$. Thus $b + 2 = (\frac{2}{k_1} + \frac{2}{k_2})b$. We conclude that $\frac{2}{k_1} + \frac{2}{k_2} > 1$; therefore k_1 and k_2 cannot both be greater than 3. Next we study the remaining cases as follows. Let us recall the fact that every interval of rank 2 is isomorphic to B_2 implies that $b \geqslant 2$.

(1) $k_2 = 1$ or $k_1 = 1$. This implies that $2b < (\frac{2}{k_1} + \frac{2}{3})b = b + 2$, so we have $b < 2$. This contradicts the fact that $b \geqslant 2$.
(2) $k_2 = 2$ or $k_1 = 2$. If $k_2 = 2$, then $2b = 2c$, so $c = b$, $a = 2$ and $k_1 = b$. Thus $B(1) = 1$, $B(2) = 2$ and $B(3) = 2b$, as well as $D(1) = 1$, $D(2) = 2$, $D(3) = 4$, and $D(4) = 4b$. The poset $T \cong \Sigma^*(P_b)$, where P_b is the lattice of a b-gon, is an Eulerian Sheffer poset with the described factorial functions. Similarly, in case $k_1 = 2$, the poset P has the same factorial functions as $\Sigma(P_b)$. That is, $B(3) = 4$, $D(3) = 2b$ and $D(4) = 4b$.
(3) $k_2 = 3$. The equation $b + 2 = a + c = (\frac{2}{k_1} + \frac{2}{3})b$ implies that $k_1 < 6$, so we need to consider the following four cases.
   (a) $k_1 = 5$. Then $b + 2 = \frac{2}{5}b + \frac{2}{3}b$, so $\frac{1}{15}b = 2$, $b = 30$, $c = 20$ and $a = 12$. Thus P has the factorial functions $B(3) = 10$, $D(3) = 3!$ and $D(4) = 5!$. The face lattice of the icosahedron is an Eulerian Sheffer poset with the same factorial functions.
   (b) $k_1 = 4$. The poset P has the same factorial functions as dual of the cubical lattice of rank 4, that is, the face lattice of an octahedron. Therefore, $B(3) = 8$, $D(3) = 3!$ and $D(4) = 2^3 \cdot 3!$.

   (c) $k_1 = 3$. The poset P has the factorial functions $B(3) = 3!$, $D(3) = 3!$ and $D(4) = 4!$. Thus P is isomorphic to B_4, that is, the face lattice of a simplex.
   (d) $k_1 = 2$. The poset P has the same factorial functions as $\Sigma(B_3)$, $B(3) = 4$, $D(3) = 3!$ and $D(4) = 2 \cdot 3!$.
(4) $k_1 = 3$. Then $b + 2 = (\frac{2}{k_1} + \frac{2}{k_2})b$ implies that $k_2 < 6$, so we have the following four cases.
   (a) $k_2 = 5$. Similar to the case $k_1 = 5$ and $k_2 = 3$, the poset P has the same factorial functions as the face lattice of a dodecahedron, $B(3) = 3!$, $D(3) = 10$ and $D(4) = 5!$.
   (b) $k_2 = 4$. Similar to the case $k_1 = 4$ and $k_2 = 3$, the poset P has the same factorial functions as the cubical lattice of rank 4. That is, $B(3) = 3!$, $D(3) = 8$ and $D(4) = 2^3 \cdot 3!$.
   (c) $k_2 = 3$. The poset P has the factorial functions $B(3) = 3!$, $D(3) = 3!$ and $D(4) = 4!$. So $P \cong B_4$.
   (d) $k_2 = 2$. Similar to the case $k_1 = 2$ and $k_2 = 3$, the poset P has the same factorial functions as $\Sigma^*(B_3)$, $B(3) = 3!$, $D(3) = 4$ and $D(4) = 2 \cdot 3!$. □

4.1. *Characterization of the factorial functions and structure of Eulerian Sheffer posets for which $B(3) = 3!$*

In this section we mainly consider Eulerian Sheffer posets with $B(3) = 3!$. As a consequence of Lemma 4.4, we know that Eulerian Sheffer posets of rank $n \geqslant 4$ with $B(3) = 3!$ have the Sheffer factorial functions $D(3) = 4, 6, 8$ and 10. Lemma 4.5 shows that for any such poset of rank $n \geqslant 6$, the Sheffer factorial function $D(3)$ can only take the values 4, 6 or 8.

In Sections 4.1.1, 4.1.2 and 4.1.3, we consider posets with $B(3) = 6$ and different cases $D(3) = 4, 6$ and 8, respectively. The question of studying the finite Eulerian Sheffer posets of rank 5 with $B(3) = 6$ and $D(3) = 10$ remains open. There is such a poset, namely the face lattice of the 120-cell with Schläfli symbol $\{5, 3, 3\}$.

Lemma 4.5. *Let P be an Eulerian Sheffer poset of rank $n \geqslant 6$ with $B(3) = 3!$. Then $D(3)$ can take only the values* 4, 6 *and* 8.

Proof. By Lemma 4.4, the Sheffer factorial function of poset P for Sheffer 3-intervals can take the following values $D(3) = 4, 6, 8$ and 10. We only need to show that the case $D(3) = 10$ is not possible. Suppose there is an Eulerian Sheffer poset P of rank of at least 6 with the factorial functions $D(3) = 10$ and $B(3) = 3!$. By Lemma 4.4, the poset P has the following factorial functions: $D(1) = 1$, $D(2) = 2$, $D(3) = 10$, $D(4) = 5!$, $B(1) = 1$, $B(2) = 2!$ and $B(3) = 3!$. Set $C(6) = E$, $C(5) = F$, where $C(k)$ is the coatom function of P. By Theorems 3.11 and 3.12, we conclude that there is an integer $\alpha > 0$ such that $B(4) = 4!$ and $B(5) = \alpha \cdot 5!$. The Euler–Poincaré relation implies that

$$1 + \sum_{k=1}^{6} (-1)^k \cdot \frac{D(6)}{D(k)B(6-k)} = 0.$$

By substituting the values in above equation, we have

$$2 = \frac{EF}{\alpha} - EF + E, \qquad \alpha(E - 2) = (\alpha - 1)EF. \tag{4.2}$$

There are two cases $\alpha = 1$ and $\alpha > 1$ to consider:

1. $\alpha = 1$. Eq. (4.2) implies that $E = 2$. However, $E \geqslant A(5) = 5$ where $A(5)$ is an atom function of B_5. This case is not possible.
2. $\alpha > 1$. By Eq. (4.2),

$$\left(\frac{\alpha}{\alpha - 1}\right)\left(\frac{E - 2}{E}\right) = F.$$

   $E \geqslant A(5) = 5$ implies that $F < 4$. On the other hand, since $F \geqslant A(4) \geqslant 4$, this case is also not possible.

