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Algebraic K-Theory and Zeta Functions of Elliptic Curves

S. Bloch*

The classical regulator formula [6, Chapter 5]

$$\lim_{s\to 1}(s-1)\zeta_K(s)=\frac{2^{r_1+r_2}\pi^{r_2}Rh}{\sqrt{|D|}\cdot w}$$

computes the residue of the zeta function of a number field at $s=1$. Various generalizations have been proposed [19], [2], [20]. Lichtenbaum, noting that $h=\#K_0(\mathcal{O}_K)$ and $m=\#K_1(\mathcal{O}_K)_{\text{tors}}$ suggested a formula relating $\zeta_K(m+1)$ to $\#K_{2m}(\mathcal{O}_K)$, $\#K_{2m+1}(\mathcal{O}_K)$ and a higher regulator R_m. Borel [4], [5], studied a regulator map

$$r_{2m+1}\colon K_{2m+1}(\mathcal{O}_K)\to \boldsymbol{R}^{d_m}$$

where

$$d_m=\left\{\begin{array}{ll} r_2, & m=2n+1>0\\ r_1+r_2, & m=2n>0\\ r_1+r_2-1, & m=0\end{array}\right\}=\text{order of zero of } \zeta_K \text{ at } s=-m.$$

He showed that r_{2m+1} embedded $K_{2m+1}(\mathcal{O}_K)/\text{torsion}$ as a lattice of maximal rank with volume a rational multiple of

$$\pi^{-d_m}\lim_{s\to -m}\zeta_K(s)(s+m)^{-d_m}\in\pi^{-d(m+1)}|D|^{1/2}\zeta_K(m+1)\cdot\boldsymbol{Q}.$$

In another direction, let E be an elliptic curve defined over a number field K, and let $L(E/K,s)$ be the "Hasse–Weil zeta function". Birch and Swinnerton–Dyer

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have conjectured [2], [20], [21] that $L(E/K, s)$ should vanish to order $\operatorname{rk} E(K)$ at $s=1$ and they have proposed a regulator formula for the first nonvanishing coefficient in the Taylor expansion.

One can envision an amalgated sum of conjectures:

{ *#?!!?#*, $L(E/K, m+1)$ }

↗ ↖

{ Lichtenbaum conjecture, Borel's work, $\zeta_K(m+1)$ } { Birch-Swinnerton Dyer Conjectures, $L(E/K, 1)$ }

↖ ↗

{ Classical regulator formula, $\zeta_K(1)$ }

Oddly enough, the fireworks at the top may be easier to deal with than the right hand side. I will sketch the construction of a regulator map $R_q\colon K_2(E) \to \boldsymbol{C}$ ($q=e^{2\pi i\tau}$), and relate in special cases the values of R_q with $L(E/K, 2)$. As an example, consider $E/\boldsymbol{Q}$ with complex multiplication by the full ring of integers $\mathcal{O}_k=\boldsymbol{Z}+\boldsymbol{Z}\tau$, k imaginary quadratic. Assume for simplicity the conductor N of E lies in $\boldsymbol{Z}$. Let χ^{Gross} denote the Grössencharakter associated to E so $L(E/\boldsymbol{Q}, s)=L(\chi^{\text{Gross}}, s)$, and let χ be the corresponding Dirichlet character on $\mathcal{O}_k$ with conductor N, so $\chi^{\text{Gross}}(\mathfrak{p})=\bar{h}\chi(h)$ if $\mathfrak{p}=(h)$. Write

$$\hat{\chi}=\frac{1}{N}\sum_{a,b=0}^{N-1}\chi(a+b\tau)\,e^{2\pi i b/N}.$$

THEOREM. *There exists* $U\in K_2(E)$ *such that*

$$L(\chi^{\text{Gross}}, 2)=\frac{\pi|\mu_k|\hat{\chi}}{i(\operatorname{Im}\tau)^2N^4}R_q(U).$$

(More generally, if $N\in k$, $\hat{\chi}$ *is replaced by a more complicated character sum.)*

For any E over a number field K, one might

CONJECTURE. $\operatorname{rk} K_2(E)=$ order of zero of $L(E/K, s)$ at $s=0$.
The conjecture, of course, presumes analytic continuation of $L(E/K, s)$.

