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On the Chebyshev polynomials

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ABSTRACT

We put special attention in this paper on the Chebyshev polynomials of the fourth kind because they are much less known and less studied than others. The representation problem of analytic functions in series of such polynomials is considered, and the important role of the Chebyshev functions of the second kind in solving them is emphasized. For analytic functions, the remainder term of Gauss quadrature rules can be represented as a contour integral with a complex kernel function. The kernel function related to the Gauss quadrature for Chebyshev polynomials of the fourth kind is especially studied on elliptic contours and the points of its maximum are specified.

Keywords: Chebyshev polynomial; function of second kind; Gauss quadrature; quadrature error estimates; series expansions.

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PRELIMINARIES

Definitions and elementary properties of the different Chebyshev polynomials are given and the corresponding functions of the second kind are discussed.

Definitions and some elementary properties

The orthogonal polynomials, called Chebyshev polynomials of the third and fourth kind (Gautschi 1992), are less well-known than the traditional first and second kind polynomials introduced as

$$T_n(x) = \cos n\theta \quad \text{and} \quad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \tag{1.1}$$

where $x = \cos\theta$, $\theta \in [0, \pi]$ (Szegö 1975, Rivlin 1990). As a consequence of (1.1) and Euler's formulas, the Chebyshev polynomials of the first and second kind can be defined by the equalities (Szegö 1975)

$$T_n(x) = \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right),$$
$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}}. \tag{1.2}$$

Some basic properties of all four kinds of Chebyshev polynomials with respect to the weight functions $(1+x)^\alpha(1-x)^\beta$ with $\alpha = \pm 1/2$ and $\beta = \pm 1/2$ are summarized in the literature (Mason 1993). In the paper cited, it is shown how well-known properties of first kind Chebyshev polynomials could be extended also to the second, third and fourth kind polynomials.

The Chebyshev polynomials $V_n(x)$ and $W_n(x)$ of the third and fourth kind, respectively, are defined on the interval $[-1, 1]$ by the formulas

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2}} \quad \text{and} \quad W_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}, \tag{1.3}$$

where $x = \cos\theta$, $\theta \in [0, \pi]$. The polynomials (1.1) are special cases of the Jacobi polynomials and contain only even or only odd powers of x according to whether n is even or odd. Thus the polynomials (1.3) are cosine polynomials in θ of degree n. These polynomials are orthogonal with respect to the non-symmetric weight functions $w_3(x) = (1+x)^{1/2}(1-x)^{-1/2}$ and $w_4(x) = (1-x)^{1/2}(1+x)^{-1/2}$, respectively, on the interval $[-1, 1]$. The orthogonal properties of these polynomials are as follows:

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}W_n(x)W_m(x)dx = \begin{cases} 0, n \neq m \\ \pi, n = m \end{cases}, \tag{1.4}$$

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}V_n(x)V_m(x)dx = \begin{cases} 0, n \neq m \\ \pi, n = m \end{cases}. \tag{1.5}$$

Many identities involving Chebyshev polynomials are paraphrases of well-known trigonometric identities. For example, the simple trigonometric formulas

$$\cos\frac{2n+1}{2}\theta\cdot\cos\theta = \frac{1}{2}\left(\cos\frac{2n+3}{2}\theta + \cos\frac{2n-1}{2}\theta\right),$$
$$\sin\frac{2n+1}{2}\theta\cdot\sin\theta = \frac{1}{2}\left(\sin\frac{2n+3}{2}\theta + \sin\frac{2n-1}{2}\theta\right)$$

lead to the recurrence relation

$$Y_{n+1}(x) = 2xY_n(x) - Y_{n-1}, \quad n = 1, 2, 3, \ldots, \tag{1.6}$$

and all Chebyshev polynomials including those of the third and fourth kind satisfy this recurrence relation. In particular, from (1.3) and (1.6), it is clear that

$$V_0(x) = 1,\ \ V_1(x) = 2x - 1,\ V_2(x) = 4x^2 - 2x - 1, \dots,$$
$$W_0(x) = 1,\ \ W_1(x) = 2x + 1,\ \ W_2(x) = 4x^2 + 2x - 1, \dots,$$

and hence, the leading coefficient of $W_n(x)$ is 2^n. In fact it is seen that the Chebyshev polynomials of the third and fourth kind are related by

