Mediterr. J. Math. 10 (2013), 57–72

Explicit Formulas and Combinatorial Identities for Generalized Stirling Numbers

Nenad P. Cakić, Beih S. El-Desouky and Gradimir V. Milovanović*

Abstract. In this paper, a modified approach to the multiparameter non-central Stirling numbers via differential operators, introduced by El-Desouky, and new explicit formulae of both kinds of these numbers are given. Also, some relations between these numbers and the generalized Hermite and Truesdel polynomials are obtained. Moreover, we investigate some new results for the generalized Stirling-type pair of Hsu and Shiue. Furthermore some interesting special cases, new combinatorial identities and a matrix representation are deduced.

Mathematics Subject Classification (2010). Primary 11B73; Secondary 05A10, 05A19.

Keywords. Multiparameter non-central Stirling numbers, generalized Stirling-type numbers, differential operators, generating function, combinatorial identities.

1. Introduction and preliminaries

Through this article we use the following notations. The falling and rising factorials are defined, respectively by

$$(x)_n = x(x-1)\cdots(x-n+1), \quad (x)_0 = 1,$$

and

$$\langle x\rangle_n = x(x+1)\cdots(x+n-1), \quad \langle x\rangle_0 = 1.$$

The generalized falling and rising factorials $(x;\bar{\alpha})_n$ and $\langle x;\bar{\alpha}\rangle_n$, associated with parameter $\bar{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ where α_j, $j = 0, 1, \ldots, n-1$, is a sequence

This paper was supported in part by the Serbian Ministry of Education and Science (Cakić and Milovanović) and by the Egyptian Ministry of Higher Education (El-Desouky).

*Corresponding author.

of real or complex numbers, are defined by

$$(x;\bar{\alpha})_n = \prod_{j=0}^{n-1}(x-\alpha_j) \quad \text{and} \quad \langle x;\bar{\alpha}\rangle_n = \prod_{j=0}^{n-1}(x+\alpha_j),$$

respectively. Note that if $\alpha_i = i\alpha$, $i = 0,1,\ldots,n-1$, then $(x;\bar{\alpha})_n$ reduces to

$$(x|\alpha)_n = t(t-\alpha)\cdots(x-(n-1)\alpha).$$

Hsu and Shiue [14] defined generalized Stirling-type pair

$$\{S^1(n,k),S^2(n,k)\} \equiv \{S(n,k;\alpha,\beta,r),S(n,k;\beta,\alpha,-r)\}$$

by the inverse relations

$$(x|\alpha)_n = \sum_{k=0}^{n} S^1(n,k)(x-r|\beta)_k \tag{1.1}$$

and

$$(x|\beta)_n = \sum_{k=0}^{n} S^2(n,k)(x+r|\alpha)_k, \tag{1.2}$$

where n is a nonnegative integer and the parameters α,β and r are real or complex numbers, with $(\alpha,\beta,r) \neq (0,0,0)$.

For the numbers $S(n,k;\alpha,\beta,r)$, we have the following recurrence relation

$$S(n+1,k;\alpha,\beta,r) = S(n,k-1;\alpha,\beta,r) + (k\beta - n\alpha + r)S(n,k;\alpha,\beta,r), \tag{1.3}$$

where $n \geq k \geq 1$.

These numbers have the vertical generating functions (with $\alpha\beta \neq 0$)

$$k!\sum_{n\geq 0} S(n,k;\alpha,\beta,r)\frac{x^n}{n!} = (1+\alpha x)^{r/\alpha}\left(\frac{(1+\alpha x)^{\beta/\alpha}-1}{\beta}\right)^k \tag{1.4}$$

and

$$k!\sum_{n\geq 0} S(n,k;\beta,\alpha,-r)\frac{x^n}{n!} = (1+\beta x)^{-r/\beta}\left(\frac{(1+\beta x)^{\alpha/\beta}-1}{\alpha}\right)^k.$$

The multiparameter non-central Stirling numbers of the first and second kind, respectively, were introduced by El-Desouky [11] with

$$(x)_n = \sum_{k=0}^{n} s(n,k;\bar{\alpha})(x;\bar{\alpha})_k \tag{1.5}$$

and

$$(x;\bar{\alpha})_n = \sum S(n,k;\bar{\alpha})(x)_k. \tag{1.6}$$

These numbers $s(n,k;\bar{\alpha})$ and $S(n,k;\bar{\alpha})$ satisfy the recurrence relations

$$s(n+1,k;\bar{\alpha}) = s(n,k-1;\bar{\alpha}) + (\alpha_k - n)s(n,k;\bar{\alpha}) \tag{1.7}$$

and

$$S(n+1,k;\bar{\alpha}) = S(n,k-1;\bar{\alpha}) + (k-\alpha_n)S(n,k;\bar{\alpha}), \tag{1.8}$$

respectively.

The multiparameter non-central Stirling numbers of the first kind have the vertical exponential generating function

$$\sum_{n\geq 0} s(n,k;\bar{\alpha})\frac{x^n}{n!} = \sum_{j=0}^{k} \frac{(1+x)^{\alpha_j}}{(\alpha_j)_k}\,, \tag{1.9}$$

where $(\alpha_k)_\ell = \prod_{j=0, j\neq k}^{\ell} (\alpha_k - \alpha_j),\ k \leq \ell$.

Also, more generalizations and extensions of Stirling numbers are given in [4] and [20]–[22].

Let $\delta = xD$, $D = d/dx$. Then we have some well-known and useful relations of this differentiation operator (see [2, 10, 16] and [19]):

(i) $\delta^n(x^\alpha) = \alpha^n x^\alpha$,

(ii) $\delta^n(u\cdot v) = \sum_{k=0}^{n} \binom{n}{k} \delta^{n-k}u\ \delta^k v$,

(iii) $F(\delta)(x^\alpha f(x)) = x^\alpha F(\delta+\alpha)f(x)$,

(iv) $F(\delta)(e^{g(x)}f(x)) = e^{g(x)}F(\delta + xg'(x))f(x)$.

