Uncovering a New L-function

Andrew R. Booker

In March of this year, my student, Ce Bian, announced the computation of some "degree 3 transcendental L-functions" at a workshop[1] at the American Institute of Mathematics (AIM). This article aims to explain some of the motivation behind the workshop and why Bian's computations are striking. I begin with a brief background on L-functions and their applications; for a more thorough introduction, see the survey article by Iwaniec and Sarnak [2].

L-functions and the Selberg Class

There are many objects that go by the name of *L-function*, and it is difficult to pin down exactly what one is. In one of his last published papers [5], the late Fields medalist A. Selberg tried an axiomatic approach, basically by writing down the common properties of the known examples. This resulted in what is generally known as the "Selberg class". Before discussing the list of axioms, it is helpful to consider a few concrete examples.

The simplest and most familiar example of an L-function is the *Riemann ζ-function*,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

which encodes information about the prime numbers. Using either formula above, one can see that $\zeta(s)$ is an analytic function of complex numbers s with $\Re(s) > 1$. However, as discovered by Riemann, ζ has an analytic continuation to the entire complex plane, with the exception of a simple pole at $s = 1$. Moreover, it satisfies a "functional equation": If

$$\gamma(s) = \Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2) \text{ and } \Lambda(s) = \gamma(s)\zeta(s)$$

then

$$\Lambda(s) = \Lambda(1-s). \tag{1}$$

By manipulating the ζ-function using the tools of complex analysis (some of which were discovered in the process), one can deduce the famous Prime Number Theorem, that there are asymptotically about $\frac{x}{\log x}$ primes $p \le x$ as $x \to \infty$.

Other L-functions reveal more subtle properties. For instance, inserting a multiplicative character $\chi : (\mathbb{Z}/q\mathbb{Z})^{\times} \to \mathbb{C}$ in the definition of the ζ-function, we get the so-called *Dirichlet L-functions*,

$$L(s,\chi) = \prod_{p \text{ prime}} \frac{1}{1-\chi(p)p^{-s}}.$$

(Here we extend the definition of χ by setting $\chi(p) = 0$ when p divides q.) It turns out that these also continue to entire functions and satisfy a functional equation similar to (1): If

$$\gamma(s,\chi) = \begin{cases} \Gamma_{\mathbb{R}}(s) & \text{if } \chi(-1) = 1, \\ \Gamma_{\mathbb{R}}(s+1) & \text{if } \chi(-1) = -1 \end{cases} \quad \text{and}$$
$$\Lambda(s,\chi) = \gamma(s,\chi)L(s,\chi)$$

then

$$\Lambda(s,\chi) = \epsilon_{\chi} q^{1/2-s}\Lambda(1-s,\overline{\chi}), \tag{2}$$

where ϵ_{χ} is a certain constant of absolute value 1 and $\overline{\chi}$ is the conjugate character. The Dirichlet L-functions encode information about primes in arithmetic progressions, which is revealed by manipulations similar to those for the ζ-function; in particular, Dirichlet's theorem says that the primes distribute themselves evenly among the invertible residue classes modulo q.

Andrew R. Booker is senior lecturer of mathematics at the University of Bristol. His email address is `andrew.booker@bristol.ac.uk`.

[1] *Computing arithmetic spectra, March 9–14, 2008.*

Another example arises from arithmetic geometry. Given an elliptic curve E of the form $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Q}$, one can consider the number of solutions, $\#E(\mathbb{F}_p)$, to its defining equation reduced modulo prime numbers p. Based on the fact that about half of the numbers mod p are squares, a simple heuristic analysis suggests that there are roughly p such solutions, and indeed a theorem of Hasse implies the sharp bound $|p - \#E(\mathbb{F}_p)| < 2\sqrt{p}$. Thus, it is natural to consider the normalized quantity $\lambda(p) = \frac{p-\#E(\mathbb{F}_p)}{\sqrt{p}}$. (As in the case of the Dirichlet L-functions, there are finitely many primes for which the definition needs some adjustment, such as those dividing the denominators of a and b. The *conductor* of E is an integer N whose prime factorization contains all of these exceptional primes.) We can then associate an L-function, called a *Hasse-Weil L-function* in this case, given by the product