$$W_n(x) = (-1)^n V_n(-x). \tag{1.7}$$

Hence, it is normally sufficient to establish properties for fourth kind Chebyshev polynomials and to deduce analogous properties for the third kind Chebyshev polynomials using essentially (1.7). Our subsequent considerations in this paper require us to point out the existing relation between the Chebyshev polynomials of the second and fourth kind (Mason 1993). If

$$u = \sqrt{\frac{1+x}{2}},\ \ x = \cos\theta,\ \ \theta \in [0, \pi],$$

then from (1.1) and (1.3) it follows immediately that

$$W_n(x) = \frac{\sin((2n+1)\arccos u)}{\sin\frac{\theta}{2}} = U_{2n}(u), \tag{1.8}$$

and likewise the relation

$$V_n(x) = u^{-1} T_{2n+1}(u) \tag{1.9}$$

holds.

The zeros of the Chebyshev polynomials are real, distinct and located in the interval $(-1, 1)$, and for $T_n(x)$ and $U_n(x)$ are given by

$$\xi_k^{(n)} = \cos\frac{2k-1}{2n}\pi \quad \text{and} \quad \eta_k^{(n)} = \cos\frac{k}{n+1}\pi,\ \ k = 1, 2, \dots, n,$$

respectively. By means of formulas (1.7) and (1.9) it is easy to conclude that the zeros of $V_n(x)$ and $W_n(x)$ are given by

$$y_k^{(n)} = \cos\frac{2k-1}{2n+1}\pi \quad \text{and} \quad x_k^{(n)} = -\cos\frac{2k-1}{2n+1}\pi,\ \ k = 1, 2, \dots, n,$$

respectively.

Functions of second kind

The recurrence Eq. (1.6), with respect to the Chebyshev polynomials of the fourth kind, has another solution given by the formula

$$Q_n(W; z) = \int_{-1}^{1} \frac{w_4(x) W_n(x)}{z - x}\, dx,\ \ n = 0, 1, 2, \dots. \tag{1.10}$$

These functions are holomorphic in the region $\mathbb{C} \setminus [-1, 1]$ and known as functions of the second kind. Indeed, in view of the above recurrence equation and the orthogonal property (1.4), one can show that for $n \geq 1$ and $z \in \mathbb{C} \setminus [-1, 1]$,

$$\frac{1}{2} Q_{n+1}(W; z) - z Q_n(W; z) + \frac{1}{2} Q_{n-1}(W; z) = -\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} W_n(x) dx = 0.$$

Each Chebyshev system of functions of the second kind satisfies the recurrence relation (1.6) of the orthogonal polynomials. As paraphrases of trigonometric integrals, certain formulas (Bateman & Erdelyi 1953 [10.11:(47) and (48)]) mean in the context of the definition (1.10) that the Chebyshev polynomials can be treated as functions of the second kind. It is not difficult to establish a similar relationship between the functions (1.10) and the Chebyshev polynomials of the third kind. Let us set

$$x = \cos\theta, \; u = \sqrt{\frac{1+x}{2}} \quad \text{and} \quad y = \cos\varphi, \; \upsilon = \sqrt{\frac{1+y}{2}}.$$

Then according to (1.8) and (1.10),

$$\begin{aligned} Q_n(W; y) &= \frac{1}{\upsilon} \int_0^1 \sqrt{1-u^2}\, U_{2n}(u) \left(\frac{1}{u+\upsilon} - \frac{1}{u-\upsilon} \right) du \\ &= \frac{1}{2\upsilon} \int_{-1}^{1} \sqrt{1-u^2} \frac{U_{2n}(u)}{u+\upsilon} du - \frac{1}{2\upsilon} \int_{-1}^{1} \sqrt{1-u^2} \frac{U_{2n}(u)}{u-\upsilon} du. \end{aligned}$$

By applying a formula (Bateman & Erdelyi 1953 [10.11:(48)]) we get the representation

$$Q_n(W; y) = \pi \upsilon^{-1} T_{2n+1}(\upsilon),$$

which because of (1.9) becomes

$$Q_n(W; y) = \pi V_n(y).$$

Finally, we note that formula (1.7) implies the equality

$$Q_n(V; z) = (-1)^{n+1} Q_n(W; -z), \tag{1.11}$$

where the function of the second kind $Q_n(V; z)$ related to $w_3(x)$ is defined likewise as in (1.10)