The paper is organized as follows. In Section 2 a modified approach via differential operator to multiparameter non-central Stirling numbers is given. Also, some relations between these numbers and the generalized Hermite and Truesdel polynomials are obtained. Moreover, we show that some of Hsu and Shiue [14] results are investigated using the results obtained by El-Desouky [11] and consequently of this paper. In Section 3, new explicit formulae for those numbers, some special cases and new combinatorial identities are derived. Finally, in Section 4, a computer program is written using Maple and executed for calculating the multiparameter non-central Stirling matrix and some special cases and matrix representation of these numbers is given.

2. A modified approach to multiparameter non-central Stirling numbers

Let the differential operator $\mathscr{D}$ be defined by $\mathscr{D} = x^\alpha \delta x^{-\alpha}$, where $\delta = xD$. Using (iii) we get $\mathscr{D}f(x) = x^\alpha \delta x^{-\alpha} f(x) = (\delta - \alpha)f(x)$, hence $\mathscr{D} = \delta - \alpha$ and by induction we get $\mathscr{D}^n = (\delta-\alpha)^n = x^\alpha \delta^n x^{-\alpha}$. Generally, we can take the following.

Definition 2.1. Let the differential operator $\mathscr{D}_n$ be defined by

$$\mathscr{D}_n = (x^{\alpha_{n-1}}\delta x^{-\alpha_{n-1}})\cdots(x^{\alpha_0}\delta x^{-\alpha_0}) = \prod_{i=0}^{n-1} x^{\alpha_i}\delta x^{-\alpha_i},\ n \geq 1$$

and $\mathscr{D}_0 = I$, the identity operator.

Then we have $\mathscr{D}_n f(x) = (\delta - \alpha_{n-1})\cdots(\delta-\alpha_0)f(x)$, i.e.,

$$\mathscr{D}_n = (\delta-\alpha_{n-1})\cdots(\delta-\alpha_0) = \prod_{i=0}^{n-1}(\delta-\alpha_i) = (\delta;\bar{\alpha})_n,$$

and so we have the operational formula $\prod_{i=0}^{n-1} x^{\alpha_i}\delta x^{-\alpha_i} = \prod_{i=0}^{n-1}(\delta-\alpha_i)$.

This leads us to define the multiparameter non-central Stirling numbers via the differential operator $\mathscr{D}_n$. Therefore, equations (1.5) and (1.6) can be represented, in terms of operational formulae, by

$$(\delta)_n = x^n D^n = \sum_{k=0}^{n} s(n,k;\bar{\alpha})(\delta;\bar{\alpha})_k = \sum_{k=0}^{n} s(n,k;\bar{\alpha})\mathscr{D}_k \tag{2.1}$$

and

$$\mathscr{D}_n = (\delta;\bar{\alpha})_n = \sum_{k=0}^{n} S(n,k;\bar{\alpha})(\delta)_k = \sum_{k=0}^{n} S(n,k;\bar{\alpha})x^k D^k, \tag{2.2}$$

respectively.

Similarly, Comtet numbers (see [4] and [20]) can be defined by

$$(\delta;\bar{\alpha})_n = \sum_{k=0}^{n} s_{\bar{\alpha}}(n,k)\delta^k$$

and

$$\delta^n = \sum_{k=0}^{n} S_{\bar{\alpha}}(n,k)(\delta;\bar{\alpha})_k.$$

Setting $\alpha_i = \alpha$, then (2.1) and (2.2), respectively, yield

$$(\delta)_n = x^n D^n = \sum_{k=0}^{n} s(n,k;\alpha)(\delta-\alpha)^k = \sum_{k=0}^{n} s(n,k;\alpha)x^\alpha \delta^k x^{-\alpha}. \tag{2.3}$$

and

$$x^\alpha \delta^n x^{-\alpha} = (\delta-\alpha)^n = \sum_{k=0}^{n} S(n,k;\alpha)(\delta)_k = \sum_{k=0}^{n} S(n,k;\alpha)x^k D^k, \tag{2.4}$$

where $s(n,k;\alpha)$ and $S(n,k;\alpha)$ are the non-central Stirling numbers of the first and second kind [11], respectively.

Acting with Eq. (2.3) on e^{-px^r}, then multiplying by e^{px^r} we obtain

$$e^{px^r}x^n D^n e^{-px^r} = \sum_{k=0}^{n} s(n,k;\alpha)e^{px^r}x^\alpha \delta^k x^{-\alpha}e^{-px^r}, \tag{2.5}$$

hence we have,

$$x^n H_n^r(x,0,p) = (-1)^n \sum_{k=0}^{n} s(n,k;\alpha)T_k^{-\alpha}(x,r,p), \tag{2.6}$$

where $H_n^r(x,\alpha,p)$ and $T_k^{-\alpha}(x,r,p)$ are the generalized Hermite and Truesdel polynomials, respectively (see [12] and [16]-[18]).

Setting $r = p = 1$ in (2.5) , or (2.6), we get

$$(-1)^n x^n = \sum_{k=0}^{n} s(n,k;\alpha)T_k^{-\alpha}(x).$$

where $T_n^{-\alpha}(x)$ are Truesdel polynomials, see [17].

Similarly, acting with Eq. (2.4) on e^{-px^r}, we obtain the inverse relations

$$T_n^{-\alpha}(x,r,p) = (-1)^n \sum_{k=0}^{n} S(n,k;\alpha) x^k (-1)^k H_k^r(x,0,p)$$

and

$$T_k^{-\alpha}(x) = \sum_{k=0}^{n} s(n,k;\alpha)(-1)^k x^k.$$

Next, we derive some new results for Hsu and Shiue numbers [14].

