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New Generalizations of Fibonacci and Lucas Sequences

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Abstract

We consider the sequences $\{f_n\}_{n=0}^{\infty}$ and $\{l_n\}_{n=0}^{\infty}$ which are generated by the recurrence relations $f_n = 2af_{n-1} + (b^2 - a)f_{n-2}$ and $l_n = 2al_{n-1} + (b^2 - a)l_{n-2}$ with the initial conditions $f_0 = 0, f_1 = 1$ and $l_0 = 2, l_1 = 2a$ where a and b are any non – zero real numbers. We obtain generating functions, Binet formulas for these two sequences and give generalizations of some well – known identities.

Mathematics Subject Classification: 11B39

Keywords: Fibonacci sequence, Lucas Sequence, Pell Sequence, Pell – Lucas Sequence

1 Introduction

Both of Fibonacci sequence and Lucas sequence are well – known sequences among integer sequences. The Fibonacci numbers satisfy the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. The Lucas numbers satisfy the same recurrence as the Fibonacci numbers, namely $L_n = L_{n-1} + L_{n-2}$, but the initial conditions $L_0 = 2$ and $L_1 = 1$.

Generating functions for the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ and the Lucas sequence $\{L_n\}_{n=0}^{\infty}$ are, respectively,

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} \text{ and } L(x) = \sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{1-x-x^2}.$$

Binet formulas for the Fibonacci numbers and the Lucas numbers are, respectively,

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}} \text{and } L_n = (1+\sqrt{5})^n + (1-\sqrt{5})^n$$

where $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ are the solutions of the equation $x^2 - x - 1 = 0$. We note that the positive root $1+\sqrt{5}$ is known as "golden ratio".

These two famous sequences delighted mathematician for centuries with their properties and applications. There are many relations and identities between these two sequences. For more details about the Fibonacci and the Lucas sequence, see [11].

Pell sequence $\{P_n\}_{n=0}^{\infty}$ and Pell – Lucas sequence $\{Q_n\}_{n=0}^{\infty}$ are also well – known sequences. Pell numbers satisfy the recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ with the initial conditions $P_0 = 0$ and $P_1 = 1$. Pell – Lucas numbers satisfy the same recurrence but the initial conditions $Q_0 = 2$ and $Q_1 = 2$.
Generating functions for the Pell sequence $\{P_n\}_{n=0}^{\infty}$ and the Pell – Lucas sequence $\{Q_n\}_{n=0}^{\infty}$ are, respectively,

$$P(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-x^2} \text{ and } Q(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{2-2x}{1-2x-x^2}.$$

Binet formulas for the Pell numbers and the Pell – Lucas numbers are, respectively,

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \text{ and } Q_n = (1+\sqrt{2})^n + (1+\sqrt{2})^n$$

where $1+\sqrt{2}$ and $1-\sqrt{2}$ are the solutions of the equation $x^2 - 2x - 1 = 0$. The positive root $1+\sqrt{2}$ is known as "silver ratio".

There are many generalizations for the Fibonacci sequences. Some authors ([9, 10, 14]) generalized the Fibonacci sequence by changing the first two terms, while others ([6, 7, 12, 13, 16, 17]) generalized it by changing the recurrence relation.

Some generalizations of Pell numbers can be found in [1, 15].

Definition 1. We define generalized sequences $\{f_n\}_{n=0}^{\infty}$ and $\{l\}_{n=0}^{\infty}$ by the following recurrence relations:

$$f_n = 2af_{n-1} + (b-a^2)f_{n-2} \quad (n \geq 2) \tag{1}$$

and

$$l_n = 2al_{n-1} + (b-a^2)l_{n-2} \quad (n \geq 2) \tag{2}$$

with initial conditions, respectively

$$f_0 = 0, f_1 = 1$$

and

$$l_0 = 2, l_1 = 2a$$

where a and b are any non – zero real numbers.