SERIES REPRESENTATIONS OF ANALYTIC FUNCTIONS

The extremal properties of the Chebyshev polynomials are essentially used in the expansion theory of real-valued functions in series of these polynomials. Particular attention is paid to the rate of convergence of the partial sums of the Chebyshev expansions (Szegö 1975, Suetin 1976, Rivlin 1990). Our principal concern in this section is the representation problem of holomorphic functions in series of the

Chebyshev polynomials of the fourth kind. We employ some methods (Rusev 1984) to emphasize the important role that the functions of the second kind (1.10) play in series expansion of analytic functions.

Christoffel–Darboux formula

For further consideration we will prove the validity of the Christoffel–Darboux formula related to Chebyshev polynomials of the fourth kind and functions of the second kind.

Lemma 2.1. *If $z \in \mathbb{C}$, $\zeta \in \mathbb{C} \setminus [-1, 1]$ and $z \neq \zeta$, then*

$$\frac{1}{\zeta - z} = \frac{1}{\pi} \sum_{n=0}^{\nu} W_n(z) Q_n(\zeta) + \frac{\Delta_\nu(z, \zeta)}{\zeta - z}, \tag{2.1}$$

where

$$\Delta_\nu(z, \zeta) = \left(-\frac{1}{2\pi}\right)(W_\nu(z) Q_{\nu+1}(\zeta) - W_{\nu+1}(z) Q_\nu(\zeta)).$$

Proof. As we have already seen, the system of functions (1.10) is also a solution of (1.6) and therefore for $\zeta \in \mathbb{C} \setminus [-1, 1]$ and $n \geq 1$ we can write

$$\left(-\frac{1}{2\pi}\right) Q_{n+1}(W; \zeta) + \frac{1}{\pi} \zeta Q_n(W; \zeta) + \left(-\frac{1}{2\pi}\right) Q_{n-1}(W; \zeta) = 0. \tag{2.2}$$

If we define

$$\Delta_n(z, \zeta) = \left(-\frac{1}{2\pi}\right)(W_n(z) Q_{n+1}(W; \zeta) - W_{n+1}(z) Q_n(W; \zeta)), \ n = 0, 1, 2, \ldots,$$

then relation (2.2) leads further to

$$\Delta_n(z, \zeta) + \frac{1}{\pi}(\zeta - z) W_n(z) Q_n(W; \zeta) - \Delta_{n-1}(z, \zeta) = 0, \ n = 1, 2, 3 \ldots .$$

Then if $\nu \geq 0$ is an integer, it follows that

$$\Delta_\nu(z, \zeta) + \frac{1}{\pi}(\zeta - z) \sum_{n=0}^{\nu} W_n(z) Q_n(W; \zeta) = \Delta(z, \zeta),$$

where $\Delta(x, \zeta) = \Delta_0(z, \zeta) + 1/\pi(\zeta - z) W_0(z) Q_0(W; z)$. In fact $\Delta(z, \zeta) \equiv 1$.

Indeed, inserting $W_0(z) = 1$ and $W_1(z) = 2z + 1$ into the definition (1.10) we obtain

$$\begin{aligned}\Delta(z, \zeta) &= \left(-\frac{1}{2\pi}\right)[Q_1(W; \zeta) - W_1(z) Q_0(W; \zeta)] + \frac{1}{\pi}(\zeta - z) Q_0(W; \zeta) \\ &= \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \left(\frac{z - x}{\zeta - x} + \frac{\zeta - z}{\zeta - x}\right) dx = \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} (W_0(x))^2 dx = 1.\end{aligned}$$

Asymptotic properties

In general there are two essentially different types of asymptotic formulas for the orthogonal polynomials depending on whether the independent variable lies in the interval of the orthogonality or it is outside of it. In view of the main goal in this section, we need information about the asymptotic behaviour as $n \to \infty$ of the Chebyshev polynomials of the fourth kind only outside of the interval $[-1, 1]$, i.e., in the region $\mathbb{C} \setminus [-1, 1]$. Since the Chebyshev polynomials are special cases of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $\alpha = \pm\frac{1}{2}$, $\beta = \pm\frac{1}{2}$, the corresponding asymptotic formula we need occurs as particular cases of the general asymptotic formulas for the Jacobi polynomials and their functions of second kind $Q_n^{(\alpha,\beta)}(x)$ (Szegö 1975 [(8.21.9) and (8.71.19)]).