Theorem 2.2. *For the special case $\alpha_i = -(r-i\alpha)/\beta$, $i = 0,1,\ldots,n-1$, we have*

$$s(n,k;\bar{\alpha}) = \beta^{k-n} S^2(n,k) \tag{2.7}$$

Proof. From equation (1.2) we get

$$\beta^n (t/\beta)\,((t/\beta)-1)\cdots((t/\beta)-(n-1)) = \sum_{k=0}^{n} S^2(n,k)(t+r|\alpha)_k.$$

Setting $t/\beta = x$, we have

$$\beta^n x(x-1)\cdots(x-(n-1)) = \sum_{k=0}^{n} S^2(n,k)(\beta x+r)(\beta x+r-\alpha)\cdots(\beta x+r-(k-1)\alpha),$$

hence

$$(x)_n = \sum_{k=0}^{n} \beta^{k-n} S^2(n,k)\,(x+r/\beta)\,(x+(r-\alpha)/\beta)\cdots(x+(r-(k-1)\alpha)/\beta)\,.$$

Comparing this equation with (1.5) yields (2.7). □

Also, it can be shown that $S(n,k;\bar{\alpha}) = \beta^{k-n} S^1(n,k)$ if $\alpha_i = (i\alpha - r)/\beta$.

Furthermore, we show that the generating function (1.4), see [14], of Hsu-Shiue numbers can be investigated from the generating function (1.9), see [11], where $\alpha_i = (r+i\beta)/\alpha$, $i = 0,1,\ldots,n-1$.

Theorem 2.3. *For the special case $\alpha_i = (r+i\beta)/\alpha$, $i = 0,1,\ldots,n-1$, the generating function* (1.9) *is reduced to the generating function* (1.4) *and*

$$S^1(n,k) = \alpha^{n-k} s(n,k;\bar{\alpha}).$$

Proof. If we start from (1.9)

$$\sum_{n\geq 0} s(n,k;\bar{\alpha})\frac{z^n}{n!} = \sum_{j=0}^{k} \frac{(1+z)^{\alpha_j}}{(\alpha_j)_k}$$

and setting $\alpha_j = r/\alpha + j\beta/\alpha$, $j = 0,1,\ldots,k$, we get

$$RHS = \sum_{j=0}^{k} \frac{(1+z)^{\frac{r}{\alpha}+j\frac{\beta}{\alpha}}}{\prod\limits_{j=0,i\neq j}^{k}\left(\left(\frac{r}{\alpha}+j\frac{\beta}{\alpha}\right)-\left(\frac{r}{\alpha}+i\frac{\beta}{\alpha}\right)\right)} = \sum_{j=0}^{k} \frac{(1+z)^{\frac{r}{\alpha}+j\frac{\beta}{\alpha}}}{\left(\frac{\beta}{\alpha}\right)^k \prod\limits_{i=0,i\neq j}^{k}(j-i)}$$

$$= \frac{(1+z)^{\frac{r}{\alpha}}}{\left(\frac{\beta}{\alpha}\right)^k} \sum_{j=0}^{k} \frac{(1+z)^{j\frac{\beta}{\alpha}}}{\prod\limits_{i=0,i\neq j}^{k}(j-i)} = \frac{(1+z)^{\frac{r}{\alpha}}}{\left(\frac{\beta}{\alpha}\right)^k} \sum_{j=0}^{k} \frac{(1+z)^{j\frac{\beta}{\alpha}}}{(-1)^{k-j}(k-j)!\,j!}$$

$$= \frac{1}{k!}\frac{(1+z)^{\frac{r}{\alpha}}}{\left(\frac{\beta}{\alpha}\right)^k} \sum_{j=0}^{k} \binom{k}{j}(-1)^{k-j}(1+z)^{j\frac{\beta}{\alpha}} = \frac{1}{k!}\frac{(1+z)^{\frac{r}{\alpha}}}{\left(\frac{\beta}{\alpha}\right)^k}\left((1+z)^{\frac{\beta}{\alpha}}-1\right)^k,$$

where we used the identity,

$$\prod_{i=0,i\neq j}^{k}(j-i) = (-1)^{k-j}(k-j)!\,j!.$$

Thus, putting $z = \alpha t$ in (1.9) we get

$$\sum_{n\geq 0} s(n,k;\bar{\alpha})\frac{\alpha^n t^n}{n!} = \frac{1}{k!}\alpha^k(1+\alpha t)^{r/\alpha}\left(\frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\right)^k,$$

hence by virtue of (1.4) we have

$$\sum_{n\geq 0} S^1(n,k)\frac{t^n}{n!} = \sum_{n\geq 0} \alpha^{n-k} s(n,k;\bar{\alpha})\frac{t^n}{n!} = \frac{1}{k!}(1+\alpha t)^{r/\alpha}\left(\frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\right)^k,$$

where $\alpha_i = (i\beta + r)/\alpha$.

In fact, this equation shows that

$$S^1(n,k) = \alpha^{n-k} s(n,k;\bar{\alpha}),$$

where $\alpha_i = (i\beta + r)/\alpha$, $i = 0,1,\ldots,n-1$. This completes the proof. □

Now, setting $\alpha_i = (i\alpha - r)/\beta$, $i = 0,1,\ldots,n-1$ in (2.2) we get the following new operational formula for Hsu-Shiue numbers:

$$\prod_{i=0}^{n-1} x^{(i\alpha-r)/\beta}\,\delta\, x^{-(i\alpha-r)/\beta} = \sum_{k=0}^{n} \beta^{k-n} S^1(n,k)(\delta)_k. \tag{2.8}$$

Remark 2.4. Note that equations (2.2) and (2.8) for $\alpha_i = 0$, $i = 0,1,\ldots,n-1$ and $\alpha = 0$, $\beta = 1$ and $r = 0$, respectively, are reduced to the well known operational representation of the ordinary Stirling numbers of the second kind

$$\delta^n = \sum_{k=0}^{n} S(n,k)x^k D^k.$$

3. New explicit formulae and combinatorial identities

We derive new explicit formulae for both kinds of the multiparametar non-central Stirling numbers.