Clearly, for $(a,b) = (1/2, 5/4), (1/2, 9/4)$ and $(1,2)$, the sequence $\{f_n\}_{n=0}^{\infty}$ reduces the Classical Fibonacci, Jacobsthal and Pell sequences, respectively, and for $(a,b) = (1/2, 5/4), (1/2, 9/4)$ and $(1,2)$, the sequence $\{l_n\}_{n=0}^{\infty}$ reduces the Classical Lucas, Jacobsthal – Lucas and Pell – Lucas sequences, respectively.

For any positive integer k, if we take $a = k/2$ and $b = (1-k^2)/4$, the sequence $\{f_n\}_{n=0}^{\infty}$ reduces the k – Fibonacci sequence which is defined in [7], and the sequence $\{l_n\}_{n=0}^{\infty}$ reduces the k – Lucas Sequence defined in [8]. Properties of the k – Fibonacci numbers can found in [2, 5]. If we take $a = 1$ and $b = k+1$ the sequence $\{f_n\}_{n=0}^{\infty}$ reduces the k – Pell sequence defined in [3], the sequence $\{l_n\}_{n=0}^{\infty}$ reduces the k – Pell – Lucas Sequence [4].

2 Generating Functions

In this section, we give generating functions for the sequences $\{f_n\}_{n=0}^{\infty}$ and $\{l_n\}_{n=0}^{\infty}$.

Theorem 2. Generating functions for the sequences $\{f_n\}_{n=0}^{\infty}$ and $\{l_n\}_{n=0}^{\infty}$ are, respectively

$$\sum_{n=0}^{\infty} f_n x^n = \frac{x}{1-2ax-(b-a^2)x^2} \tag{3}$$

and

$$\sum_{n=0}^{\infty} l_n x^n = \frac{2-2ax}{1-2ax-(b-a^2)x^2}. \tag{4}$$

Proof. We define $f(x) = \sum_{n=0}^{\infty} f_n x^n$ and $l(x) = \sum_{n=0}^{\infty} l_n x^n$. Then, we have

$$f(x) = x + 2ax^2 + \sum_{n=3}^{\infty} f_n x^n, \tag{5}$$

$$-2axf(x) = -2ax^2 - \sum_{n=3}^{\infty} 2af_{n-1}x^n \tag{6}$$

and

$$(a^2-b)x^2 f(x) = \sum_{n=3}^{\infty} (a^2-b) f_{n-2} x^n. \tag{7}$$

If we sum Eq. (5), (6) and (7), and use the recurrence relation (1), we get

$$[1-2ax-(b-a^2)x^2]f(x)=x.$$

So, the last equation gives the Eq.(3). Similarly, Eq.(4) can be obtained. ■

3 Binet Formulas

The following theorem gives Binet formulas for the sequences $\{f_n\}_{n=0}^{\infty}$ and $\{l_n\}_{n=0}^{\infty}$.

Theorem 3. The n^{th} term of the sequences $\{f_n\}_{n=0}^{\infty}$ and $\{l_n\}_{n=0}^{\infty}$ are, respectively,

$$f_n=\frac{\alpha^n-\beta^n}{\alpha-\beta} \tag{8}$$

and

$$l_n=\alpha^n+\beta^n \tag{9}$$

where $\alpha=a+\sqrt{b}$ and $\beta=a-\sqrt{b}$ are roots of the equation $x^2-2ax-(b-a^2)=0$.

Proof. We can write Eq.(3) as

$$\sum_{n=0}^{\infty} f_n x^n=\frac{x}{(\alpha x-1)(\beta x-1)}.$$

Partial fraction decomposition of the right-hand side on the last equation is

$$\frac{x}{(\alpha x-1)(\beta x-1)}=-\frac{1}{2\sqrt{b}}\frac{1}{(\alpha x-1)}+\frac{1}{2\sqrt{b}}\frac{1}{(\beta x-1)}.$$

This equation gives

$$\begin{aligned}\sum_{n=0}^{\infty} f_n x^n&=\frac{1}{2\sqrt{b}}\frac{1}{(1-\alpha x)}-\frac{1}{2\sqrt{b}}\frac{1}{(1-\beta x)}\\&=\frac{1}{2\sqrt{b}}\sum_{n=0}^{\infty}\alpha^n x^n-\frac{1}{2\sqrt{b}}\sum_{n=0}^{\infty}\beta^n x^n\\&=\sum_{n=0}^{\infty}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)x^n.\end{aligned}$$

So, we have Eq.(8). Eq.(9) can be obtained similarly. ■

4 Main Results

In this section, we give generalizations of some well – known identities for the classical Fibonacci and Lucas sequences.