$$L(s, E) = \prod_{p \text{ prime}} \frac{1}{1 - \lambda(p)p^{-s} + \chi_N(p)p^{-2s}},$$

where $\chi_N(p) = 0$ if p divides N and 1 otherwise. Again from the definition and Hasse's bound we see that $L(s, E)$ is analytic for $\Re(s) > 1$. However, the Shimura-Taniyama-Weil conjecture, now a theorem of Wiles et al., implies that $L(s, E)$ continues to an entire function and satisfies a functional equation: If

$$\gamma(s, E) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) \quad \text{and}$$
$$\Lambda(s, E) = \gamma(s, E)L(s, E)$$

then

$$\Lambda(s, E) = \epsilon_E N^{1/2-s}\Lambda(s, E),$$

for a certain $\epsilon_E \in \{\pm 1\}$. As is now well known, the analytic properties of these L-functions lie at the heart of Wiles' proof of Fermat's Last Theorem.

So what do these examples have in common? One difficulty with Selberg's approach is that it is not obvious which properties should be considered intrinsic and which not, and there is no general agreement on that point. A few things are clear, however:

- An L-function should be given by an *Euler product*. In all known cases, it takes the form

$$L(s) = \prod_{p \text{ prime}} \frac{1}{f_p(p^{-s})},$$

where f_p is a polynomial of a fixed degree $r \geq 1$, with the exception of at most finitely many primes p, for which the degree can be smaller than usual. Moreover, the product should converge absolutely for s with $\Re(s) > 1$, and thus define an analytic function in that region.

- $L(s)$ should continue to an entire function, with the exception of at most finitely many poles on the line $\Re(s) = 1$. Moreover, the analytically continued function should satisfy a functional equation of the following form: If

$$\gamma(s) = \prod_{j=1}^{r} \Gamma_{\mathbb{R}}(s + \mu_j) \quad \text{and}$$
$$\Lambda(s) = \gamma(s)L(s),$$

for certain complex constants μ_j, then

$$\Lambda(s) = \epsilon N^{1/2-s}\overline{\Lambda(1-\bar{s})},$$

where N is a positive integer and ϵ is a constant of absolute value 1. (Note that the above examples are all of this form; in particular, $\Lambda(s, \overline{\chi}) = \overline{\Lambda(\bar{s}, \chi)}$.)

It is also expected that all of the functions in the Selberg class satisfy an analogue of the *Riemann hypothesis*, i.e., all zeros of the completed function $\Lambda(s)$ should have real part $\frac{1}{2}$. It would seem sensible to include that as an axiom, but for the fact that not a single instance of the Riemann hypothesis is yet known to be true!

Langlands' Philosophy

Another common feature of the examples above is that they are all generating functions for sequences that occur naturally in number theory. In order to extract the information that they contain, one needs to know the nice analytic properties (i.e., analytic continuation and functional equation) that the generating functions possess. However, as is apparent from the example of elliptic curves, those properties can be difficult to establish. Fortunately, we have a source of L-functions with good analytic properties, known as *automorphic forms* or *modular forms*. The problem is thus reduced to showing that an L-function of arithmetic interest is equal to one arising from an automorphic form; this is in fact what Wiles et al. proved in their resolution of the Shimura-Taniyama-Weil conjecture.

The study of automorphic forms is a discipline in its own right, called the *Langlands program*, after R. P. Langlands, who was arguably the first to understand the scope for applying them to number theory. In particular, Langlands predicted the existence of certain "functorial transfers" between different types of automorphic forms, which may be viewed as a (largely conjectural) set of rules governing the expected equalities of L-functions.