Theorem 3.1. *The numbers $S(n,k;\bar{\alpha})$, have the following new explicit formula*

$$S(n,k;\bar{\alpha}) = \sum_{I_{n-1}=n-k} \binom{-\alpha_0}{i_0}\binom{1-\alpha_1-i_0}{i_1}\cdots\binom{n-1-\alpha_{n-1}-i_0-\cdots-i_{n-2}}{i_{n-1}},$$

where $i_j \in \{0,1\}$, $j \in \{0,1,\ldots,n-1\}$, and $I_{n-1} := i_0+i_1+\cdots+i_{n-1}$.

Proof. The statement for $k=0$ gives that

$$S(n,0;\bar{\alpha}) = (-1)^n \alpha_0\alpha_1\cdots\alpha_{n-1},$$

which agrees with the definition of $S(n,k;\bar{\alpha})$ (see [11]).

Also, if $i_{n-1} \in \{0,1\}$, we have that

$$\begin{aligned} S(n,k;\bar{\alpha}) = &\sum_{I_{n-2}=(n-1)-(k-1)} \binom{-\alpha_0}{i_0}\binom{1-\alpha_1-i_0}{i_1}\cdots \\ &\qquad \times \binom{n-2-\alpha_{n-2}-i_0-\cdots-i_{n-3}}{i_{n-2}} \\ &+ \sum_{I_{n-2}=(n-1)-k} [n-1-\alpha_{n-1}-(n-k-1)]\binom{-\alpha_0}{i_0}\binom{1-\alpha_1-i_0}{i_1}\cdots \\ &\qquad \times \binom{n-2-\alpha_{n-2}-i_0-\cdots-i_{n-3}}{i_{n-2}}, \end{aligned}$$

i.e.,

$$S(n,k;\bar{\alpha}) = S(n-1,k-1;\bar{\alpha}) + (k-\alpha_{n-1})S(n-1,k;\bar{\alpha}).$$

Therefore, by (1.8) and induction we get the desired result. □

Remark 3.2. We used $\alpha_0 = 0 \Leftrightarrow i_0 = 0$.

Theorem 3.3. *The multiparameter non-central Stirling numbers of the first kind have the following new explicit expression*

$$s(n,k;\bar{\alpha}) = \sum_{\substack{i_1+\cdots+i_n=k \\ i_j\in\{0,1\}}} \binom{i_1+\alpha_{i_1}}{1-i_1}\binom{i_2+\alpha_{i_1+i_2}-1}{1-i_2}\cdots\binom{i_n+\alpha_{i_1+\cdots+i_n}-n+1}{1-i_n}. \tag{3.1}$$

Proof. For $k=0$ we have

$$s(n,0;\bar{\alpha}) = \alpha_0(\alpha_0-1)\cdots(\alpha_0-n+1) = (\alpha_0)_n$$

and if $i_n \in \{0,1\}$, we have

$$s(n,k;\bar{\alpha}) = \sum_{i_1+\cdots+i_{n-1}=k-1} \binom{i_1+\alpha_{i_1}}{1-i_1} \cdots \binom{i_{n-1}+\alpha_{i_1+\cdots+i_{n-1}}-n+2}{1-i_{n-1}}$$
$$+ \sum_{i_1+\cdots+i_{n-1}=k} (\alpha_{i_1+\cdots+i_{n-1}}-n+1) \binom{i_1+\alpha_{i_1}}{1-i_1} \binom{i_{n-1}+\alpha_{i_1+\cdots+i_{n-1}}-n+2}{1-i_{n-1}}$$
$$= s(n-1,k-1;\bar{\alpha}) + (\alpha_k - n + 1)s(n-1,k;\bar{\alpha}),$$

where $i_1+\cdots+i_{n-1}=k$. By virtue of (1.7), this completes the proof. □

It is worth noting that setting $\alpha_i = (r+i\beta)/\alpha$, $i \in \{0,1,\ldots,n-1\}$ in the recurrence relation (1.7) for multiparameter non-central Stirling numbers of the first kind, we can get the recurrence relation (1.3), and hence

$$s(n,k;\bar{\alpha}) = \alpha^{k-n} S^1(n,k).$$

Furthermore, Corcino, Hsu and Tan [9] mentioned that the multiparameter non-central Stirling numbers are related with the generalized Stirling-type pair of Hsu-Shiue (see also [23]) by $S(n,k;1,\alpha,0) = s(n,k;\bar{\alpha})$ and $S(n,k;\alpha,1,0) = S(n,k;\bar{\alpha})$ for special case $\alpha_i = i\alpha$, $i = 0,1,\ldots,n-1$.