We conclude that there is no Eulerian Sheffer poset of rank at least 6 with $D(3) = 10$ and $B(3) = 3!$, as desired. □

4.1.1. Characterization of the factorial functions of Eulerian Sheffer posets of rank $n \geqslant 4$ for which $B(3) = 3!$ and $D(3) = 8$

In this subsection, we study the factorial functions of Eulerian Sheffer posets of rank $n \geqslant 4$ for which $B(3) = 3!$ and $D(3) = 8$. Theorem 4.6 characterizes the factorial functions of such posets of even rank. However, the question of characterizing the factorial functions of Eulerian Sheffer posets of odd rank $n = 2m + 1 \geqslant 4$ with $B(3) = 3!$ and $D(3) = 8$ remains open.

Theorem 4.6. *Let P be an Eulerian Sheffer poset of even rank $n = 2m + 2 \geqslant 4$ with $B(3) = 3!$ and $D(3) = 8$. Then P has the same factorial functions as C_n, the cubical lattice of rank n, that is, $D(k) = 2^{k-1}(k-1)!$, $1 \leqslant k \leqslant n$ and $B(k) = k!$ for $1 \leqslant k \leqslant n - 1$.*

In order to prove Theorem 4.6, we establish the following Lemmas 4.7 and 4.9.

Lemma 4.7. *Let Q be an Eulerian Sheffer poset of odd rank $2m + 1$, $m \geqslant 2$, with $B(3) = 3!$. Then the coatom function of Q must satisfy at least one of the following inequalities: $C(n) \neq 2(n-1)$ for $2 \leqslant n \leqslant 2m$ and $C(2m + 1) \neq 4m + 1$.*

Proof. We proceed by contradiction. Assume Q is such a poset which does not satisfy any of the stated inequalities. Theorem 3.11 implies that Q has the binomial factorial function $B(k) = k!$ for $1 \leqslant k \leqslant 2m$. By Eq. (2.2) we enumerate the elements of ranks 1, $2m - 1$ and $2m$ in this Sheffer poset.

Let $\{a_1, \dots, a_{4m+1}\}$, $\{e_1, \dots, e_{(4m+1)(2m-1)}\}$ and $\{x_1, \dots, x_t\}$ denote the sets of elements of rank $2m$, $2m - 1$ and 1 in Q, respectively, where $t = \frac{4m+1}{2m} \cdot 2^{2m-1}$. The idea of the proof is as follows: We show that for any a_i and a_j, there is at least one element of rank $2m - 2$, e_k, $1 \leqslant k \leqslant (4m + 1)(2m - 1)$, which is covered by both a_i and a_j. In addition, we know that for every e_l there is exactly one pair a_i and a_j such that e_l is covered by them. Hence, the number of the disjoint pairs of elements of rank $2m$ in the poset Q is at most the number of elements of rank $2m - 1$. That is, $(4m + 1)(2m - 1) \geqslant (4m + 1)(2m)$ which leads us to contradiction.

For each element y of rank at least 2, let $S(y)$ be the set of atoms in $[\hat{0}, y]$. Set $A_j = S(a_j)$ for each element a_j of rank $2m$ and also $E_j = S(e_j)$ for each element e_j of rank $2m - 1$. Eq. (2.2) implies that $|S(y)| = 2^{r-1}$ for any element y of rank $2 \leqslant r \leqslant 2m$.

Assume that $A_i \cap A_j \neq \emptyset$ for some $1 \leqslant i, j \leqslant 4m + 1$. We claim that in case $A_i \neq A_j$, there is a unique element $\alpha_{i,j}$ which is covered by a_i and a_j.

Consider $x_a \in A_i \cap A_j$, then the interval $[x_a, \hat{1}]$ is isomorphic to B_{2m}. Therefore, there is an element $\alpha_{i,j}$ which is covered by both a_i and a_j. Clearly, by properties of the boolean lattice, $\alpha_{i,j}$ is unique in the case that x_a is the only element in $A_i \cap A_j$. Assume that there is $x_b \in A_i \cap A_j$ so that $x_b \neq x_a$. Thus, there is an element $\beta_{i,j} \in [x_b, \hat{1}]$ which is covered by both a_i and a_j. In case $\alpha_{i,j} \neq \beta_{i,j}$, we can see $S(\alpha_{i,j})$ and $S(\beta_{i,j})$ are disjoint sets, therefore we have $A_i = A_j = S(\alpha_{i,j}) \cup S(\beta_{i,j})$. This contradicts the fact $A_i \neq A_j$.

Informally speaking, when $A_i \cap A_j \neq \emptyset$ and $A_i \neq A_j$, we denote the unique $\alpha_{i,j}$ which is covered by both a_i and a_j by $a_i \wedge a_j$. In this case $A_i \cap A_j$ is the set of atoms which are below $a_i \wedge a_j$. Thus,

$$|A_i \cap A_j| = \left|S(a_i \wedge a_j)\right| = 2^{\operatorname{rank}(a_i \wedge a_j)-1} = 2^{2m-2}. \tag{4.3}$$

We claim that for all pairs i and j satisfying $i \neq j$ and $1 \leqslant i, j \leqslant 4m + 1$, the intersection of the atom sets satisfies $A_i \cap A_j \neq \emptyset$. Suppose this claim does not hold. Then, there exist two different s and l such that $|A_s \cap A_l| = 0$ where $1 \leqslant s, l \leqslant 4m + 1$. Since $|A_s| + |A_l| < t$, there is a set $A_k = S(a_k)$ such that $A_k \cap (\{x_1, \dots, x_t\} - A_s \cup A_l) \neq \emptyset$ for $1 \leqslant k \leqslant 4m+1$. Therefore we have $|A_l \cap A_k|, |A_s \cap A_k| \leqslant 2^{2m-2}$. Furthermore, since $|\{x_1, \dots, x_t\}| = t = \frac{4m+1}{2m} \cdot 2^{2m-1}$, $|A_l| = |A_s| = |A_k| = 2^{2m-1}$ and $|A_l \cap A_s| = 0$, we conclude that

$$\left|A_k \cap \left(\{x_1, \ldots, x_t\} - A_s \cup A_l\right)\right| \leqslant \left|\{x_1, \ldots, x_t\} - A_s \cup A_l\right| = \frac{2^{2m-1}}{2m}. \tag{4.4}$$