Let $u = u(z)$ be that inverse of Joukowski transformation $z = \frac{1}{2}(u + u^{-1})$, for which $|u(z)| > 1$.

Theorem 2.2. *Let α and β be arbitrary real numbers. There exists a sequence of analytic functions $\varphi_\nu(u) = \varphi_\nu(\alpha, \beta; u)$ which are real for real z and regular for $|u| > 1$ and $|u| = 1$, $u \neq \pm 1$, such that*

$$u^{-n} P_n^{(\alpha,\beta)}(z) = \sum_{\nu=0}^{p-1} \varphi_\nu(u) n^{-\nu-\frac{1}{2}} + O(n^{-p-\frac{1}{2}})(n \to \infty)$$

uniformly for $|u| \geq R$, $R > 1$.

Corollary 2.3. *If $z \in \mathbb{C}[-1, 1]$ and $n \geq 1$, the following asymptotic formula holds*

$$W_n(z) = A(z) n^{-\frac{1}{2}} [u(z)]^n \{1 + p_n(z)\}, \tag{2.3}$$

where $A(z) \neq 0$, $\{p_n(z)\}_{n \geq 0}$ are holomorphic functions in $\mathbb{C} \setminus [-1, 1]$ and $p_n(z)$ tends uniformly to zero on every compact subset of this region as $n \to \infty$.

When $n \to \infty$ and z is bounded, the asymptotic behaviour of the Jacobi functions of the second kind is in some sense "reciprocal" to that of the polynomials.

Theorem 2.4. *Let α and β be arbitrary real numbers such that $\alpha, \beta, \alpha + \beta + 1 \neq -1, -2, \ldots$. Then if $z \in \mathbb{C} \setminus [-1, 1]$ and $n \geq 1$, the following asymptotic formula holds*

$$Q^{(\alpha,\beta)}(z) = Q^{(\alpha,\beta)}(z) n^{-\frac{1}{2}} [u(z)]^{-n-1} \{1 + q_n^{(\alpha,\beta)}(z)\},$$

where $Q^{(\alpha,\beta)}(z) \neq 0$, $\{q_n^{(\alpha,\beta)}(z)\}_{n \geq 0}$ are holomorphic in the region $\mathbb{C} \setminus [-1, 1]$ and $q_n^{(\alpha,\beta)}(z)$ tends uniformly to zero on every compact subset of this region as $n \to \infty$.

Corollary 2.5. *If $z \in \mathbb{C} \setminus [-1, 1]$ and $n \geq 1$, the following asymptotic formula holds*

$$Q_n(W; z) = B(z) n^{\frac{1}{2}} [u(z)]^{-n-1} \{1 + q_n(z)\}, \tag{2.4}$$

where $B(z) \neq 0$, $\{q_n(z)\}_{n \geq 0}$ are holomorphic in $z \in \mathbb{C} \setminus [-1, 1]$ and $q_n(z)$ tends uniformly to zero on every compact subset of this region as $n \to \infty$.

Series convergence

The asymptotic formulas (2.3) and (2.4) provide complete information about the regions and mode of convergence of series in the Chebyshev polynomials of the fourth kind, as well as of the series in the corresponding functions of the second kind. If $1 < \rho < \infty$ let us denote by Γ_ρ the image of the circle $|u| = \rho$ by the Joukowski transformation $z = \frac{1}{2}(u + u^{-1})$ i.e., $\Gamma_\rho = \{z \in \mathbb{C} : |u(z)| = \rho\}$ where $u = u(z)$ is the inverse of the Joukowski function for which $|u(z)| > 1$, $z \in \mathbb{C} \setminus [-1, 1]$. It is clear that Γ_ρ is an ellipse with focuses at the ± 1 and with the sum of its semiaxis equal to ρ. We use the abbreviations

$$E(\rho) := \operatorname{int} \Gamma_\rho \; (1 < \rho < \infty) \quad \text{and} \quad E^*(\rho) = \mathbb{C} \setminus \overline{E(\rho)}.$$

The asymptotic formulas (2.3) and (2.4) lead to the following analogues of the well-known Abel's lemma and Cauchy–Hadamard's formula in the power series theory.