Corollary 3.4. *A new explicit expression for generalized Stirling numbers (Hsu-Shiue numbers type (1.3)) is given by*

$$S(n,k;\alpha,\beta,r) = \beta^{n-k} \sum_{\substack{I_{n-1}=n-k \\ i_j\in\{0,1\}}} \binom{r/\beta}{i_0} \binom{1+(r-\alpha)/\beta-i_0}{i_1} \cdots$$
$$\cdots \binom{n-1+(r-(n-1)\alpha)/\beta-i_0-\cdots-i_{n-2}}{i_{n-1}}$$

or in the modified form

$$S(n,k;\alpha,\beta,r) = \sum_{\substack{I_{n-1}=n-k \\ i_j\in\{0,1\}}} \binom{r}{i_0} \binom{r+\beta-\alpha-\beta i_0}{i_1} \binom{r+2(\beta-\alpha)-\beta(i_0+i_1)}{i_2} \cdots$$
$$\cdots \binom{r+(n-1)(\beta-\alpha)-\beta(i_0+i_1+\cdots+i_{n-2})}{i_{n-1}},$$

where we use $i_0 = 0 \Leftrightarrow r = 0$.

Proof. The proof of this result follows directly from Theorem 3.1. Namely, by setting $\alpha_i = (i\alpha - r)/\beta$, $i = 0,1,\ldots,n-1$, we get the statement. □

The previous corollary implies new explicit expressions for the usual Stirling numbers of the second and first kind

$$S(n,k;0,1,0) = S(n,k) = \sum_{\substack{i_1+\cdots+i_{n-1}=n-k \\ i_j\in\{0,1\}}} \binom{1}{i_1} \binom{2-i_1}{i_2} \cdots \binom{n-1-i_1-\cdots-i_{n-2}}{i_{n-1}}$$

and

$$S(n,k;1,0,0) = s(n,k) = \sum_{\substack{i_1+\cdots+i_{n-1}=n-k \\ i_j\in\{0,1\}}} \binom{-1}{i_1}\binom{-2}{i_2}\cdots\binom{-(n-1)}{i_{n-1}}$$
$$= (-1)^{n-k} \sum_{\substack{i_1+\cdots+i_{n-1}=n-k \\ i_j\in\{0,1\}}} \binom{i_1}{i_1}\binom{1+i_2}{i_2}\cdots\binom{n-2+i_{n-1}}{i_{n-1}},$$

respectively.

Remark 3.5. From the above corollary we obtain the explicit formula for the numbers $F_{\alpha,\gamma}(n,k)$ (see [7]), defined by the triangular recurrence relation

$$F_{\alpha,\gamma}(n+1,k) = F_{\alpha,\gamma}(n,k-1) + (\gamma - n\alpha)F_{\alpha,\gamma}(n,k),$$

where $F_{\alpha,\gamma}(0,0) = 1$ and $F_{\alpha,\gamma}(n,k) = 0$ when $n < 0$, $k < 0$ and $n < k$. Namely, for $\beta = 0$ and $r = \gamma$, the modified form of $S(n,k,\alpha,\beta,r)$ in Corollary 3.4 reduces to

$$F_{\alpha,\gamma}(n,k) = S(n,k;\alpha,0,\gamma) = \sum_{\substack{I_{n-1}=n-k \\ i_j\in\{0,1\}}} \binom{\gamma}{i_0}\binom{\gamma-\alpha}{i_1}\cdots\binom{\gamma-(n-1)\alpha}{i_{n-1}}.$$

Note that some properties and identities on $S(n,k;\alpha,\beta,r)$ cannot be obtained for $\beta = 0$, which is a reason why the explicit formula, i.e., the modified form (from Corollary 3.4), is very convenient to use.

Corcino et al. [8] defined the (r,β)-Stirling numbers, denoted by $\left\langle {n \atop k} \right\rangle_{\beta,r}$ using the following recurrence relation

$$\left\langle {n+1 \atop k} \right\rangle_{\beta,r} = \left\langle {n \atop k-1} \right\rangle_{\beta,r} + (k\beta + r)\left\langle {n \atop k} \right\rangle_{\beta,r}.$$

It is easy to conclude that $\left\langle {n \atop k} \right\rangle_{\beta,r} = S(n,k,0,\beta,r)$. Thus, we can obtain an explicit formula for (r,β)-Stirling numbers from Corollary 3.4

$$\left\langle {n \atop k} \right\rangle_{\beta,r} = S(n,k;0,\beta,r)$$
$$= \sum_{\substack{I_{n-1}=n-k \\ i_j\in\{0,1\}}} \binom{r/\beta}{i_0}\binom{1+r/\beta-i_0}{i_1}\cdots\binom{n-1+r/\beta-i_0-\cdots-i_{n-2}}{i_{n-1}}.$$

Moreover, El-Desouky [11, Theorem 2.1] derived the following explicit formula for $s(n,k;\bar{\alpha})$

$$s(n,k;\bar{\alpha}) = \sum_{r=k}^{n} (-1)^{n-r}\frac{n!}{r!}\sum \frac{1}{i_1\cdots i_r}\sum_{j=0}^{k}\frac{{\alpha_j}^r}{(\alpha_j)_k}, \tag{3.2}$$

where, in the second sum, the summation extends over all ordered n-tuples of integers $(i_1,i_2,\ldots i_r)$ satisfying the condition $i_1+i_2+\cdots+i_r = n$ and $i_j \geq 0$, $j = 1,2,\ldots,r$.

From (3.1) and (3.2), we obtain the following new combinatorial identity

$$\sum_{\substack{i_1+\cdots+i_n=n-k\\ i_j\in\{0,1\}}} \binom{i_1+\alpha_{i_1}}{1-i_1}\binom{i_2+\alpha_{i_1+i_2}-1}{1-i_2}\cdots\binom{i_n+\alpha_{i_1+\cdots+i_n}-n+1}{1-i_n}$$

$$=\sum_{r=k}^{n}(-1)^{n-r}\frac{n!}{r!}\sum\frac{1}{i_1\cdots i_r}\sum_{j=0}^{k}\frac{{\alpha_j}^r}{(\alpha_j)_k}.$$

Remark 3.6. Now, it is worth noting that all special cases, (i)–(xi), derived in [14], are special cases of the multiparameter non-central Stirling numbers.