Theorem 4. For every integer *n*, we have

$$f_{-n} = -\frac{1}{(a^2-b)^n} f_n, \tag{10}$$

and

$$l_{-n} = \frac{1}{(a^2-b)^n} l_n. \tag{11}$$

The proof of Theorem 4 can be made easily by using the Binet formulas (8) and (9).

Theorem 5. (Catalan's identity) For every integers *n* and *r*, we have

$$f_{n+r}f_{n-r} - f_n^2 = -(a^2-b)^{n-r}f_r^2, \tag{12}$$

and

$$l_{n+r}l_{n-r} - l_n^2 = 4b(a^2-b)^{n-r}f_r^2. \tag{13}$$

Proof. Using the Binet formula (8), we get

$$\begin{aligned} f_{n+r}f_{n-r} - f_n^2 &= \frac{1}{(\alpha-\beta)^2}[(\alpha^{n+r}-\beta^{n+r})(\alpha^{n-r}-\beta^{n-r}) - (\alpha^n-\beta^n)^2] \\ &= -\frac{1}{(\alpha-\beta)^2}[\alpha^{n+r}\beta^{n-r} + \alpha^{n-r}\beta^{n+r} - 2\alpha^n\beta^n] \\ &= -\frac{(\alpha\beta)^{n-r}}{(\alpha-\beta)^2}[\alpha^{2r} - 2\alpha^r\beta^r + \beta^{2r}] \\ &= -(\alpha\beta)^{n-r}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2 \\ &= -(a^2-b)^{n-r}f_r^2. \end{aligned}$$

The proof of Eq. (13) is similar. ■

If we take $r = 1$ in Theorem 5 and use the fact that $f_1 = 1$, we find the following:

Corollary 6. (Cassini's identity) For every integer *n*, we have

$$f_{n+1}f_{n-1} - f_n^2 = -(a^2-b)^{n-1}$$

and

$$l_{n+1}l_{n-1} - l_n^2 = 4b(a^2-b)^{n-1}.$$

Theorem 5 give us generalized Gelin – Cesaro identity, also.

Corollary 7. (Gelin Cesaro identity) For every integer *n*, we have

$$f_{n-2}f_{n-1}f_{n+1}f_{n+2} - f_n^4 = f_n^2(a^2-b)^{n-2}(-5a^2+b) + 4a^2(a^2-b)^{2n-3}$$

and

$$l_{n-2}l_{n-1}l_{n+1}l_{n+2} - l_n^4 = l_n^2(a^2-b)^{n-2}(20a^2b-4b^2) + 64a^2b^2(a^2-b)^{2n-3}.$$

Theorem 8. (d'Ocagne's identity) For every integers *m* and *n*, we have

$$f_mf_{n+1} - f_nf_{m+1} = (a^2-b)^nf_{m-n} \tag{14}$$

And

$$l_m l_{n+1} - l_n l_{m+1} = -4b(a^2-b)^n f_{m-n}. \tag{15}$$

Proof. Using the Binet formula (8), we have

$$\begin{aligned}
&f_m f_{n+1} - f_n f_{m+1}\\
&\quad = \frac{1}{(\alpha-\beta)^2}\,[(\alpha^m-\beta^m)(\alpha^{n+1}-\beta^{n+1}) - (\alpha^n-\beta^n)(\alpha^{m+1}-\beta^{m+1})]\\
&\quad = \frac{1}{(\alpha-\beta)^2}\,[\alpha^{m+1}\beta^n - \alpha^m\beta^{n+1} + \alpha^n\beta^{m+1} - \alpha^{n+1}\beta^m]\\
&\quad = \frac{(\alpha\beta)^n}{(\alpha-\beta)^2}\,[(\alpha-\beta)\alpha^{m-n} - (\alpha-\beta)\beta^{m-n}]\\
&\quad = (\alpha\beta)^n f_{m-n}.
\end{aligned}$$