Maass Forms

After accounting for all of these equalities, it turns out that there are far more automorphic forms, each with an associated L-function, than there are L-functions that can be properly interpreted as generating functions. Thus, there are many so-called "transcendental L-functions", which are associated to automorphic forms but not obviously connected to number theory. The most classical examples are known as *Maass forms*, after H. Maass, who was the first to construct them. These are functions f on the hyperbolic upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$, equipped with the Riemannian metric $ds^2 = \frac{dx^2+dy^2}{y^2}$, which are modular, in the sense that

$$f\left(\frac{az+b}{cz+d}\right) = f(z) \tag{3}$$

for all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$, the group of orientation-preserving isometries of $\mathbb{H}$.

The prototypical case is $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, for which an even Maass form[2] has a Fourier series expansion of the type

$$f(z) = \sum_{n=1}^{\infty} \lambda(n)\sqrt{y}K_{ir}(2\pi ny)\cos(2\pi nx), \tag{4}$$

where r and $\lambda(n)$ are certain real constants and K_{ir} is the classical K-Bessel function. Thus, to describe a Maass form completely, one need only specify the numbers r and $\lambda(n)$. For any choice of these data, f is an eigenfunction of the hyperbolic Laplace operator $\Delta = \mathrm{div} \circ \mathrm{grad} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, of eigenvalue $\frac{1}{4} + r^2$. However, there is only a discrete set of values of r and $\lambda(n)$ for which (4) is invariant under $\mathrm{SL}(2,\mathbb{Z})$. Given such a form, the associated L-function is the series

$$L(s,f) = \sum_{n=1}^{\infty} \lambda(n)n^{-s},$$

which converges absolutely for $\Re(s) > 1$. However, as a consequence of the $\mathrm{SL}(2,\mathbb{Z})$-invariance of f, $L(s,f)$ continues to an entire function and satisfies the following functional equation: If

$$\gamma(s,f) = \Gamma_{\mathbb{R}}(s+ir)\Gamma_{\mathbb{R}}(s-ir) \quad \text{and}$$
$$\Lambda(s,f) = \gamma(s,f)L(s,f)$$

then

$$\Lambda(s,f) = \Lambda(1-s,f).$$

Moreover, by choosing a suitable basis for the $(\frac{1}{4}+r^2)$-eigenspace[3] of Δ, one can always ensure that the coefficients $\lambda(n)$ are multiplicative as a function of n, which is equivalent to the existence of an Euler product formula

$$L(s,f) = \prod_{p \text{ prime}} \frac{1}{1-\lambda(p)p^{-s}+p^{-2s}}.$$

Thus, these functions belong to the Selberg class.

With a few exceptions arising from instances of the Langlands functoriality conjectures, no explicit examples of Maass forms are known. In fact, the data associated to a typical Maass form are believed to be transcendental, which explains the terminology above; thus, in a sense one can never know a typical form exactly, and all constructions of them are approximate and numerical. (This is in contrast to the more classical *holomorphic modular forms*, which have algebraic data and can be described by explicit formulas.)

Applications

That begs the question why number theorists should be interested in these functions at all. The answer lies in the relatively recent realization that in order to deduce information about a single L-function (an algebraic one, say), it is often beneficial to embed it in a "family" of similar L-functions and study the whole set of them at once.[4] Many examples and applications of this notion are discussed in the survey paper by Michel [3]. I highlight two recent examples here.

- Hilbert's 11th problem asks which algebraic integers in a number field are represented by values of a fixed quadratic form (e.g., which ones are sums of three squares?). It was only recently solved completely by J. Cogdell, I. Piatetski-Shapiro, and P. Sarnak [1], utilizing the full spectral theory of Maass forms over number fields.
- The following problem was raised at the AIM workshop in March:

  Given a large number $X > 0$, how quickly can one determine the structure of the ideal class groups of the quadratic fields $\mathbb{Q}(\sqrt{d})$ for $0 < d < X$?