Corcino, Hsu and Tan [9] proved that

$$S^1(n,k)=\frac{n!\,\alpha^n}{k!\,\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{(\beta/\alpha)j+(r/a)}{n}. \tag{3.3}$$

From (3.3) and Corollary 3.4, we have the combinatorial identity

$$\frac{n!\alpha^n}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{(\beta/\alpha)j+(r/a)}{n}=\beta^n\sum_{\substack{I_{n-1}=n-k\\ i_j\in\{0,1\}}}\binom{r/\beta}{i_0}\times$$

$$\times\binom{1+(r-\alpha)/\beta-i_0}{i_1}\cdots\binom{n-1+(r-(n-1)\alpha)/\beta-i_0-\cdots-i_{n-2}}{i_{n-1}}.$$

Hongquan Yu [23, Theorem 4 and Corollary 5] proved that

$$S(p+l,k;\beta,\alpha,0)\equiv 0 \pmod{p}, \tag{3.4}$$

where p is a prime number, k and l are integers such that $l+1<k<p$ and

$$S(p,k;\alpha,\beta,r)\equiv 0 \pmod{p}, \tag{3.5}$$

where α, β and r are integers and p is a prime, $1<k<p$.

By virtue of (3.4), (3.5) and Corollary 3.4, we obtain the combinatorial identities

$$\alpha^{p+l-k}\sum_{\substack{i_1+\cdots+i_{p+l-1}=p+l-k\\ i_j\in\{0,1\}}}\binom{1-\beta/\alpha}{i_1}\binom{2(1-\beta/\alpha)-i_1}{i_2}\cdots$$

$$\times\binom{(p+l-1)(1-\beta/\alpha)-i_1-\cdots-i_{p+l-2}}{i_{p+l-1}}\equiv 0 \pmod{p},\ l+1<k<p,$$

and

$$\beta^{p-k}\sum_{\substack{I_{p-1}=p-k\\ i_j\in\{0,1\}}}\binom{r/\beta}{i_0}\binom{1+(r-\alpha)/\beta-i_0}{i_1}\cdots$$

$$\times\binom{p-1+(r-(p-1)\alpha)/\beta-i_0-\cdots-i_{p-2}}{i_{p-1}}\equiv 0 \pmod{p},\ 1<k<p.$$

The degenerate weighted Stirling numbers, denoted by $S(n,k,\lambda|\theta)$, are the pair $\langle\theta,1,\lambda\rangle$ (see [13]).

Howard [13] derived the following explicit formula

$$S(n,k,\lambda|\theta)=\frac{1}{k!}\sum_{l=0}^{k}(-1)^{k+l}\binom{k}{l}(\lambda+l|\theta)_n. \tag{3.6}$$

Next, we find new explicit formulae for the following special cases of Stirling numbers.

Corollary 3.7. *The degenerate weighted Stirling numbers $S(n,k,\lambda|\theta)$ have the following explicit formula*

$$S(n,k,\lambda|\theta)=\sum_{\substack{I_{n-1}=n-k,\\ i_j\in\{0,1\}}}\binom{\lambda}{i_0}\binom{\lambda-\theta+1-i_0}{i_1}\binom{\lambda+2(1-\theta)-i_0-i_1}{i_2}\cdots \times\binom{\lambda+(n-1)(1-\theta)-(i_0+i_1+i_2+\cdots+i_{n-2})}{i_{n-1}}. \tag{3.7}$$

Proof. The proof follows by setting $\alpha=\theta$, $\beta=1$ and $r=\lambda$ in the modified form of Corollary 3.4. □

From (3.6) and (3.7) we have the new combinatorial identity

$$\begin{aligned}\sum_{\substack{I_{n-1}=n-k\\ i_j\in\{0,1\}}}&\binom{\lambda}{i_0}\binom{\lambda-\theta+1-i_0}{i_1}\binom{\lambda+2(1-\theta)-i_0-i_1}{i_2}\cdots\\ &\times\binom{\lambda+(n-1)(1-\theta)-(i_0+i_1+\cdots+i_{n-2})}{i_{n-1}}\\ &=\frac{1}{k!}\sum_{l=0}^{k}(-1)^{k+l}\binom{k}{l}(\lambda+l|\theta)_n.\end{aligned}$$

Also, setting $\lambda=0$ in (3.7) (see [13, Lemma 2.1]), we have a new explicit formula for degenerate Stirling numbers

$$\begin{aligned}S(n,k|\theta)\quad=\quad&\sum_{\substack{i_1+\cdots+i_{n-1}=n-k\\ i_j\in\{0,1\}}}\binom{(1-\theta)}{i_1}\binom{2(1-\theta)-i_1}{i_2}\cdots\\ &\times\binom{(n-1)(1-\theta)-(i_1+i_2+\cdots+i_{n-2})}{i_{n-1}}.\end{aligned}$$

In the special case $\theta=\lambda=0$ we obtain the identity

$$\begin{aligned}\sum_{\substack{i_1+\cdots+i_{n-1}=n-k\\ i_j\in\{0,1\}}}&\binom{1}{i_1}\binom{2-i_1}{i_2}\cdots\binom{(n-1)-(i_1+i_2+\cdots+i_{n-2})}{i_{n-1}}\\ &=\frac{1}{k!}\sum_{l=0}^{k}(-1)^{k+l}\binom{k}{l}l^n.\end{aligned}$$

From (1.1) we get

$$\alpha^n(t/\alpha)((t/\alpha)-1)\cdots((t/\alpha)-(n-1))=\sum_{k=0}^{n}S^1(n,k)(t-r|\beta)_k.$$