By Eq. (4.4), we have $|A_l \cap A_k|, |A_s \cap A_k| \neq 0$ and $A_k \neq A_l, A_s$. Therefore, Eq. (4.3) implies that $|A_l \cap A_k| = 2^{2m-2}$ and $|A_s \cap A_k| = 2^{2m-2}$. We have seen that $|A_s \cap A_k| = 2^{2m-2}$, $|A_l \cap A_k| = 2^{2m-2}$ and $|A_k| = 2^{2m-1}$. Moreover, since we assumed $A_s \cap A_l = \emptyset$, we conclude that $A_k = A_s \cup A_l$. On the other hand $A_k \cap (\{x_1, \ldots, x_t\} - A_s \cup A_l) \neq \emptyset$, which is not possible when $A_k = A_s \cup A_l$. This contradicts our assumption. Therefore $|A_i \cap A_j| \neq 0$ for $1 \leqslant i, j \leqslant 4m+1$. So for every pair of elements a_i and a_j, there is an atom $x_h \in A_i \cap A_j$. As above $[x_h, \hat{1}] \cong B_{2m}$, so there is at least one element of rank $2m-2$ in this interval, denoted by e_k, $1 \leqslant k \leqslant (4m+1)(2m-1)$, and it is covered by both a_i and a_j. In addition, for every element e_l of rank $2m-1$ in the poset Q the interval $[e_l, \hat{1}]$ is isomorphic to B_2. As a consequence, for every e_l there is exactly one pair a_i, a_j such that e_l is covered by a_i and a_j. Hence, the number of the disjoint pairs of elements of rank $2m$ in the poset Q is at most the number of elements of rank $2m-1$. That is, $(4m+1)(2m-1) \geqslant (4m+1)(2m)$ which is not possible. This is a contradiction. So there is no poset Q with the described factorial and coatom functions, as desired. □

Lemma 4.7 implies the following.

Corollary 4.8. *Let P be an Eulerian Sheffer poset of rank $2m+2$, $m \geqslant 2$, with $B(k) = k!$, for $1 \leqslant k \leqslant 2m$. Then the coatom function of P must satisfy at least one of the following inequalities: $C(n) \neq 2(n-1)$, $2 \leqslant n \leqslant 2m$, $C(2m+1) \neq 4m+1$ and $C(2m+2) \neq 4(2m+1)$.*

Lemma 4.9. *Let Q be an Eulerian Sheffer poset of rank $2m+2$, $m \geqslant 2$, with $B(k) = k!$ for $1 \leqslant k \leqslant 2m$. Then the coatom function of Q must satisfy at least one of the following inequalities: $C(n) \neq 2(n-1)$, $2 \leqslant n \leqslant 2m$, $C(2m+1) \neq 4m-1$ and $C(2m+2) \neq \frac{4}{3}(2m+1)$.*

Proof. We proceed by contradiction. Suppose Q is such a poset of rank $2m+2$ with the described factorial functions. The idea of the proof is as follows: We show that there is an element c of rank $2m-2$ in poset Q, so that the interval $[c, \hat{1}]$ has two elements of rank 1 which are covered by two elements of rank 2. On the other hand $[c, \hat{1}] \cong B_3$ and this is not possible in B_3. This leads us to contradiction.

We enumerate the elements of rank $2m+2-k$ in Q as follows:

$$\frac{D(2m+2)}{B(k)D(2m+2-k)} = \frac{C(2m+2)\cdots C(2m+2-k+1)}{k!}. \tag{4.5}$$

Thus, $\{a_1, \ldots, a_{\frac{4}{3}(2m+1)}\}$ and $\{e_1, \ldots, e_{\frac{4}{6}(2m+1)(4m-1)}\}$ are the sets of elements of rank $2m+1$ and $2m$ in Q, respectively. For every element e_i of rank $2m$, the interval $[e_i, \hat{1}]$ is isomorphic to B_2. So each element of rank $2m$ is covered by exactly two different elements of rank $2m+1$.

There are exactly $\frac{4}{6}(2m+1)(4m-1)$ elements of rank $2m$ in Q, and we also know that there are $(\frac{4}{6}(2m+1))(\frac{4}{3}(2m+1)-1)$ different pairs of coatoms $\{a_i, a_j\}$ in Q, $1 \leqslant i < j \leqslant \frac{4}{3}(2m+1)$. We conclude there are at least two different coatoms a_k, a_l such that they both cover two different elements e_i, e_j of rank $2m$. The interval $T = [\hat{0}, a_k]$ has the binomial factorial function $B_T(k) = k!$ for $1 \leqslant k \leqslant 2m$ and coatom function $C_T(n) = 2(n-1)$ for $2 \leqslant n \leqslant 2m$ and $C_T(2m+1) = 4m-1$.

Let $\{y_1, \ldots, y_t\}$ be the set of atoms in the poset T where $t = \frac{(4m-1)}{2m} \cdot 2^{2m-1}$. Thus $A_k = \{y_1, \ldots, y_t\}$. Set $E_j = S(e_j)$, $E_i = S(e_i)$, so $E_j, E_i \subset A_k$. By Eq. (2.2) $|E_i| = |E_j| = 2^{2m-1}$, therefore $|E_i| + |E_j| > |A_k|$. We conclude that there is at least one atom $y_1 \in T$ which is below the elements e_i, e_j and a_k. Proposition 3.8 implies that $[y_1, a_k] \cong B_{2m}$. By the boolean lattice properties, there is an element c of rank $2m-2$ in the interval $[y_1, a_k]$ such that c is covered by the elements e_i and e_j. By Proposition 3.8, we have $[c, \hat{1}] \cong B_3$. Consider the interval $[c, \hat{1}]$. Let a_k and a_l be two elements of rank 2 in this interval which both cover two elements e_i and e_j of rank 1. This contradicts the fact that $[c, \hat{1}] \cong B_3$.

We conclude that $[c, \hat{1}] \ncong B_3$, giving the desired contradiction. Hence there is no poset Q with the described conditions. $\square$

The following lemma can be obtained by applying the proof of Lemma 4.8 in [4].

Lemma 4.10. *Let P and P' be two Eulerian Sheffer posets of rank $2m+2$, $m \geqslant 2$, such that their binomial factorial functions and coatom functions agree up to rank $n \leqslant 2m$. That is, $B(n) = B'(n)$ and $C(n) = C'(n)$, where $m \geqslant 2$. Then the following equation holds*:

$$\frac{1}{C(2m+1)}\left(1 - \frac{1}{C(2m+2)}\right) = \frac{1}{C'(2m+1)}\left(1 - \frac{1}{C'(2m+2)}\right). \tag{4.6}$$

Proof of Theorem 4.6. In order to prove the theorem, we inductively show that the Eulerian Sheffer poset P and C_{2m+2}, the cubical lattice of rank $2m+2$ have the same coatom functions.