Theorem 2.6. *(a) Abel's lemma: If the series*

$$\sum_{n=0}^{\infty} a_n W - n(z) \tag{2.5}$$

is convergent at a point $z_0 \in \mathbb{C} \setminus [-1, 1]$ then it is absolutely uniformly convergent on every compact subset of $E(\rho)$, where $\rho = |u(z_0)|$.

(b) Cauchy–Hadamard's formula: If

$$R = \max\{1, [\limsup_{n\to\infty} \sqrt[n]{|a_n|}]^{-1}\},$$

then the series (2.5) *is absolutely uniformly convergent on every compact subset of $E(R)$ and diverges in $E^*(R) \setminus \{\infty\}$.*

Theorem 2.7. *(a) Abel's lemma: If the series*

$$\sum_{n=0}^{\infty} b_n Q_n(z) \tag{2.6}$$

is convergent at a point $z_0 \in \mathbb{C} \setminus [-1, 1]$, then it is absolutely uniformly convergent on every closed subset of $E^*(\rho)$, where $\rho = |u(z_0)|$.

(b) Cauchy–Hadamard's formula: If

$$R = \max\{1, [\limsup_{n\to\infty} \sqrt[n]{|b_n|}]\}$$

then the series (2.6) *is absolutely uniformly convergent on every closed subset of $E^*(R)$ and diverges in $E(R) \setminus [-1, 1]$.*

Series expansions of analytic functions

By means of the asymptotic properties and the Christoffel–Darboux formula for the Chebyshev polynomials of the fourth kind and the associated functions of the second kind, we are able to solve the representation problem of analytic functions in series of the form (2.5) and (2.6). We start with two auxiliary statements related to complex functions having Cauchy-type integral representation.

Lemma 2.8. *Let* $1 < \rho < \infty$ *and* φ *be a complex function absolutely integrable in the ellipse* Γ_ρ. *Then the function*

$$\Phi(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\varphi(\zeta)}{\zeta - z}\,d\zeta, \quad z \in \mathbb{C} \setminus \Gamma_\rho, \tag{2.7}$$

can be represented in $E(\rho)$ *by a series* (2.5) *with coefficients*

$$a_n = \frac{1}{2\pi^2 i}\int_{\Gamma_\rho} \varphi(\zeta)Q_n(\zeta)d\zeta, \quad n \geq 0. \tag{2.8}$$

Proof. Let us multiply the Christoffel–Darboux formula (2.1) by $(2\pi i)^{-1}\varphi(\zeta)$ and integrate along Γ_ρ. Then the Christoffel–Darboux formula (2.1) provides that for every $z \in \mathbb{C} \setminus \Gamma_\rho$,

$$\Phi(z) = S_\nu(z) + R_\nu(z), \tag{2.9}$$

where

$$S_\nu(z) = \sum_{n=0}^{\nu} a_n W_n(z)$$

with coefficients a_n given by (2.8) and

$$R_\nu(z) + \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\varphi(\zeta)\Delta_\nu(z,\zeta)}{\zeta - z}\,d\zeta.$$

If $z \in E(\rho) \setminus [-1, 1]$, the asymptotic formulas (2.3) and (2.4) as well as Stirling's formula yield that

$$|R_\nu(z)| = O\left(\rho^{-\nu}|u(z)|^\nu \int_{\Gamma_\rho} |\varphi(\zeta)|ds\right)(\nu \to \infty).$$

But if $z \in E(\rho) \setminus [-1, 1]$, then $|u(z)| < \rho$ and hence, $\lim_{\nu\to\infty} R_\nu(z) = 0$. From (2.9) we get the desired representation of the function (2.7) in the region $E(\rho) \setminus [-1, 1]$ in a series of the kind (2.5) with coefficients defined by formula (2.8).