REMARK. There is a basic analogy between $K_{i+1}(k)$ and $K_i(E)$ where k is a field and E is an elliptic curve. In fact, $K_{i+1}(k)$ is contained in the relative K-group $K_i(\boldsymbol{P}^1_k, \{0, \infty\})$ which can be thought of (perhaps not too literally as I do not know if excision holds in this case) either as K_i of the degenerate nodal elliptic curve or as "K_i with compact supports" of $\boldsymbol{G}_m$. To use this dictionary to formulate a Birch–Lichtenbaum–Swinnerton-Dyer conjecture is perhaps premature. A more accessible problem might be to formulate and prove an analogue for $K_1(E)$ of the exact sequence due to Moore

$$K_2(k)\to\coprod_{\text{places of }k}\mu_{k_v}\to\mu_k\to 0.$$

Let E be an elliptic curve over C, A an abelian group. Given divisors $\delta=\sum n_i(a_i)$ and $\delta'=\sum m_j(b_j)$ on E, define

$$\delta^- * \delta' = \sum n_i m_j (b_j - a_i),$$
$$C(E)^* \otimes C(E)^* \to \coprod_{x\in E} Z,\ f\otimes g \mapsto (f)^- * (g).$$

A set theoretic function $D\colon E \to A$ extends to $D\colon \coprod_{x\in E} Z \to A$. D is a *Steinberg function* if $D\big((f)^- * (1-f)\big)=0$ for any $f\in C(E)^*$, $f\not\equiv 0, 1$. A Steinberg function induces a map $K_2\big(C(E)\big)\to A$.

Let R be the semilocal ring at $\{0, \infty\}$ on P^1_C, $I\subset R$ the radical. Using the group structure on $P^1-\{0,\infty\}$ one defines $(f)^- * (g)\in \coprod_{x\in C^*} Z$ for $f\in 1+I$, $g\in C(P^1)^*$. A function $D\colon C^*\to A$ is a relative Steinberg function if $D\big((f)^- * (1-f)\big)=0$ for $f\in 1+I$. Using work of Keune [10] one shows that a relative Steinberg symbol induces a map $K_2(R, I)\to A$.

The key transcendental object is the single valued(!) function

$$D(x) = \log|x|\cdot \arg(1-x) - \operatorname{Im}\int_0^x \log(1-t)\frac{dt}{t}$$

first discovered by D. Wigner. The functional properties of $D(x)$ seem unbelievably rich: ((i) and (ii) below are joint work with Wigner)

THEOREM. (i) $D(x) = -D(x^{-1}) = -D(1-x) = -D(\bar{x})$. $D(0)=D(1)=D(\infty)=0$.

(ii) *For* $g\in \mathrm{SL}_2(C)$, *let* $\bar{g}\in \mathrm{SL}_2(C)/B\cong P^1_C$. *Let* $\{\bar{g}_1, \ldots, \bar{g}_4\}$ *denote the cross ratio. Then* $D(\{\bar{g}_1, \ldots, \bar{g}_4\})$ *is a measurable* 3-*cocycle on* $\mathrm{SL}_2(C)$. *If* $V(\bar{g}_1, \ldots, \bar{g}_4)$ *denotes the volume of the geodesic tetrahedron in the Poincaré upper half space with vertices* $\bar{g}_i\in P^1$ *lying at* ∞, *then* $D(\{\bar{g}_1, \ldots, \bar{g}_4\}) = \pm\frac{2}{3}V(\bar{g}_1, \ldots, \bar{g}_4)$.

(iii) $D(x)$ *is a relative Steinberg function and so induces*

$$D\colon K_3(C) \to K_2(P^1_C, \{0, \infty\}) \to K_2(R, I) \to R.$$

(iv) *Write* $E=C^*/q^Z$ *with* $|q|<1$. *The series*

$$D_q(x) = \sum_{n=-\infty}^{\infty} D(xq^n)$$

converges. D_q *is a continuous Steinberg function on* E *and induces a map* $K_2(E)\to K_2\big(C(E)\big)\to R$.