Let $C(k)$ and $C'(k) = 2(k-1)$ respectively be the coatom functions of the Eulerian Sheffer poset P and C_{2m+2} for $2 \leqslant k \leqslant 2m+2$. We only need to show that $C(k) = C'(k) = 2(k-1)$ for $2 \leqslant k \leqslant 2m+2$. We prove this claim by induction on m. By Lemma 4.4, an Eulerian Sheffer poset of even rank 4 with $B(3) = 3!$ and $D(3) = 8$ has the same factorial function as C_4. Therefore, $C(4) = C'(4) = 6$ and the claim holds for $m = 1$.

Suppose $m \geqslant 2$. By the induction hypothesis $C(k) = C'(k) = 2(k-1)$ for $2 \leqslant k \leqslant 2m$. Set $F = C(2m+1)$ and $E = C(2m+2)$. Theorem 3.12 implies that $B(k) = k!$ for $1 \leqslant k \leqslant 2m$ and there is a positive integer α such that $B(2m+1) = \alpha(2m+1)!$. We know that $D(k) = 2^{k-1}(k-1)!$ for $1 \leqslant k \leqslant 2m$, so $D(2m+1) = F2^{2m-1}(2m-1)!$ and $D(2m+2) = EF2^{2m-1}(2m-1)!$. Since P is an Eulerian Sheffer poset, the Euler–Poincaré relation implies that

$$1 + \sum_{k=1}^{2m+2} \frac{(-1)^k D(2m+2)}{D(k)B(2m+2-k)} = 0. \tag{4.7}$$

By substituting the values of the factorial functions, we have

$$2 - E + \frac{EF}{2}\left[\frac{1}{2m} - \frac{1}{2m(2m+1)} + \frac{2^{2m}}{2m(2m+1)} - \frac{2^{2m}}{2\alpha m(2m+1)}\right] = 0. \tag{4.8}$$

Thus,

$$E\left(1 - F\left(\frac{2\alpha m + (\alpha-1)2^{2m}}{4\alpha m(2m+1)}\right)\right) = 2. \tag{4.9}$$

In case $\alpha \geqslant 2$, it is easy to verify that

$$\left(\frac{2\alpha m + (\alpha-1)2^{2m}}{4\alpha m(2m+1)}\right) > \frac{1}{2m}. \tag{4.10}$$

We know that $F \geqslant A(2m) \geqslant 2m$, so the left-hand side of Eq. (4.9) becomes negative in this case. Therefore, $\alpha = 1$ and the posets P and C_{2m+2} have the same binomial factorial functions. Since $2m+1 = A(2m+1) \leqslant C(2m+2) < \infty$, Lemma 4.10 implies that $4m-1 \leqslant C(2m+1) = F \leqslant 4m+1$. Since $\alpha = 1$, Eq. (4.9) implies that $2 - E + \frac{EF}{4m+2} = 0$. Thus E and F must satisfy one of the following cases:

(1) $F = 4m-1$ and $E = \frac{4}{3}(2m+1)$.
(2) $F = 4m$ and $E = 4m+2$.
(3) $F = 4m+1$ and $E = 4(2m+1)$.

As we have discussed in Corollary 4.8 and Lemma 4.9, the cases (1) and (3) are not possible. Case (2) occurs in the cubical lattice of rank $2m+2$, C_{2m+2}. Thus, the poset P has the same factorial functions as C_{2m+2}, as desired. $\square$

Classification of the factorial functions of Eulerian Sheffer posets of odd rank $n = 2m + 1 \geqslant 5$ with $B(3) = 6$ and $D(3) = 8$ remains open. Let α be a positive integer and set $Q_\alpha = \boxplus^\alpha(C_{2m+1})$. It can be seen that Q_α is an Eulerian Sheffer poset and it has the following factorial functions: $D(k) = 2^{k-1}(k-1)!$ for $1 \leqslant k \leqslant n-1$, $D(n) = \alpha \cdot 2^{n-1}(n-1)!$ and $B(k) = k!$ for $1 \leqslant k \leqslant n-1$. We ask the following question:

Question. Let P be an Eulerian Sheffer poset of odd rank $n = 2m + 1 \geqslant 5$ with $B(3) = 6, D(3) = 8$. Is there a positive integer α such that the poset P has the same factorial functions as poset $Q_\alpha = \boxplus^\alpha(C_{2m+1})$, where C_{2m+1} is a cubical lattice of rank $2m + 1$?

4.1.2. Characterization of the structure of Eulerian Sheffer posets of rank $n \geqslant 3$ for which $B(3) = 3!$, and $D(3) = 3! = 6$

In this section, we prove the following:

Theorem 4.11. *Let P be an Eulerian Sheffer poset of rank $n \geqslant 3$ with $B(3) = D(3) = 3! = 6$ for 3-intervals. P satisfies one of the following cases*:

(i) *There is an integer $k \geqslant 1$ such that $P \cong \boxplus^k(B_n)$, where n is odd.*
(ii) *$P \cong B_n$, where n is even.*

Proof. We proceed by induction on n. Lemmas 4.3 and 4.4 imply that this theorem holds for $n = 3$ and 4. Assuming that Theorem 4.11 holds for $n \leqslant m$, we wish to show that it also holds for $n = m + 1 \geqslant 5$. This problem divides into the following cases:

(i) $n = m + 1$ is odd. Consider any of the posets Q obtained by adding $\hat{0}$ and $\hat{1}$ to a connected component of $P - \{\hat{0}, \hat{1}\}$. So Q is an Eulerian Sheffer poset with $B(3) = D(3) = 3! = 6$. By the induction hypothesis, every interval of rank $k \leqslant m$ is isomorphic to B_k. So the Sheffer and binomial factorial functions of Q and the boolean lattice of rank $m + 1$ agree up to rank $m = n - 1$. Therefore Q and also P are binomial posets. Theorem 3.12 implies that there is a positive integer k such that $P \cong \boxplus^k(B_n)$, as desired.
(ii) $n = m + 1$ is even. We proceed by induction on n, the rank of P. Let $C(k)$ and $C'(k) = k$ be the coatom functions of the posets P and B_n, respectively, where $k \leqslant n$. By the induction hypothesis $C(k) = C'(k)$ for $k \leqslant n - 2$. So, Lemma 4.10 implies that

$$\frac{1}{C(n-1)}\left(1 - \frac{1}{C(n)}\right) = \frac{1}{C'(n-1)}\left(1 - \frac{1}{C'(n)}\right). \tag{4.11}$$

By the induction hypothesis, there is a positive integer α such that $C(n-1) = \alpha(n-1)$. Moreover, we know that $C'(n-1) = n - 1$ and $C'(n) = n$. Eq. (4.11) implies that $\alpha = 1$ and $C(n) = n$, so the poset P has the same factorial functions as B_n and $P \cong B_n$, as desired. □

4.1.3. Characterization of the structure of Eulerian Sheffer posets of rank $n \geqslant 4$ for which $B(3) = 3!$ and $D(3) = 4$

Let P be an Eulerian Sheffer poset of rank $n \geqslant 4$, with $B(3) = 3!$ and $D(3) = 4$. In this section we show that in the case $n = 2m + 2$, the poset P satisfies $P \cong \Sigma^*(\boxplus^\alpha(B_{2m+1}))$ for some integer $\alpha \geqslant 1$, and in the case $n = 2m + 1$, $P \cong \boxplus^\alpha(\Sigma^*(B_{2m}))$, for some integer $\alpha \geqslant 1$.