Lemma 2.9. *Let* $1 < \rho < \infty$ *and* Ψ *be a complex function absolutely integrable on* Γ_ρ. *Then the function*

$$\Psi(z) = -\frac{1}{2\pi^2 i}\int_{\Gamma_\rho} \frac{\Psi(\zeta)}{\zeta - z}\,d\zeta, \quad z \in \mathbb{C} \setminus \Gamma_\rho$$

can be represented in $E^*(\rho)$ *in a series of the kind* (2.6) *with coefficients*

$$b_n = \frac{1}{2\pi^2 i} \int_{\Gamma_\rho} \Psi(\zeta) W_n(\zeta) d\zeta, \quad n \geq 0.$$

Proof. If $1 < r < \rho$ is arbitrary, this series is absolutely uniformly convergent on Γ_r and therefore in $E(r)$, i.e., this series represents $\Phi(z)$ in the whole region $E(\rho)$. The interchanging ζ with z implies the desired assertion.

The standard employment of the Cauchy integral formula and the above lemmas lead to the following statements.

Theorem 2.10. *Let* $1 < \rho \leq \infty$ *and* f *be a complex function holomorphic in* $E(\rho)$. *Then* f *can be represented in* $E(\rho)$ *by a series of the kind* (2.5) *with coefficients*

$$a_n = \frac{1}{2\pi^2 i} \int_{\Gamma_\rho} f(\zeta) Q_n(\zeta) d\zeta, \quad n \geq 0,$$

where $1 < r < \rho$.

Theorem 2.11. *Let* $1 \leq \rho < \infty$ *and* f *be a complex function holomorphic in* $E^*(\rho)$ *such that* $f(\infty) = 0$. *Then* f *can be represented in* $E^*(\rho)$ *by a series of the kind* (2.6) *with coefficients*

$$b_n = \frac{1}{2\pi^2 i} \int_{\Gamma_\rho} f(\zeta) W_n(\zeta) d\zeta, \quad n \geq 0,$$

where $1 < r < \rho$.

In general, the orthogonal expansions have the property of uniqueness. For the particular case of Chebyshev polynomials of the fourth kind, by means of the orthogonal property (1.4) and Theorem 2.6 (*b*) one can easily prove the following statement.

Theorem 2.12. *Let* $1 < \rho \leq \infty$. *If a complex function* f *has for* $z \in E(\rho)$ *a representation*

$$f(z) = \sum_{n=0}^{\infty} a_n W_n(z),$$

then f *is a holomorphic in* $E(\rho)$ *and*

$$a_n - \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} f(x) W_n(x) dx, \quad n \geq 0.$$

In particular, if $f(x) \equiv 0$, *then* $a_n = 0$ *for every* $n \geq 0$ *holds.*

THE REMAINDER TERM OF GAUSSIAN QUADRATURE

For analytic functions the remainder term of Gaussian quadrature rules can be represented as a contour integral on an ellipse with a complex kernel function, which is given as the quotient of the functions of the second kind $Q_n(P; z)$ and the orthogonal polynomials $P_n(z)$ itself. The error estimates are practical and useful, if it is known where the maximum value of the kernel is attained on the ellipse considered.

Gaussian quadrature

Consider the n-point Gauss quadrature formula

$$\int_{-1}^{1} w(x)f(x)dx = \sum_{\nu=1}^{n} w_\nu f(x_\nu) + R_n f \tag{3.1}$$

with respect to some nonnegative weight function $w(x)$ and with algebraic degree of precision $k = 2n - 1$, *i.e.*, $R_n f = 0$, whenever f is a polynomial of degree less or equal to $2n - 1$. The nodes x_ν are the zeros of the n-th degree polynomial $P_n(x)$, orthogonal with respect to $w(x)$ on the interval $[-1, 1]$.

Remainder terms for quadrature formulas are traditionally expressed in terms of some high-order derivative of the function $f(x)$ involved. This provides a serious disadvantage in cases where such derivatives are unknown, do not exist or are too complicated to be computed. Likewise it is also difficult to compare formulas of different algebraic degrees of precision. That is why derivative-free estimates were developed by the theory of analytic functions and by application of functional analysis (Davis & Rabinowitz 1984 [p. 300–336]). For integrands $f(x)$ having an analytic extension in a region D containing $[-1, 1]$, the remainder term $R_n f$ can be expressed as a contour integral

$$R_n f = \frac{1}{2\pi i}\int_\Gamma K_n(z)f(z)dz,$$

where Γ is a closed contour in D with length $L(\Gamma)$ surrounding $[-1, 1]$, and $K_n(z)$ is the complex kernel function. Then an error estimate is given by

$$|R_n f| \le \frac{1}{2\pi} L(\Gamma) \max_{z\in\Gamma} |K_n(z)| \max_{z\in\Gamma} |f(z)|.$$

While the second maximum depends only on f, the main question is how to determine the maximum of $|K_n(z)|$ on Γ, which depends on the quadrature formula.