Setting $t/\alpha = x$,

$$\alpha^n x(x-1)\cdots(x-(n-1)) = \sum_{k=0}^{n} S^1(n,k)(\alpha x - r)(\alpha x - r - \beta)\cdots(\alpha x - r - (k-1)\beta),$$

hence

$$(x)_n = \sum_{k=0}^{n} \alpha^{k-n} S^1(n,k)(x - (r/\alpha)|(\beta/\alpha))_k. \tag{3.8}$$

From (1.1) and (3.8), we obtain $\alpha^{k-n} S^1(n,k) = S(n,k;1,r/\alpha,\beta/\alpha)$, whence

$$S(n,k;\alpha,\beta,r) = \alpha^{n-k} S(n,k;1,\beta/\alpha,r/\alpha). \tag{3.9}$$

Similarly, we can prove that

$$S(n,k;\alpha,\beta,r) = \beta^{n-k} S(n,k;\alpha/\beta,1,r/\beta). \tag{3.10}$$

Indeed (3.9) and (3.10) (see [1]) can be investigated from (1.4).

Using (3.10), as well as the fact that the degenerate weighted Stirling numbers $S(n,k,\lambda|\theta)$ are the pair $\langle \theta, 1, \lambda \rangle$, then the numbers $S(n,k;\alpha,\beta,r)$ can be represented in terms of $S(n,k,\lambda|\theta)$ where

$$S(n,k;\alpha,\beta,r) = \beta^{n-k} S(n,k,(r/\beta)|(\alpha/\beta)). \tag{3.11}$$

Remark 3.8. In fact, (3.11) agrees with Corollary 3.4.

Furthermore, Munagi [15] defined $S_b(n,k)$, the B-Stirling numbers of the second kind, by

$$S_b(n,k) = S_b(n-1,k-1) + (k+b-1)S_b(n-1,k), \qquad n \geq k,$$

$S_b(n,0) = \delta_{n,0}$, $S_b(n,1) = b^{n-1}$, where $\delta_{i,j}$ is the Kronecker delta.

Corollary 3.9. *The B-Stirling numbers of the second kind, $S_b(n,k)$, have the following new explicit formula*

$$\begin{aligned} S_b(n+1,k+1) &= S(n,k;0,1,b) \\ &= \sum_{\substack{I_{n-1}=n-k \\ i_j\in\{0,1\}}} \binom{b}{i_0}\binom{b+1-i_0}{i_1}\binom{b+2-(i_0+i_1)}{i_2}\cdots \\ &\qquad \times \binom{b+n-1-(i_0+i_1+\cdots+i_{n-2})}{i_{n-1}}, \end{aligned} \tag{3.12}$$

where $b \geq 0$ and we use $i_0 = 0 \Leftrightarrow b = 0$.

Proof. The proof follows by setting $\alpha_i = -b$, $i = 0,1,\ldots,n-1$, in Theorem 3.1 (or setting $\alpha = 0$, $\beta = 1$ and $r = b$ in Corollary 3.4). □

This show that the B-Stirling numbers of the second kind is a special case of the multiparameter non-central Stirling numbers and consequently of Hsu-Shiue numbers.

From (3.12) and [15, Corollary 6.2] we obtain the new combinatorial identity

$$\sum_{\substack{I_{n-1}=n-k \\ i_j\in\{0,1\}}} \binom{b}{i_0}\binom{b+1-i_0}{i_1}\cdots\binom{b+n-1-(i_0+i_1+\cdots+i_{n-2})}{i_{n-1}} = \sum_{j=0}^{k}\frac{(-1)^{k-j}(b+j)^n}{j!(k-j)!}.$$

4. The matrix representation

Let s,S; $s(\overline{\alpha}),S(\overline{\alpha})$ and S^1,S^2 be $(n+1)\times(n+1)$ lower triangular matrices, where s and S are the matrices whose entries are the Stirling numbers of the first and second kinds (i.e., $s=[s_{ij}]_{i,j\geq 0}$ and $S=[S_{ij}]_{i,j\geq 0}$); $s(\overline{\alpha})$ and $S(\overline{\alpha})$ are the matrices whose entries are the multiparameter non-central Stirling numbers of the first and second kinds (i.e., $s(\overline{\alpha})=[s(\overline{\alpha})_{ij}]_{i,j\geq 0}$ and $S(\overline{\alpha})=[S(\overline{\alpha})_{ij}]_{i,j\geq 0}$) and S^1,S^2 are the matrices whose entries are the generalized Stirling-type pair of Hsu and Shiue (i.e., $S^1=[S^1_{ij}]_{i,j\geq 0}$ and $S^2=[S^2_{ij}]_{i,j\geq 0}$), respectively.

The multiparameter non-central Stirling numbers of the first and second kinds, Eqs. (2.1) and (2.2), can be represented in a matrix form as

$$\tilde{\delta}=s(\overline{\alpha})\widetilde{\mathscr{D}} \quad \text{and} \quad \widetilde{\mathscr{D}}=S(\overline{\alpha})\tilde{\delta},$$

respectively, where $\tilde{\delta}=((\delta)_0,(\delta)_1,\ldots,(\delta)_n)^T$ and $\widetilde{\mathscr{D}}=(\mathscr{D}_0,\mathscr{D}_1,\ldots,\mathscr{D}_n)^T$.

A computer program is written using Maple and executed for calculating the multiparameter non-central Stirling matrix of the first kind $s(\overline{\alpha})$ (and $S(\overline{\alpha})=s^{-1}(\overline{\alpha})$ of the second kind) and the generalized Stirling-type pair of Hsu and Shiue as a special case when $\alpha_i=(r+i\beta)/\alpha$.