Theorem 4.12. *Let P be an Eulerian Sheffer poset of even rank $n = 2m + 2 \geqslant 4$ with $B(3) = 3!$ and $D(3) = 4$. Then $P \cong \Sigma^*(\boxplus^\alpha(B_{2m+1}))$, where $\alpha = \frac{B(2m+1)}{(2m+1)!}$ is a positive integer for $n \geqslant 6$ and $\alpha = 1$ for $n = 4$. Consequently the poset P has the following binomial and Sheffer factorial functions.*

(i) $B(k) = k!$ *for* $1 \leqslant k \leqslant 2m$, *and* $B(2m + 1) = \alpha(2m + 1)!$,
(ii) $D(1) = 1$, $D(k) = 2(k - 1)!$ *for* $2 \leqslant k \leqslant 2m + 1$, *and* $D(2m + 2) = 2\alpha(2m + 1)!$.

Proof. We prove the theorem by induction on m. By Theorem 3.12, we know that there is a positive integer α such that P has the binomial factorial function $B(2m+1) = \alpha(2m+1)!$ and $B(k) = k!$, for $1 \leqslant k < n = 2m+1$.

As we see in Lemma 4.4, the case $m = 1$ implies that $\alpha = 1$ when $B(3) = 3!$ and $D(3) = 4$. By applying Lemma 4.4, it can be seen that the poset P has the same factorial functions as $\Sigma^*(B_3)$. Therefore, the poset P has two atoms and its binomial 3-intervals are isomorphic to B_3. We conclude that $P \cong \Sigma^*(B_3)$ and so the theorem holds for $m = 1$.

In the case $m > 1$, suppose that the poset P has the binomial factorial function $B(k) = k!$ for $1 \leqslant k \leqslant 2m$, and $B(2m+1) = \alpha(2m+1)!$. By Theorem 3.12, the poset $Q \cong \boxplus^\alpha(B_{2m+1})$ is the only Eulerian binomial poset of rank $2m+1$ with the binomial factorial function $B(k) = k!$ for $1 \leqslant k \leqslant 2m$ and $B(2m+1) = \alpha(2m+1)!$, where α is a positive integer. Set $P' = \Sigma^*(Q) = \Sigma^*(\boxplus^\alpha(B_{2m+1}))$. It can be seen that P' is an Eulerian Sheffer poset of rank $2m+2$ with coatom function $C'(2m+2) = \alpha(2m+1)$ and $C'(k) = (k-1)$ for $3 \leqslant k \leqslant 2m+1$, as well as $C'(2) = 2$.

We wish to show that the poset P has the same coatom function as the poset P'. Since $B(k) = k!$ for $k \leqslant 2m$, by the induction hypothesis the coatom function of P is $C(k) = k-1$ for $3 \leqslant k \leqslant 2m$ and $C(2) = 2$. By substituting the values of $C'(2m+2)$ and $C'(2m+1)$ in Eq. (4.6) of Lemma 4.10, we have

$$\frac{1}{C(2m+1)}\left(1 - \frac{1}{C(2m+2)}\right) = \frac{1}{2m}\left(1 - \frac{1}{\alpha(2m+1)}\right). \tag{4.12}$$

The poset P has the binomial factorial function $B(2m+1) = \alpha(2m+1)!$, where α is a positive integer, and $B(k) = k!$ for $1 \leqslant k < 2m+1$. We conclude that $A(2m+1) = \alpha(2m+1)$ and $A(2m) = 2m$. So $C(2m+2) \geqslant A(2m+1) = \alpha(2m+1)$, as well as $C(2m+1) \geqslant A(2m) = 2m$.

We claim that Eq. (4.12) implies that $C(2m+1) = 2m$ and $C(2m+2) = \alpha(2m+1)$. Assume that there are $k, s \geqslant 1$ so that $C(2m+1) = 2m+k$ and $C(2m+2) = \alpha(2m+1)+s$. (Note that $k = 0$ implies $s = 0$ and vice versa.) Eq. (4.12) implies that

$$\frac{2m}{2m+k} = \left(\frac{\alpha(2m+1)-1}{\alpha(2m+1)}\right)\left(\frac{\alpha(2m+1)+s}{\alpha(2m+1)+s-1}\right). \tag{4.13}$$

It is easy to verify that in case $k, s \geqslant 1$, the right-hand side of Eq. (4.13) is always greater than the left-hand side. Thus, $k, s = 0$ and $C(2m+1) = 2m$ as well as $C(2m+2) = \alpha(2m+1)$. By the induction hypothesis, $D(k) = 2(k-1)!$ for $2 \leqslant k \leqslant 2m$. Since $C(2m+1) = 2m$ as well as $C(2m+2) = \alpha(2m+1)$, we conclude that P has the same factorial functions as the poset $P' = \Sigma^*(\boxplus^\alpha(B_{2m+1}))$.

Applying Eq. (2.2), the poset P has $\frac{D(2m+2)}{B(2m+1)} = 2$ elements of rank 1. Call them $\hat{0}_1$ and $\hat{0}_2$. Using Eq. (2.2), the number of elements of rank $1 \leqslant k \leqslant 2m+1$ in the intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ is

$$\frac{\alpha(2m+1)!}{k!(2m+1-k)!}. \tag{4.14}$$

The intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ both have the factorial function, $B(k) = k!$ for $1 \leqslant k \leqslant 2m$ and $B(2m+1) = \alpha(2m+1)!$. It can be seen that the intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ satisfy the Euler–Poincaré relation and so these intervals are Eulerian and binomial. Applying Theorem 3.12, one sees that both intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ are isomorphic to the poset $\boxplus^\alpha(B_{2m+1})$. Since the poset P has the same factorial functions as poset $P' = \Sigma^*(\boxplus^\alpha(B_{2m+1}))$, Eq. (2.2) yields that the number of elements of rank $k+1$ in P is the same as the number of elements of rank k in the intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ for $1 \leqslant k \leqslant 2m+1$, that is

$$\frac{\alpha(2m+1)!}{k!(2m+1-k)!}. \tag{4.15}$$

In summary, we have

(1) $[\hat{0}_1, \hat{1}] \cong [\hat{0}_2, \hat{1}] \cong Q \cong \boxplus^\alpha(B_{2m+1})$.