Since $\alpha\beta = a^2 - b$, we obtain Eq.(14). Eq.(15) can be proven similarly. ■

Taking $n \to -n$ in Theorem 8 and using Theorem 4, we have the followings:

Corollary 9. For every integers *m* and *n*, we have

$$f_{m+n} = f_{m+1}f_n - (a^2-b)f_m f_{n-1} \tag{16}$$

and

$$4bf_{m+n} = l_{m+1}l_n - (a^2-b)l_m l_{n-1}. \tag{17}$$

Theorem 3 gives the following:

Corollary 10. For every integer *n*, we have

$$f_n l_n = f_{2n}.$$

Theorem 11. For every integers *m* and *n*, we have

$$f_n l_m = f_{n+m} + (a^2-b)^m f_{n-m}.$$

Proof.

$$\begin{aligned}
f_n l_m = \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)(\alpha^m-\beta^m) &= \frac{1}{(\alpha-\beta)}(\alpha^{n+m} + \alpha^n\beta^m - \alpha^m\beta^n - \beta^{n+m})\\
&= \frac{\alpha^{n+m}-\beta^{n+m}}{\alpha-\beta} + (\alpha\beta)^m\left(\frac{\alpha^{n-m}-\beta^{n-m}}{\alpha-\beta}\right)\\
&= f_{n+m} + (a^2-b)^m f_{n-m}.
\end{aligned}$$ ■

Theorem 12. For every integer *n*, we have

$$l_{n+1} - (a^2+b)l_{n-1} = 4abf_{n-1} \tag{18}$$

and

$$l_n - al_{n-1} = 2bf_{n-1}. \tag{19}$$

Proof. For the first equation, we write

$$\begin{aligned}
l_{n-1} + l_{n+1} &= \alpha^{n-1} + \beta^{n-1} + \alpha^{n+1} + \beta^{n+1}\\
&= \alpha^{n-1}(1+\alpha^2) + \beta^{n-1}(1+\beta^2)
\end{aligned}$$

$$= \alpha^{n-1}\left(a^2 + 2a\sqrt{b} + b + 1\right) + \beta^{n-1}\left(a^2 - 2a\sqrt{b} + b + 1\right)$$
$$= (a^2 + b + 1)(\alpha^{n-1} + \beta^{n-1}) + 2a\sqrt{b}(\alpha^{n-1} + \beta^{n-1})$$
$$= (a^2 + b + 1)(\alpha^{n-1} + \beta^{n-1}) + 4ab\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right)$$
$$= (a^2 + b + 1)l_{n-1} + 4abf_{n-1}.$$

The last equation gives Eq.(18). If we use the recurrence relation (2) in Eq.(18), we obtain Eq.(19), easily. ■

Theorem 13. For every integer *n*, we have

$$f_{n+1} - (a^2 + b)f_{n-1} = al_{n-1}$$

and

$$f_n - a\,f_{n-1} = \frac{l_{n-1}}{2}.$$

Theorem 13 can be proven with a similar way to Theorem 12.

Theorem 14. For every integer *n*, we have

$$l_n^2 = 4bf_n^2 + 4(a^2 - b)^n. \tag{20}$$

Proof. By using the recurrence relation (1), we get

$$f_n^2 = \frac{1}{(\alpha - \beta)^2}(\alpha^{2n} - 2\alpha^n\beta^n + \beta^n).$$