Given a number field k and an embedding $\sigma\colon k\to C$ one gets $D_\sigma\colon K_3(k)\to R$. One builds in this way the Borel regulator for K_3. The function $J(x)=\log|x|\cdot\log|1-x|$ is also a relative Steinberg function, although the map on $K_2(R, I)$ factors

$$K_2(R, I) \xrightarrow{\text{tame}} \coprod_{x\in C^*} C^* \xrightarrow{\log|\,|\cdot\log|\,|} R$$

and hence is trivial on $K_3(C)$.

In the elliptic case define

$$J_q(x) = \sum_{n=0}^{\infty} J(xq^n) - \sum_{n=1}^{\infty} J(x^{-1}q^n).$$

Given divisors $(f)=\sum n_i(a_i)$, $(g)=\sum m_j(b_j)$ on E we can choose lifting $(\tilde{f})=\sum n_i(\alpha_i)$, $(\tilde{g})=\sum m_j(\beta_j)$ to divisors on C^* such that $\sum n_i=\sum m_j=0$, $\prod \alpha_i^{n_i}=\prod \beta_j^{m_j}=1$.

THEOREM. *The expression*

$$J_q\{f, g\} = J_q((\tilde{f})^- * (\tilde{g})) = \sum n_i m_j J_q(\alpha_i^{-1}\beta_j)$$

is well defined independent of the choices. Moreover,

$$J_q\{f, 1-f\} = 0$$

so there is an induced map $J_q\colon K_2(C(E))\to \mathbf{R}$.

Define, finally

$$R_q = J_q + iD_q\colon\ K_2(E) \to K_2(C(E)) \to \mathbf{C}.$$

Assume now E defined over an arbitrary field k of characteristic 0. Recall that the sequence

$$K_2(E) \to K_2(k(E)) \xrightarrow[\text{symbol}]{\text{tame}} \coprod_{x\in E} k(x)^*$$

is exact. To study the image of R_q we construct elements in Ker (tame) as follows: let f, g be functions on E and assume the divisors (f), (g) are supported on the points of order N of E. Assume for simplicity these points of order N are defined over k. Then there exist $c_i\in k^*, f_i\in k(E)^*$ such that

$$S_{f,g} = \{f, g\}^N \cdot \prod_i \{f_i, c_i\}\in \operatorname{Ker}(\text{tame}).$$

R_q is trivial on symbols with one entry constant, so when $k\hookrightarrow \mathbf{C}$, $R_q(S_{f,g})$ is well defined. Let ϱ have a pole of order 1 at every $x\in E_N$, $x\neq 0$, and a zero of order N^2-1 at 0. Let $x\in E_N$ and let f_x have a zero of order N at x and a pole of order N at 0. Define $S_x=S_{\varrho,f_x}$. If, for example, $E=\mathbf{C}/\mathbf{Z}+\mathbf{Z}\tau$, one finds

$$R_q(S_{(a+b\tau)/N}) = \frac{(\operatorname{Im}\tau)^2N^3}{\pi} \sum_{m,n=-\infty}^{\infty}{}' \frac{\sin(2\pi((an-bm)/N))}{(m+n\tau)^2(m+n\bar{\tau})}.$$

REMARKS. (i) The $S_{f,g}$ are analogous to cyclotomic units. They are available when certain torsion points of the curve are rational over k. I do not expect they generate Ker (tame) in general.

(ii) The techniques discussed in this report are *ad hoc*. One could try to give a general regulator construction by interpreting the higher K-groups of a variety as relative algebraic cycles, e.g. $K_1(C) \cong$ picard variety of P^1_C relative to $\{0, \infty\}$. The *Akel–Jacoby* map would associate to these cycle points in a relative *Griffiths intermediate jacobian.* Factoring out by the maximal compact subgroup of this torus yields invariants in a real vector space which frequently inherits a complex structure from Hodge theory.

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University of Chicago
Chicago, Illinois 60637, U.S.A.