(2) The number of elements of rank $k+1$ in the poset P is the same as the number of elements of rank k in the intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$, $1 \leqslant k \leqslant 2m+1$.
(3) The poset P has only two atoms $\hat{0}_1$ and $\hat{0}_2$.

Statements (1), (2) and (3) imply that $P \cong P' = \Sigma^*(\boxplus^\alpha(B_{2m+1}))$, as desired. $\square$

Theorem 4.13. *Let P be an Eulerian Sheffer poset of odd rank $n = 2m+1 \geqslant 5$ with $B(3) = 6$ and $D(3) = 4$. Then $P \cong \boxplus^\alpha(\Sigma^*(B_{2m}))$ for some positive integer α.*

Proof. We obtain the poset Q by adding $\hat{0}$ and $\hat{1}$ to a connected component of $P - \{\hat{0}, \hat{1}\}$. In order to prove the theorem, we wish to show that $Q \cong \Sigma^*(B_{2m})$. It is easy to see that Q is an Eulerian Sheffer poset and also that P and Q have the same factorial functions and coatom function up to rank $2m$. That is, $B_Q(k) = B_P(k)$ and $D_Q(k) = D_P(k)$ for $1 \leqslant k \leqslant 2m$. We use $B(k)$, $D(k)$, $C(k)$, $A(k)$ to denote the factorial functions and the coatom function and atom function of Q.

By Theorem 3.11, the poset Q has the binomial factorial function $B(k) = k!$ for $1 \leqslant k \leqslant 2m$. We have $C(2m+1) \geqslant A(2m) = 2m$. Since every interval of rank 2 in Q is isomorphic to B_2, it has at least two coatoms. For every coatom a_i in Q, Theorem 4.12 implies that the interval $[\hat{0}, a_i]$ is isomorphic to $\Sigma^*(\boxplus^\alpha(B_{2m-1}))$. By considering the factorial functions we conclude that $\alpha = 1$ and so $[\hat{0}, a_i] \cong \Sigma^*(B_{2m-1})$.

Since Q is obtained by adding $\hat{0}$ and $\hat{1}$ to a connected component of $P - \{\hat{0}, \hat{1}\}$, we conclude that there are at least two particular coatoms a_1 and a_2 such that there is an element $c \in [\hat{0}, a_1], [\hat{0}, a_2]$ where $c \neq \hat{0}$. By considering the factorial functions of the interval $[c, \hat{1}]$, Theorems 3.11 and 3.12 imply that there is a positive integer k such that $[c, \hat{1}] \cong B_k$. Therefore, there is an element b of rank $k-2$ in $[c, \hat{1}]$ such that $b = a_1 \wedge a_2$. The element b is also an element of rank $2m-2$ in Q. The interval $[\hat{0}, b]$ is a subinterval of $[\hat{0}, a_1]$, so we have $[\hat{0}, b] \cong \Sigma^*(B_{2m-2})$. We conclude that the interval $[\hat{0}, b]$ only has two atoms, say x_1 and x_2. Since $[\hat{0}, a_1] \cong [\hat{0}, a_2] \cong \Sigma^*(B_{2m-1})$, the intervals $[\hat{0}, a_1]$ and $[\hat{0}, a_2]$ only have two atoms x_1 and x_2.

Define a graph G_Q as follows: vertices of G_Q are coatoms of poset Q and two vertices (coatoms) a_i and a_j are adjacent in G_Q if and only if there is an element $d \neq \hat{0}$ such that $d \in [\hat{0}, a_i]$ and $d \in [\hat{0}, a_j]$. Since Q is obtained by adding $\hat{0}$ and $\hat{1}$ to a connected component of $P - \{\hat{0}, \hat{1}\}$, G_Q is a connected graph. Thus, every coatom of rank $2m$ in Q is above only two atoms x_1 and x_2 in Q. Hence the number of elements of rank 1 in poset Q is 2, and by Eq. (2.2) we have

$$\frac{C(2m+1)D(2m)}{B(2m)} = 2. \tag{4.16}$$

Thus, $C(2m+1) = 2m$ and also Q has the same factorial functions as $\Sigma^*(B_{2m})$. By the same argument as Theorem 4.12, we conclude that $Q \cong \Sigma^*(B_{2m})$. So $P \cong \boxplus^\alpha(\Sigma^*(B_{2m}))$ for some positive integer α, as desired. $\square$

4.2. Characterization of the structure and factorial functions of Eulerian Sheffer posets of rank $n \geqslant 5$ with $B(3) = 4$

In this section, we characterize Eulerian Sheffer posets of rank $n \geqslant 5$ with $B(3) = 4$. Let P be an Eulerian Sheffer poset of rank $n \geqslant 5$ with $B(3) = 4$. It can be seen that the poset P satisfies one of the following cases:

1. The poset P has the binomial factorial function $B(k) = 2^{k-1}$, where $1 \leqslant k \leqslant n-1$;
2. n is even and there is a positive integer $\alpha > 1$ such that poset P has the binomial factorial function $B(k) = 2^{k-1}$ for $1 \leqslant k \leqslant n-2$ and $B(n-1) = \alpha \cdot 2^{n-1}$ for some positive integer α.

As a consequence of Theorems 3.11 and 3.12 in [4], we can characterize posets in the case (i). The main result of this section, Theorem 4.16, deals with the case (ii). It shows that if the Eulerian Sheffer

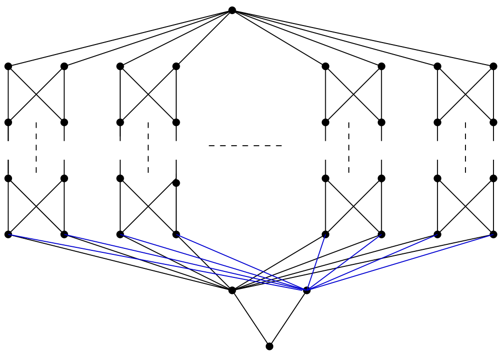


Fig. 3. $P = \Sigma^*(\boxplus^\alpha(T_{2m+1}))$.

poset P of rank $n = 2m+2 \geqslant 6$ has the binomial factorial function $B(k) = 2^{k-1}$ for $1 \leqslant k \leqslant 2m$ and $B(2m+1) = \alpha \cdot 2^{2m}$ for some positive integer α, then $P \cong \Sigma^*(\boxplus^\alpha(T_{2m+1}))$. See Fig. 3.

Given two ranked posets P and Q, define the *rank product* $P * Q$ by

$$P * Q = \{(x, z) \in P \times Q\colon \rho_P(x) = \rho_Q(z)\}.$$

Define the order relation by $(x, y) \leqslant_{P*Q} (z, w)$ if $x \leqslant_P z$ and $y \leqslant_Q w$. The rank product is also known as the Segre product; see [2].

The next theorem is a consequence of [4, Theorem 3.11].