Then, we find

$$4bf_n^2 = q_{2n} + 4(a^2 - b)^m. \tag{21}$$

Similarly, by using the recurrence relation (2), we obtain

$$l_n^2 = l_{2n} + 2(a^2 - b)^n. \tag{22}$$

Eqs. (21) and (22) give the theorem. ■

Theorem 15. For every integer *n*, we have

$$f_n f_{n+1} = \frac{1}{4b}[l_{2n+1} - 2a(a^2 - b)^n] \tag{23}$$

and

$$l_n l_{n+1} = l_{2n+1} + 2a(a^2 - b)^n. \tag{24}$$

Proof. Eq. (1) gives

$$f_n f_{n+1} = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right)$$
$$= \frac{1}{(\alpha - \beta)^2}[\alpha^{2n+1} + \beta^{2n+1} - \alpha^n\beta^n(\alpha + \beta)].$$

The last equality gives Eq.(23). The other can be proven similarly. ■

Using two equations in Theorem 15, we find the following:

Corollary 16. For every integers *n*, we have

$$4bf_nf_{n+1} + l_nl_{n+1} = 2l_{2n+1.}$$

References

[1] N. Bicknell, "A primer on the Pell sequence and related sequence", *Fibonacci Quarterly*, **32** (5) (1975), 345 – 349.

[2] C. Bolat, H. Kose, "On the Properties of *k* – Fibonacci Numbers", *Int. J. Contemp. Math. Sciences*, **22** (5) (2010), 1097-1105.

[3] P. Catarino, "On Some Identities and Generating Functions for *k* – Pell Numbers", *Int. J. of Math. Analysis*, **7** (38) (2013), 1877 – 1884. http://dx.doi.org/10.12988/ijma.2013.35131

[4] P. Catarino, P. Vasco, "On some Identities and Generating Functions for k-Pell-Lucas sequence", *Applied Mathematical Sciences*, **7** (98) (2013), 4867-4873. http://dx.doi.org/10.12988/ams.2013.37379

[5] P. Catarino, "On Some Identities for *k* – Fibonacci Sequence", *Int. J. Contemp. Math. Sciences*, **9** (1) (2014). http://dx.doi.org/10.12988/ijcms.2014.311120

[6] M. Edson, O. Yayenie, "A New Generalization of Fibonacci Sequences and Extended Binet's Formula", *Integers*, **9** (2009), 639 – 654. http://dx.doi.org/10.1515/INTEG.2009.051

[7] S. Falcon, A. Plaza, "The k-Fibonacci Sequence and the Pascal 2-Triangle", *Chaos, Solitons and Fractals*, **33** (2007), 38-49. http://dx.doi.org/10.1016/j.chaos.2006.10.022

[8] S. Falcon, "On the *k* – Lucas Numbers", *Int. J. Contemp. Math. Sciences*, **6** (21) (2011), 1039 – 1050.

[9] A.F. Horadam, "A Generalized Fibonacci Sequence", *Amer. Math. Monthly*, **68** (1961) 455 – 459.

[10] S.T. Klein, "Combinatorial Representation of Generalized Fibonacci Numbers", *Fibonacci Quarterly*, **29** (1991), 124 – 131.

[11] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.

[12] A.T. Krassimir, A.C. Liliya, S.D. Dimitar, "A New Perspective to the Generalization of the Fibonacci Sequence", *Fibonacci Quarterly*, **23** (1) (1985), 21 – 28.

[13] G.Y. Lee, J.S. Kim, H.K. Shin, "The Binet Formula and Representations of *k*-Generalized Fibonacci Numbers", *Fibonacci Quarterly*, **39** (2) (2001), 158 – 164.

[14] J.C. Pond, "Generalized Fibonacci Summations", *Fibonacci Quarterly*, **6** (1968), 97 – 108.

[15] A.G. Shannon, A.F. Horadam, *Generalized Pell Numbers and Polynomials, Application of Fibonacci Numbers*, Volume 9: Proceedings of the Tenth International Research Conference on Fibonacci Numbers and Their Application, Kluwer Academic Publishers, 213 – 224, 2004.

[16] G. Sburlati, "Generalized Fibonacci Sequence and Linear Congruences", *Fibonacci Quarterly*, **40** (2002), 446 – 452.

[17] L.J. Zai, L.J. Sheng, "Some Properties of the Generalization of the Fibonacci Sequence", *Fibonacci Quarterly*, **25** (2) (1987), 111 – 117.

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