  If one is allowed to assume the truth of the (generalized) Riemann hypothesis then there are known algorithms for computing such class groups very quickly—in "essentially linear time" $O_\varepsilon(X^{1+\varepsilon})$ for each $\varepsilon > 0$. The catch is that one cannot be sure that the results of the computation are correct without the Riemann hypothesis. However, given a fast algorithm for computing the eigenvalues and Fourier coefficients of Maass forms, such as the one discussed below, it turns out that one can certify the results of the computation unconditionally, again in essentially linear time. This has been implemented in practice by M. Jacobson et al., and is currently the

[2] *Meaning even as a function of x. There are also odd forms, which have cosine replaced by sine in their Fourier expansions.*

[3] *The eigenspaces are conjectured to be simple, and all numerical evidence to date supports this conclusion. Thus, if f is normalized so that $\lambda(1) = 1$ then $\lambda(n)$ is automatically multiplicative.*

[4] *While the applications to L-functions are recent, this method of attack was already familiar to number theorists from Deligne's proof of the Weil conjectures, including a finite analogue of the Riemann hypothesis that remains some of our best evidence to date in favor of the version for L-functions.*

fastest known method of computing real quadratic class groups!

The Workshop

In recognition of the important role that computation plays concerning automorphic forms and L-functions, the AIM workshop in March was an attempt to unify and extend the known computational techniques. More precisely, it addressed the following question:

To what extent can we

(A) *compute the data (i.e., Laplacian eigenvalues and Fourier coefficients) of automorphic forms, and*

(B) *prove theorems about them?*

There are good reasons for separating the question into two parts; while there is a long history of computations of this type, most notably algorithms for Maass forms due to H. Stark and D. Hejhal, the issue of *rigorously proving* the correctness of the computations has only recently been addressed. Moreover, in many situations, such as when giving numerical evidence for a conjecture, rigorous results are not necessary and may even be overkill. With that in mind, the sort of answer that we seek to this question is again best illustrated by the case of $\mathrm{SL}(2,\mathbb{Z})$, for which we have the following:

(A) Some 50,000 values of r have been computed approximately (to 6 or 7 decimal place precision), with heuristic justifications of their correctness, including some very large values (which are computationally more difficult). This is work of H. Then, based on an algorithm of Hejhal.

(B) The first 2000 r-values (in increasing order of size) have been rigorously computed to better than 40 decimal place accuracy. The eigenspaces turn out to be simple, and for each one the first several Fourier coefficients have been rigorously computed. This is work in preparation with A. Strömbergsson, based on joint work with A. Venkatesh. Moreover, in this case we have the following theoretical result:

Theorem. *Given $\Lambda, D \geq 0$, there is an algorithm that will compute all discrete eigenvalues of Δ on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ in $[0,\Lambda]$ to within 10^{-D} in polynomial time in Λ, D. Up to standard heuristic assumptions (the simplicity of the spectrum, in particular) that can be checked at run time, this also yields the first several Fourier coefficients of each form.*

These results represent the best that one can hope for in terms of computation. However, it would be misleading to describe them as typical; nearly all success so far has been limited to the classical Maass forms and holomorphic modular forms. Indeed, "most" of the automorphic forms of interest have never been observed directly. I describe some of these more general forms in the next section.

Higher Degree Automorphic Forms

An alternative way to realize the upper half plane is as the quotient of its isometry group by the stabilizer of any point, i.e., $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$. This, in turn, is isomorphic to the projective quotient $\mathrm{GL}(2,\mathbb{R})/\mathrm{O}(2,\mathbb{R})\cdot\mathbb{R}^\times$, and any element of the latter group has (by the *Iwasawa decomposition*) a unique representative of the form $\left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right)$ with $y > 0$, which is associated via these identifications to the element $x + iy \in \mathbb{H}$. Expressing the elements in this form has the advantage that the linear fractional transformations in (3) turn into left-multiplication by group elements.