Theorem 4.14. *Let P be an Eulerian Sheffer poset of rank $n \geqslant 4$ with the binomial factorial function $B(k) = 2^{k-1}$ for $1 \leqslant k \leqslant n-1$. Then the poset P and its coatom function $C(k)$ satisfy the following conditions*:

- (i) *$C(3) \geqslant 2$, and a length 3 Sheffer interval is isomorphic to a poset of the form $P_{q_1,\dots,q_r}$, as described before.*
- (ii) *$C(2k) = 2$, for $\lfloor \frac{n}{2} \rfloor \geqslant k \geqslant 2$ and the two coatoms in a length $2k$ Sheffer interval cover exactly the same element of rank $2k-2$.*
- (iii) *$C(2k+1) = h$ is an even positive integer for $\lfloor \frac{n-1}{2} \rfloor \geqslant k \geqslant 2$. Moreover, the set of h coatoms in a Sheffer interval of length $2k+1$ partitions into $\frac{h}{2}$ pairs, $\{c_1, d_1\}, \{c_2, d_2\}, \dots, \{c_{\frac{h}{2}}, d_{\frac{h}{2}}\}$, such that c_i and d_i cover the same two elements of rank $2k-1$.*

The next theorem is a consequence of [4, Theorem 3.12].

Theorem 4.15. *Let P be an Eulerian Sheffer poset of rank $n > 4$ with the binomial factorial function $B(k) = 2^{k-1}$, $1 \leqslant k \leqslant n-1$ and the coatom function $C(k)$, $1 \leqslant k \leqslant n$. Then a Sheffer k-interval $[\hat{0}, y]$ of P factors in the rank product as $[\hat{0}, y] \cong (T_{k-2} \cup \{\hat{0}, \widehat{-1}\}) * Q$, where $T_{k-2} \cup \{\hat{0}, \widehat{-1}\}$ denotes the butterfly interval of rank $k-2$ with two new minimal elements attached sequentially, and Q denotes a poset of rank k such that*

- (i) *each element of rank 2 through $k-1$ in Q is covered by exactly one element,*
- (ii) *each element of rank 1 in Q is covered by exactly two elements,*
- (iii) *each element of even rank 4 through $2\lfloor \frac{k}{2} \rfloor$ in Q covers exactly one element,*
- (iv) *each element of odd rank r from 5 through $2\lfloor \frac{k}{2} \rfloor + 1$ in Q covers exactly $\frac{C(r)}{2}$ elements, and*
- (v) *each 3-interval $[\hat{0}, x]$ in Q is isomorphic to a poset of the form $P_{q_1,\dots,q_r}$, where $q_1 + \cdots + q_r = C(3)$.*

In the following theorem we study the only remaining case (ii).

Theorem 4.16. *Let P be an Eulerian Sheffer poset of even rank $n = 2m+2 > 4$ with the binomial factorial function $B(k) = 2^{k-1}$ for $1 \leqslant k \leqslant 2m$, and $B(2m+1) = \alpha \cdot 2^{2m}$, where $\alpha > 1$ is a positive integer. Then $P \cong \Sigma^*(\boxplus^\alpha(T_{2m+1}))$.*

Proof. Let $D(k)$, $1 \leqslant k \leqslant 2m+2$, and also $B(k)$, $1 \leqslant k \leqslant 2m+1$, be the Sheffer and binomial factorial functions of the poset P, respectively. The Euler–Poincaré relation for an interval of length $2m+2$ is

$$1+\sum_{k=1}^{2m+2}(-1)^k\cdot\frac{D(2m+2)}{D(k)B(2m+2-k)}=0. \tag{4.17}$$

As is discussed in Eq. (3.2) of [4], the above Euler–Poincaré relation for the interval of even rank $2m+2$ can also be stated as follows:

$$\frac{2}{D(2m+2)}+\sum_{k=1}^{2m+1}\frac{(-1)^k}{D(k)B(2m+2-k)}=0. \tag{4.18}$$

By expanding the left side of Eq. (4.18), we have:

$$\frac{(-1)}{\alpha\cdot 2^{2m}}+\sum_{k=2}^{2m+2}\frac{(-1)^k}{D(k)\cdot 2^{2m+2-k-1}}=0. \tag{4.19}$$

Here, Eq. (4.18) for Sheffer $2m$-intervals can be stated as follows,

$$\sum_{k=1}^{2m}\frac{(-1)^k}{D(k)\cdot 2^{2m-1-k}}=0. \tag{4.20}$$

Thus,

$$\frac{1}{2^{2m}}=\sum_{k=2}^{2m}\frac{(-1)^k}{D(k)\cdot 2^{2m+1-k}}. \tag{4.21}$$

It follows from Eqs. (4.19) and (4.21) that

$$\frac{-1}{\alpha\cdot 2^{2m}}+\frac{1}{2^{2m}}+\frac{-1}{D(2m+1)}+\frac{2}{D(2m+2)}=0. \tag{4.22}$$

Let k be the number of atoms in a Sheffer interval of size $2m+1$ and $c=C(2m+2)$. We thus have $D(2m+1)=k\cdot 2^{2m-1}$ and $D(2m+2)=c\cdot k\cdot 2^{2m-1}$. Therefore

$$\frac{1}{2^{2m}}-\frac{1}{\alpha\cdot 2^{2m}}=\frac{1}{k\cdot 2^{2m-1}}-\frac{1}{\frac{c\cdot k}{2}\cdot 2^{2m-1}}. \tag{4.23}$$

Therefore,

$$\frac{1}{2}-\frac{1}{2\alpha}=\frac{1}{k}-\frac{1}{\frac{c}{2}\cdot k}. \tag{4.24}$$

Comparing coatom and atom functions of the Sheffer and binomial intervals, we have $k \geqslant 2$ as well as $c \geqslant 2\alpha$.