For example if $0\leq n\leq 3$, then

$$s(\overline{\alpha})=\begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha_0 & 1 & 0 & 0 \\ \alpha_0(\alpha_0-1) & \alpha_0+\alpha_1-1 & 1 & 0 \\ \alpha_0(\alpha_0-1)(\alpha_0-2) & \alpha_0\alpha_1-3\alpha_0+\alpha_0^2-3\alpha_1+\alpha_1^2+2 & \alpha_0+\alpha_1+\alpha_2-3 & 1 \end{bmatrix}$$

and

$$S^1=\begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r(r-\alpha) & 2r+\beta-\alpha & 1 & 0 \\ r(r-\alpha)(r-2\alpha) & 3r^2-6r\alpha+3\beta r+\beta^2-3\beta\alpha+2\alpha^2 & 3r+3\beta-3\alpha & 1 \end{bmatrix}.$$

Remark 4.1. Note that there are some missing terms in [23, Table 1].

Similarly, the degenerate weighted Stirling matrix $S(\lambda|\theta)$, the degenerate Stirling matrix $S(\theta)$ and the Stirling matrix of the second kind S are given by

$$S(\lambda|\theta)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ \lambda(\lambda-\theta) & 2\lambda+1-\theta & 1 & 0 \\ \lambda(\lambda-\theta)(r-2\theta) & 3\lambda^2-6\lambda\theta+3\lambda+1-3\theta+2\theta^2 & 3\lambda+3-3\theta & 1 \end{bmatrix},$$

$$S(\theta)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1-\theta & 1 & 0 \\ 0 & (1-\theta)(1-2\theta) & 3(1-\theta) & 1 \end{bmatrix},$$

and

$$S=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix},$$

respectively, as special cases.

Acknowledgements. The authors would like to thank the anonymous referee for helpful comments and suggestions.

References

[1] D. Branson, *Stirling numbers and Bell numbers: their role in combinatorics and probability*. Math. Sci. **25** (2000), 1–31.

[2] Whei-Ching C. Chan, Kung-Yu Chen and H. M. Srivastava, *Certain classes of generating functions for the Jacobi and related hypergeometric polynomials*. Comput. Math. Appl. **44** (2002), 1539-1556.

[3] Wenchang Chu and Chuanam Wei, *Set partitions with restrictions*. Discrete Math. **308** (2008), 3163–3168.

[4] L. Comtet, *Nombres de Stirling generaux et fonctions symetriques*. C. R. Acad. Sci. Paris. (Ser. A) **275** (1972), 747–750.

[5] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*. D. Reidel Publishing Company, Dordrecht, Holand, 1974.

[6] R. B. Corcino, *Some theorems on generalized Stirling numbers*. Ars Combin. **60** (2001) 273–286.

[7] R. B. Corcino and M. Herrera, *Limit of the differences of the generalized factorial, its q- and p,q-analogue*. Util. Math. **72** (2007), 33–49.

[8] R. B. Corcino, C. B. Corcino and R. Aldema, *Asymptotic normality of the (r,β)-Stirling numbers*. Ars Combin. **81** (2006), 81–96.

[9] R. B. Corcino, L. C. Hsu and E. L. Tan, *A q-analogue of generalized Stirling numbers*. Fibonacci Quart. **44** (2) (2006), 154–156.

[10] G. Dattoli, M. X. He, and P. E. Ricci, *Eigenfunctions of Laguerre-type operators and generalized inevolution problems*. Math. Comput. Modelling **42** (2005), 1263–1268.

[11] B. S. El-Desouky, *The multiparameter non-central Stirling numbers*. Fibonacci Quart. **32** (3) (1994), 218–225.

[12] H. W. Gould and A. T. Hopper, *Operational formulas connected with two generalizations of Hermite polynomials*. Duke Math. J. **29** (1962), 51–63.

[13] F. T. Howard, *Degenerate weighted Stirling numbers*. Discrete Math. **57** (1985), 45–58.

[14] L. C. Hsu and P. J-Sh. Shiue, *A unified approach to generalized Stirling numbers*. Adv. in Appl. Math. **20** (1998), 366–384.

[15] A. O. Munagi, *Labeled factorization of integers*. Electron. J. Combin. **16** (2009), R50.

[16] P. N. Shrivastava, *On the polynomials of Truesdel type*. Publ. Inst. Math. (Beograd) (N.S.) **9 (23)** (1969), 43–46.

[17] R. P. Singh, *On generalized Truesdel polynomials*. Riv. Mat. Univ. Parma, (2) **8** (1967), 345–353.

[18] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*. Ellis Horwood Limited, Chichester, 1984.

[19] O. V. Viskov and H. M. Srivastava, *New approaches to certain identities involving differential operators*. J. Math. Anal. Appl. **186** (1994), 1–10.

[20] C. G. Wagner, *Generalized Stirling and Lah numbers*. Discrete Math. **160** (1996), 199–218.

[21] C. G. Wagner, *Partition statistics and q-Bell numbers* $(q = -1)$. J. Integer Seq. **7** (2004), Article 04.1.1.

[22] H. S. Wilf, *Generatingfunctionology*. Academic Press Inc., New York, 1994.

[23] H. Yu, *A generalization of Stirling numbers*. Fibonacci Quart. **36** (3) (1998), 252–258.

Nenad P. Cakić
Department of Mathematics
Faculty of Electrical Engineering
University of Belgrade
P.O. Box 35-54
11120 Belgrade
Serbia
e-mail: cakic@etf.rs

Beih S. El-Desouky
Department of Mathematics
Faculty of Science
Mansoura University
35516 Mansoura
Egypt
e-mail: b_desouky@yahoo.com

Gradimir V. Milovanović*[1]
Mathematical Institute
Serbian Academy of Science and Arts
Kneza Mihaila 36
P.O. 367
11001 Belgrade
Serbia
e-mail: gvm@mi.sanu.ac.rs

[1] *Corresponding author.