Another advantage is that it is now clear how to generalize the upper half plane and automorphic forms to higher-dimensional spaces; one simply replaces $\mathrm{SL}(2,\mathbb{R})$ by a Lie group, $\mathrm{SO}(2,\mathbb{R})$ by a maximal compact subgroup, and Γ by a co-finite (with respect to Haar measure) discrete subgroup. For example, "degree 3 hyperbolic space" is the quotient

$$\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R}) \cong \mathrm{GL}(3,\mathbb{R})/\mathrm{O}(3,\mathbb{R})\cdot\mathbb{R}^\times,$$

and the Iwasawa decomposition in this case says that any element has a unique representative of the form $z = xy$, where

$$x = \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1\end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix},$$

with $x_{12}, x_{13}, x_{23}, y_1, y_2 \in \mathbb{R}$ and $y_1, y_2 > 0$. Taking $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ gives rise to degree 3 automorphic forms, which are functions on the double coset space $\mathrm{SL}(3,\mathbb{Z})\backslash\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$. Such forms have a Fourier expansion akin to (4), but with a two-parameter set of coefficients, $\lambda(n,m) \in \mathbb{C}$:

(5)

$$f(z) = \sum_{g\in\Gamma^2_\infty\backslash\Gamma^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\lambda(n,m)}{nm}W_{u,v}\left(\begin{pmatrix} nm & & \\ & m & \\ & & 1\end{pmatrix} gz\right).$$

Here Γ^2 consists of all integer matrices of the form $\left(\begin{smallmatrix} a & b & \\ c & d & \\ & & 1\end{smallmatrix}\right)$ with determinant ± 1, $\Gamma^2_\infty \subset \Gamma^2$ is the subgroup of unipotent ones, and $W_{u,v}$ is "Jacquet's Whittaker function". The latter is a suitable replacement for the classical K-Bessel function and cosine from (4); in degree 3 it has two parameters, $u, v \in \mathbb{R}$, which are analogous to the parameter r.

The associated L-function in this case is given by the Dirichlet series

$$L(s,f) = \sum_{n=1}^{\infty}\lambda(n,1)n^{-s},$$

converging absolutely for $\Re(s) > 1$. Again it is always possible to choose the form so that we have an Euler product formula, this time of the form

(6)
$$L(s,f)=\prod_{p \text{ prime}}\frac{1}{1-\lambda(p,1)p^{-s}+\overline{\lambda(p,1)}p^{-2s}-p^{-3s}}.$$

Moreover, it turns out that f is determined by its L-function, in the following sense. Let $\tilde{f}$ be the "dual" form, for which (u,v) is replaced by (v,u) and the $\lambda(n,m)$ by their complex conjugates. Then we have the following identity, due to Bump:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\lambda(n,m)}{n^{s_1}m^{s_2}}=\frac{L(s_1,f)L(s_2,\tilde{f})}{\zeta(s_1+s_2)}.$$

In particular, this shows that all $\lambda(n,m)$ are determined by the 1-dimensional sequence $\lambda(n,1)$. Thus, working with the L-function eliminates some of the redundancy that is present in the Fourier expansion (5).

The Converse Theorem

One can also define so-called "twisted" L-functions, as follows. Let χ be a Dirichlet character of conductor q. Then we define $L(s,f\times\chi)=\sum_{n=1}^{\infty}\lambda(n,1)\chi(n)n^{-s}$. Further, if χ has parity $a\in\{0,1\}$, meaning $\chi(-1)=(-1)^a$, then we set

$$\gamma(s,u,v,\chi)=\Gamma_{\mathbb{R}}\left(s+a-i\frac{2u+v}{3}\right)$$
$$\Gamma_{\mathbb{R}}\left(s+a+i\frac{u-v}{3}\right)\Gamma_{\mathbb{R}}\left(s+a+i\frac{u+2v}{3}\right)$$

and $\Lambda(s,f\times\chi)=\gamma(s,u,v,\chi)L(s,f\times\chi)$.