The kernel function satisfies

$$K_n(z) = \frac{Q_n(z)}{P_n(z)}, \tag{3.2}$$

where $Q_n(z)$ is a function of the second kind related to the polynomial $P_n(x)$ defined as

$$Q_n(z) = \int_{-1}^{1} \frac{w(x)P_n(x)}{z - x} dx, \quad z \in \mathbb{C} \setminus [-1, 1].$$

In all the literature on the subject, $\max_{z\in\Gamma}|K_n(z)|$ is either bounded from above, or estimated asymptotically for large n (or large z, or both). In cases when Γ is a circle $|z| = \rho$, $\rho > 1$, the problem for evaluating the $\max_{z\in\Gamma}|K_n(z)|$ is solved for a large class of weight functions (including the Jacobi weight function ($w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha > -1, \beta > -1$) and the maximum of the kernel function can be expressed exactly as either $K_n(\rho)$ or $|K_n(-\rho)|$, of which values can be evaluated accurately and efficiently by recursion (Gautschi & Varga 1983). For elliptic contours

$$\Gamma_\rho = \{z : z = \tfrac{1}{2}(u + u^{-1}),\ u = \rho e^{i\theta},\ \rho > 1,\ \theta \in [0, 2\pi]$$

the problem is considerably more difficult. To deduce realistic error estimates, the kernel function $K_n(z)$ has to be estimated or computed carefully on the ellipse Γ_ρ. In the case of Jacobi weight functions as $\alpha = \beta = \pm 1/2$ and $\alpha = -1/2$, $\beta = 1/2$, explicit representations of $K_n(z)$ on Γ_ρ are obtained and the maximum points on the ellipse are located (Gautschi & Varga 1983, Gautschi *et al.* 1990). The maximum value of $|K_n(z)|$ is taken on the points of intersection between the ellipse and the real or imaginary axis, i.e., on $\pm\frac{1}{2}(\rho + \rho^{-1})$ or $\pm\frac{i}{2}(\rho - \rho^{-1})$. The remainder term of Gauss–Lobatto quadrature rules for the Chebyshev weight functions of the second, third and fourth kind are also studied (Schira 1996).

In this section we deal with the kernel function behaviour on elliptic contours Γ_ρ for Gauss quadrature related to the Chebyshev weight function of the fourth kind with $\alpha = 1/2$ and $\beta = 1/2$. We offer straightforward proof of the fact that the kernel function of the remainder in the Gauss quadrature under consideration attains its maximal value at the crossing point of Γ_ρ with the negative real semiaxis. This result completely matches those already established (Schira 1996 [Theorem 3.2]) for the case of the Lobatto–Chebyshev quadrature rule.

The maximum of the kernel function on an ellipse

Consider the Gauss quadrature (3.1) for Chebyshev polynomials of the fourth kind. We use essentially the relations (1.2) and (1.8) that if

$$z = \tfrac{1}{2}(u + u^{-1}),\ u = \rho^{i\theta},\ \rho > 1,\ \theta \in [0, 2\pi],$$

it leads to the representation

$$W_n(z) = \frac{u^{n+1} - u^{-n}}{u - 1}. \tag{3.3}$$

By (1.10) and a special equation (Gradshteyn & Ryzhik 1965 [Eq. 3.613.1]) we obtain

$$Q_n(z) = \frac{\pi}{\sqrt{z^2-1}}\left[\left(z - \sqrt{z^2-1}\right)^n - \left(z - \sqrt{z^2-1}\right)^{n+1}\right] = \frac{2\pi}{u^n(u+1)}. \tag{3.4}$$

Therefore, from (3.2), (3.3) and (3.4) it follows that

$$K_n(z) = \frac{2\pi(u-1)}{(u+1)u^n(u^{n+1} - u^{-n})}.$$

In particular,

$$|K_n(z)| = \frac{\sqrt{2}\pi\sqrt{a_1(\rho) - \cos\theta}}{\rho^{n+1/2}\sqrt{[a_1(\rho) + \cos\theta][a_{2n+1}(\rho) - \cos(2n+1)\theta]}}, \tag{3.5}$$

where $z \in \Gamma_\rho$ and $a_k(\rho) = \frac{1}{2}(\rho^k + \rho^{-k}),\ k \geq 1$.