In case $k \geqslant 4$, we have

$$\frac{1}{2}-\frac{1}{k}>\frac{2}{c\cdot k}-\frac{1}{2\alpha} \tag{4.25}$$

which is not possible. By Theorem 4.14, $C(2m+1)=k$ is an even number and the case $k=3$ is also not possible. Therefore, we conclude that $k=2$ and so $c=2\alpha$. Thus, every Sheffer j-interval has two atoms for $1 \leqslant j \leqslant 2m+1$. We thus have $D(k)=2B(k-1)=2^{k-1}$ for $2 \leqslant k \leqslant 2m+1$ as well as $D(2m+2)=\alpha\cdot 2^{2m+1}$. Let $\hat{0}_1$ and $\hat{0}_2$ be atoms of P. By Theorem 3.12, both of the intervals $[\hat{0}_1,\hat{1}]$ and $[\hat{0}_2,\hat{1}]$ are isomorphic to the poset $Q=\boxplus^{\alpha}(T_{2m+1})$. It follows from Eq. (2.2) that the number of elements of rank $k-1$ in the intervals $[\hat{0}_1, \hat{1}] \cong [\hat{0}_2, \hat{1}] \cong Q$ is the same as the number of elements of rank k in poset P and it can be computed as follows,

$$\frac{D(2m+2)}{D(k)B(2m+2-k)} = \frac{B(2m+1)}{B(k)B(2m+1-k)}. \tag{4.26}$$

We know that $\hat{0}_1$ and $\hat{0}_2$ are the only atoms in P, so by the above fact we conclude that $P \cong \Sigma^* Q = \Sigma^*(\boxplus^\alpha(T_{2m+1}))$, as desired. $\square$

5. Finite Eulerian triangular posets

As we discussed before, a larger class of posets to consider are triangular posets. For definitions regarding triangular posets, see Section 2. An Eulerian example of a triangular poset is the face lattice of the 4-dimensional regular polytope known as the 24-cell. In the following theorem, we characterize the Eulerian triangular posets of rank $n \geqslant 4$ such that $B(k, k+3) = 6$ for $1 \leqslant k \leqslant n-3$.

Theorem 5.1. *Let P be an Eulerian triangular poset of rank $n \geqslant 4$ such that its factorial function satisfies $B(k, k+3) = 6$ for every $0 \leqslant k \leqslant n-3$. Then P can be characterized as follows*:

1. *For n odd, there is an integer $\alpha \geqslant 1$ such that $P \cong \boxplus^\alpha(B_n)$.*
2. *For n even, the poset P is isomorphic to $P \cong B_n$.*

Proof. We proceed by induction on the rank n of the poset P.

- $n = 4$. An Eulerian triangular poset of rank 4 is also a Sheffer poset. Since $B(1, 4) = 6$, by Lemma 4.4 we conclude that $P \cong B_4$.
- $n = 2m+1$. By the induction hypothesis, every interval of rank $k \leqslant 2m$ in P is isomorphic to B_k. Hence P is a Sheffer poset and Theorem 4.11 implies that $P \cong \boxplus^\alpha(B_n)$, where $\alpha \geqslant 1$ is a positive integer.
- $n = 2m+2$. Let r and t be the number of elements of rank 1 and $2m+1$ in P. By the induction hypothesis, there are positive integers k_t and k_r such that $B(1, 2m+2) = k_t(2m+1)!$ and $B(0, 2m+1) = k_r(2m+1)!$. Therefore, $B(0, 2m+2) = tk_r(2m+1)! = rk_t(2m+1)!$ and also $B(n, n+k) = k!$, where $1 \leqslant k \leqslant 2m+1-n$ and $n \geqslant 1$. The Euler–Poincaré relation for interval of size $2m+2$ states the following,

$$1 + \sum_{k=1}^{2m+2} \frac{(-1)^k B(0, 2m+2)}{B(0,k)B(k, 2m+2)} = 0. \tag{5.1}$$

By substituting the values in Eq. (5.1), we have

$$1 + tk_r\left(\sum_{k=2}^{2m} \frac{(-1)^k(2m+1)!}{k!(2m+2-k)!}\right) + \frac{-tk_r(2m+1)!}{k_t(2m+1)!} + \frac{-tk_r(2m+1)!}{k_r(2m+1)!} + 1 = 0. \tag{5.2}$$

Eq. (5.2) leads us to

$$2 - t\left(\frac{k_r}{k_t} + \frac{k_r}{k_r}\right) + tk_r\left(\sum_{k=2}^{2m}\left(\frac{(-1)^k(2m+1)!}{k!(2m+2-k)!}\right)\right) = 0, \tag{5.3}$$

so,

$$2 = t\left(\frac{k_r}{k_t} + \frac{k_r}{k_r} + \frac{-(k_r)(4m+2)}{2m+2}\right). \tag{5.4}$$

Without loss of generality, let us assume that $k_r \geqslant k_t \geqslant 1$. Therefore,

$$2 = t\left(\frac{k_r}{k_t} + \frac{k_r}{k_r} + \frac{-(k_r)(4m+2)}{2m+2}\right) \leqslant t\left(k_r + 1 - \left(\frac{4m+2}{2m+2}\right)k_r\right)$$
$$\leqslant t\left(1 - \frac{2m}{2m+2}k_r\right). \tag{5.5}$$

The right-hand side of the above equation is positive only if $k_r = 1$. So $k_r = 1$ and since $k_r \geqslant k_t \geqslant 1$, we conclude that $k_t = 1$. Therefore, $2 = t\frac{2}{2m+2}$ and so $t = 2m+2$. Similarly, we conclude that $r = 2m+2$. Thus, P has the same factorial function as B_{2m+2} and by Proposition 3.8, this poset is isomorphic to B_{2m+2}, as desired. □

6. Conclusions and remarks

An interesting research problem is to classify the factorial functions of Eulerian triangular posets. It is also interesting to classify Eulerian triangular posets with specific factorial functions on their smaller intervals. In Theorem 5.1, we characterize the Eulerian triangular posets of rank $n \geqslant 4$ such that $B(k, k+3) = 6$, for $1 \leqslant k \leqslant n-3$.

The following result of Stanley (see [7, Lemma 8]) may be of relevance: A graded poset P is a boolean lattice if every 3-interval is a boolean lattice and for every $[x, y]$ of rank at least 4 the open interval (x, y) is connected. Using Stanley's result, it might be possible to obtain different proofs for Theorems 3.11, 3.12, 4.11 and 5.1.

This research is motivated by the above result of Stanley. We characterize Eulerian binomial and Sheffer posets by considering the factorial functions of 3-intervals. The project of studying Eulerian Sheffer posets is almost complete. Only the following cases remain to be studied:

- Finite Eulerian Sheffer posets of odd rank with $B(3) = 6$, $D(3) = 8$. In this case we ask the following question: Let P be an Eulerian Sheffer poset of odd rank $n = 2m+1 \geqslant 5$ with $B(3) = 6$, $D(3) = 8$. Is there a positive integer k such that P has the same factorial functions as the poset $Q_k = \boxplus^k(C_{2m+1})$?
- Characterization of the finite Eulerian Sheffer posets of rank 5 with $B(3) = 6$, $D(3) \neq 4, 6, 8$. There is such a poset with $D(3) = 10$, namely the face lattice of the 120-cell with Schläfli symbol $\{5, 3, 3\}$.

Acknowledgments

I would like to thank Richard Stanley for suggesting this research problem. I also would like to thank Richard Ehrenborg, Margaret A. Readdy, Richard Stanley and Seth Sullivant for helpful discussions and comments and thank Craig Desjardins for reading a draft of this paper. I am grateful to two anonymous referees for many helpful suggestions.

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