It follows from the fact that f is an automorphic form that all of the twisted L-functions have nice analytic properties. In particular, $\Lambda(s,f\times\chi)$ has analytic continuation to an entire function and satisfies the following functional equation relating f to its dual and χ to its conjugate:

(7) $$\Lambda(s,f\times\chi)=\epsilon_\chi^3 q^{3(1/2-s)}\Lambda(1-s,\tilde{f}\times\overline{\chi}),$$

where ϵ_χ is the factor from (2).

Remarkably, these analytic properties actually *characterize* the degree 3 automorphic forms; precisely, we have the follow result:

Theorem (Jacquet, Piatetski-Shapiro, Shalika). *Suppose $L(s,f)$ is an Euler product of the form given in* (6) *such that all complete twisted L-functions $\Lambda(s,f\times\chi)$ extend to entire functions of finite order and satisfy* (7). *Then $L(s,f)$ is the L-function of a degree 3 automorphic form.*

Similar results are known for automorphic forms of arbitrary degree, although the set of objects that one must twist by grows with the degree. (For instance, for degree 4 one has to twist by all degree 2 automorphic forms, including the Maass forms.) Collectively these results are known simply as the "converse theorem". They give strong support to Langlands' philosophy that automorphic forms are the right source for L-functions with nice analytic properties, and they have found many applications in the Langlands program. For instance, the theorem above is the key point in the proof of one of the first cases of Langlands' conjectures to be confirmed, called the *Gelbart-Jacquet lift*; given a Maass form with parameters r and $\lambda(p)$, it associates a degree 3 automorphic form with parameters $u=v=2r$ and $\lambda(p,1)=\lambda(p)^2-1$. This application in turn motivated the proof of the theorem.

Bian's Computations

Besides its theoretical uses, the converse theorem also points to a method for computing automorphic forms, as follows. For his computations, Bian considered the following smooth sums, which are linear functionals on a sequence $\lambda(n)$ of complex numbers, with parameters $u,v,X>0$ and χ a Dirichlet character:

(8) $$S(\{\lambda(n)\},u,v,X,\chi)=\frac{1}{\sqrt{X}}\sum_{n=1}^{\infty}\lambda(n)\chi(n)F_{u,v}(n/X,\chi),$$

where $F_{u,v}(y,\chi)=\frac{1}{2\pi i}\int_{\Re(s)=1}\gamma(s,u,v,\chi)y^{-s}\,ds$. This F is related to Jacquet's Whittaker function $W_{u,v}$, and it has properties similar to those of the K-Bessel function, i.e., it oscillates for small y, but eventually settles down (at a point depending on u and v) and tends rapidly to 0 as $y\to\infty$. Thus, with the cost of a small error, the series in (8) can be truncated at a point roughly proportional to X. If one imagines choosing the $\lambda(n)$ randomly from a fixed distribution of mean 0, then the sum is the result of taking a random walk of length X in the complex plane; the central limit theorem predicts that such a sum typically has size on the order of $\sqrt{X}$. Thus, for a random choice of the $\lambda(n)$'s, S should typically be of size 1.

However, if the $\lambda(n)$ happen to be the coefficients of the L-function of an automorphic form, then S has a very different behavior. Precisely, the analytic properties and functional equation (7) of the twisted L-functions are equivalent, by Mellin inversion, to the identity

(9) $$S(\{\lambda(n,1)\},u,v,X,\chi)=\epsilon_\chi^3\overline{S(\{\lambda(n,1)\},u,v,q^3/X,\chi)}.$$

For X much larger than q^3, the right-hand side is a short sum, and thus S is very small. Moreover, (9) gives a linear equation relating the real and imaginary parts of the $\lambda(n,1)$, which can be tested for any X at least as large as the point of symmetry, $q^{3/2}$. In particular, if we consider only the central point $X=q^{3/2}$ for every Dirichlet character χ of conductor $q\le Q$, then we get a system of equations involving roughly $Q^{3/2}$ unknowns. The key point is that there are asymptotically about $\frac{18}{\pi^4}Q^2$ such characters as $Q \to \infty$. Thus, if Q is large enough then we will have an overdetermined system, and for a given choice of parameters u and v we can test the consistency of the system of equations by computing the least squares solution.