The following inequality (Gautschi & Varga 1983 [Lemma 5.2])

$$\frac{a_1(\rho)a_{2n+1}(\rho) - 1}{a_2(\rho) - 1} > \left(\frac{2n+1}{2}\right)^2, \quad n \geq 1,\ \rho > 1 \tag{3.6}$$

plays a key role in the proof of our main result.

Theorem 3.1. *If* $w(x) = (1-x)^{1/2}(1+x)^{-1/2}$ *on* $(-1, 1)$, *then*

$$\max_{z\in\Gamma_\rho} |K_n(z)| = K_n\left(-\frac{1}{2}(\rho + \rho^{-1})\right).$$

Proof. To show that the expression on the right hand side in (3.5), considered as a function of $\theta \in [0, \pi]$, attains its maximum only at $\theta = \pi$, it is equivalent to establishing the validity of the inequality

$$\frac{a_1 - \cos\theta}{(a_1 + \cos\theta)[a_{2n+1} - \cos(2n+1)\theta]} < \frac{a_1 + 1}{(a_1 - 1)(a_{2n+1} + 1)}, \quad 0 \leq \theta < \pi,$$

where $a_k = a_k(\rho), k \geq 1$. The standard simplifications and the introduction of half angles yield the equivalent inequality

$$\frac{a_1(a_{2n+1} + 1)}{a_1 + 1} - (a_1 - 1)\frac{\cos^2\frac{2n+1}{2}\theta}{2\cos^2\frac{\theta}{2}} - \cos^2\frac{2n+1}{2}\theta > 0. \tag{3.7}$$

By (1.3) and the induction method it is easy to see that

$$|V_n(x)| = \left|\frac{\cos\frac{2n+1}{2}\theta}{\cos\frac{\theta}{2}}\right| \leq 2n+1, \quad n = 1, 2, 3 \ldots .$$

Then it is clear that the left-hand side in (3.7) is larger than or equal to

$$\frac{a_1(a_{2n+1} + 1)}{a_1 + 1} - \frac{1}{2}(a_1 - 1)(2n+1)^2 - 1 = 2(a_1 - 1)\left\{\frac{a_1 a_{2n+1} - 1}{a_2 - 1} - \left(\frac{2n+1}{2}\right)^2\right\}$$

which is strictly positive by (3.6).

Finally, the relation (1.11) and Theorem 3.1 lead to a well-known result that the maximum of $|K_n(z)|$ on Γ_ρ in the case of Gaussian quadrature for the Chebyshev polynomials of the third kind is attained on the positive real semiaxis (Gautschi & Varga 1983 [Theorem 5.3]).

Remark. Integrals with the four different kinds of Chebyshev weight functions can always be reduced to an integral with Chebyshev weight function of the first kind. Thus each of these integrals can be computed by the Gauss–Chebyshev quadrature of the first kind. Some numerical experiments show that for special integrals the different kinds of Gauss–Chebyshev quadratures provide better numerical results.

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خلاصه

في هذا البحث نعطي اهتماما خاصا إلى كثيرات حدود تشيبشف من النوع الرابع لأن هـــذا النوع أقل معرفة ودراسة من الأنواع الأخرى. يتضمن البحث صياغة مسائلة الدوال التحليليـــة في متسلسلة من كثيرات الحدود هذه والدور الهام لدوال تشيبشف من النوع الثاني في حل هــذه المسائلة. من أجل الدوال التحليلية يعبر عن الحد الباقي من مكتمل قوانيـــــن جـــاوس بكفـــاف تكاملي مع دالة مركبة ذات نواه. لقد درست خصيصا دالة النواة المنسوبة إلى مكتمل جـــــاوس على كفاف زائدى من أجل كثيرات حدود تشيبشف من النوع الرابع وحددت نقاطها العظمى.