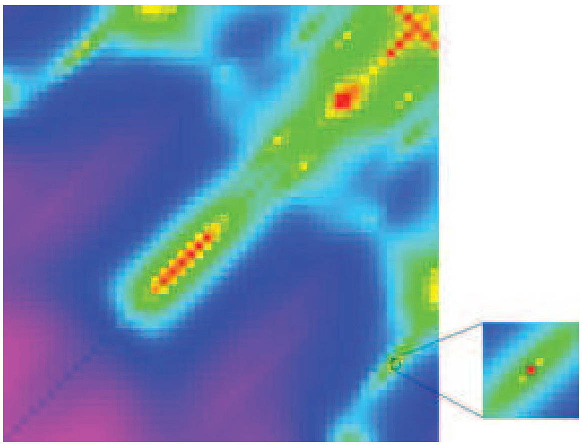

Figure 1. Bian's initial scan and first example.

Bian's computations followed this approach, though he used more values of X in order to reduce Q and relied on other known data about the possible locations of (u, v). In particular, using a completely different method, S. Miller [4], who was the first to prove existence of the degree 3 forms in question (non-constructively), had earlier ruled out values of u and v that are both smaller than about 10. We also know the first several Gelbart-Jacquet lifts, which occur on the line $u = v$; the first has $u = v = 19.06739\ldots$. Thus, Bian elected to search the square region with $10 \le u, v \le 20$. With parameters of that size, it turns out that for each choice of u and v one ends up with a (non-sparse) system of equations in about 10,000 real variables. One important practical point is that solving such systems is now well within the capabilities of a standard desktop PC; it seems unlikely that it could have been done a decade ago[5].

Above is an image of Bian's initial scan, which contains about 2,500 sample points. The "hot" areas indicate places where the system of equations is close to consistent. The inset image shows a zoomed and rescaled version around the warm point near the lower right-hand corner, which was Bian's first example. This indeed turns out to correspond to an automorphic form, with parameters $(u, v) \approx (18.902415, 11.761250)$, as do three other points that Bian zoomed in on before the workshop[6].

Checking the Results

One of the ironies of our lack of concrete examples of higher degree automorphic forms is that we have many conjectures about objects that have never been observed. However, when a purported example presents itself, these conjectures make it easy to tell whether the example is genuine or not.

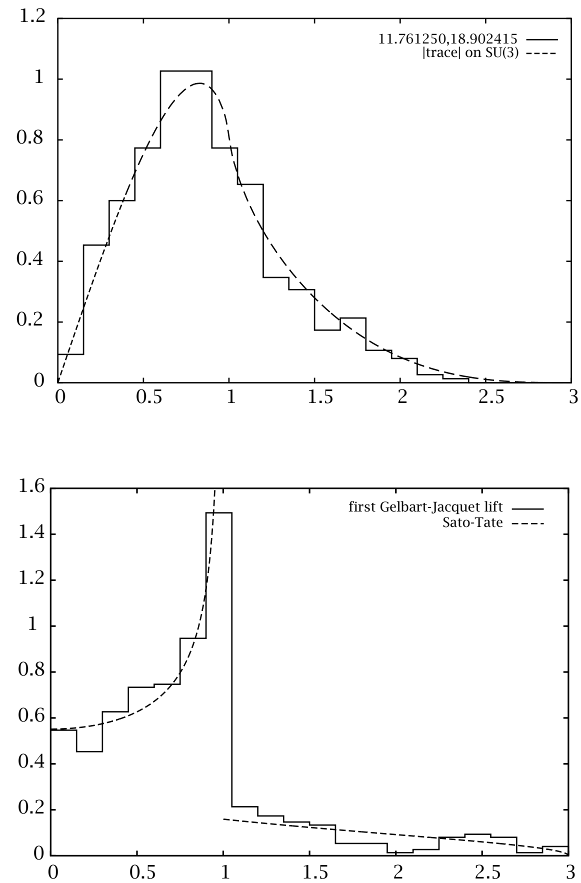

Figure 2. First 500 $|\lambda(p, 1)|$ for Bian's first example vs. distribution of absolute trace on $SU(3, \mathbb{C})$ **(top) and for first Gelbart-Jacquet lift vs. Sato-Tate distribution (bottom).**

Firstly, as in the case of Maass forms for $SL(2, \mathbb{Z})$, the spectrum for these degree 3 forms is thought to be simple, meaning that there is at most one form for each pair (u, v). One consequence is that the Dirichlet coefficients of any form will automatically be multiplicative; in particular, all $\lambda(n, 1)$ are determined by the ones for n prime.

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[5] *However, after seeing Bian's results at the workshop, David Farmer, Sally Koutsoliotas, and Stefan Lemurell presented a similar method that seems to detect degree 3 forms with substantially smaller systems of equations. It remains to be seen which method, or combination of the two, will be preferable in the long run.*

[6] *The very hot points on or near the $u = v$ line arise from another image of degree 2 forms, known as* Eisenstein series*; like the Gelbart-Jacquet lifts, these are well understood. The points of greatest interest lie off of the line of symmetry.*

This turns out to be true (numerically, to several decimal places) for Bian's examples, a fact that was never imposed in the scanning process, which used only linear algebra.

A second, more thorough test is of the distribution of $\lambda(p,1)$ for prime p. The Langlands conjectures imply that as $p \to \infty$, $\lambda(p,1)$ has the same distribution as that of the trace of matrices chosen randomly from $SU(3,\mathbb{C})$ according to Haar measure. The top portion of Figure 2 compares a histogram of values of $|\lambda(p,1)|$ from Bian's first example, for the first 500 primes, against the distribution of the absolute trace on $SU(3,\mathbb{C})$. The bottom shows a similar picture for the first Gelbart-Jacquet lift. The difference in behavior of these two examples is striking, but easy to explain; the Fourier coefficients of the lifted form are determined by those of the underlying Maass form, and hence their distribution is a distorted version of the distribution of the trace on $SU(2,\mathbb{C})$, the so-called *Sato-Tate distribution*, rather than that of a "generic" degree 3 automorphic form.

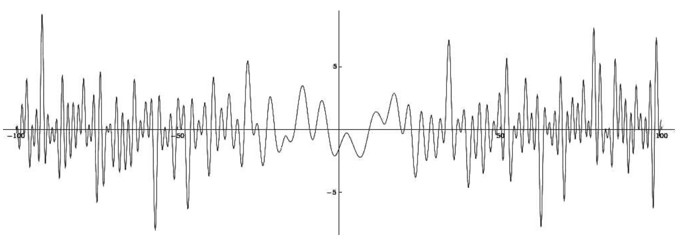

Figure 3. Plot of $L(s,f)$ along $\Re(s) = \frac{1}{2}$.

Yet a third prediction is the Riemann hypothesis. In an impressive display of computational prowess, M. Rubinstein tested this in real time at the workshop using Bian's data, as Bian and I were speaking. Figure 3 shows a graph of $L(\frac{1}{2}+it,f)$ for Bian's first example, with the phase divided out to make it real-valued, as computed by Rubinstein. The picture confirms that the first several zeros are in the expected location.

Given that Bian's examples pass all of these tests, there is very little room for doubt that he has computed genuine degree 3 automorphic forms. Nevertheless, there is still no *proof* of this. In essence, the computation of degree 3 forms has now reached the point where that of degree 2 Maass forms stood for over twenty years. The rigorous verifications and passage to degree 4 and higher should keep us busy for a few more!

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