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Hankel determinants of sums of consecutive Motzkin numbers

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ABSTRACT

We use combinatorial methods to evaluate Hankel determinants for the sequence of sums of consecutive t-Motzkin numbers. More specifically, we consider the following determinant:

$$\det\left(m^t_{i+j+r} + m^t_{i+j+r+1}\right)_{0\leqslant i,j\leqslant n-1},$$

where t is a real number and m^t_k is the total weight of all paths from $(0, 0)$ to $(k, 0)$ that stay above the x-axis and use up and down steps of weight one and level steps of weight t.

1. Introduction

The Motzkin number sequence, $1, 1, 2, 4, 9, 21, 51, \ldots$, enumerates many different combinatorial objects and has been the subject of several studies [1,10]. A primer on combinatorial objects counted by the Motzkin numbers should begin with one of the exercises from Stanley's combinatorial volume [19, Exercise 6.38]. The generating function, $M(z) = \sum_{k=0}^{\infty} m_k z^k$, where m_k is the k-th Motzkin number, is given by

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}$$

and satisfies $M(z) = 1 + zM(z) + z^2M^2(z)$. The Motzkin numbers are of particular interest because they often appear in variations of counting problems involving the ubiquitous Catalan number

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sequence, $c_k = \frac{1}{k+1}\binom{2k}{k}$. There are many noteworthy relationships between the Motzkin and Catalan numbers documented throughout the literature. As one of many path interpretations, c_k is the number of paths from $(0, 0)$ to $(2k, 0)$ staying above the x-axis and using steps of the form $(1, 1)$ (an *up* or northeast step) and $(1, -1)$ (a *down* or southeast step), whereas m_k is the number of paths from $(0, 0)$ to $(k, 0)$ staying above the x-axis and using up steps, down steps and steps of the form $(1, 0)$ (a *level* step). In this paper, we will refer to a weighted version of the Motzkin numbers known as t-Motzkin numbers, denoted by $\{m_k^t\}_{k\geqslant 0}$, where t is a real number. One can describe m_k^t as the total weight of all paths from $(0, 0)$ to $(k, 0)$ that stay above the x-axis and use up steps $(1, 1)$ of weight one, down steps $(1, -1)$ of weight one and level steps $(1, 0)$ of weight t. (The weight of a path is the product of the weights of all its steps.) Note that $m_k^1 = m_k$ and $m_{2k}^0 = c_k$. One can also show that $m_k^2 = c_{k+1}$ through a bijection to subdiagonal paths [15].

In this paper, we explore Hankel determinants associated with the t-Motzkin numbers. Given a sequence of real numbers $\{a_k\}_{k\geqslant 0}$, we define an $n \times n$ Hankel determinant for $\{a_k\}$ as

$$\det(a_{i+j})_{0\leqslant i,j\leqslant n-1} = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & & \cdots & a_{2n-2} \end{vmatrix}.$$

Hankel determinants for generalized Catalan numbers have been considered extensively, for example in [2,8,12,13,18], and Hankel determinants for the Motzkin number sequence, in particular, are already documented in the literature. In particular, it is well-known that for every nonnegative integer n,

$$\det(m_{i+j})_{0\leqslant i,j\leqslant n-1} = 1.$$

See for instance [21, Chapter 4] or [1]. Here, we will also consider Hankel determinants for the sequence $\{a_{k+r}\}_{k\geqslant 0}$, where the parameter r is a fixed nonnegative integer, that is,

$$\det(a_{i+j+r})_{0\leqslant i,j\leqslant n-1} = \begin{vmatrix} a_r & a_{r+1} & a_{r+2} & a_{r+3} & \cdots & a_{r+n-1} \\ a_{r+1} & a_{r+2} & a_{r+3} & a_{r+4} & \cdots & a_{r+n} \\ a_{r+2} & a_{r+3} & a_{r+4} & a_{r+5} & \cdots & a_{r+n+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{r+n-1} & a_{r+n} & \cdots & & \cdots & a_{r+2n-2} \end{vmatrix}.$$

For example, in the case of Motzkin numbers where $r = 1$, it turns out that

$$\det(m_{i+j+1})_{0\leqslant i,j\leqslant n-1} = \begin{cases} 1 & \text{if } n \equiv 1, 4 \pmod 6 \\ 0 & \text{if } n \equiv 2, 5 \pmod 6 \\ -1 & \text{if } n \equiv 0, 3 \pmod 6. \end{cases}$$

This result appears as a special case of Theorem 29 in [12] and with several other extensions, using the Gessel–Viennot–Lindström method, in [20,16].

In this paper, our interest is evaluating Hankel determinants for the sequence of sums of consecutive t-Motzkin numbers, $\{m_k^t + m_{k+1}^t\}_{k\geqslant 0}$. More precisely, we are interested in determinants of the form

$$\det\left(m_{i+j+r}^t + m_{i+j+r+1}^t\right)_{0\leqslant i,j\leqslant n-1} \tag{1}$$

where r is a nonnegative integer. The two main results of this article are the evaluation of (1) for the case when $r = 0$ and $r = 1$, which appear in Theorems 3.2 and 4.4, respectively. We consider Hankel determinants of this type partly because such determinants have already been considered for generalized Catalan numbers and produced interesting results [4,6,7,9,13,17]. Hence, the questions considered here are quite natural in light of the intimate relationship between Catalan and Motzkin numbers.

2. Theory

All determinants in this paper will be evaluated using the well-known Gessel–Viennot–Lindström method (G–V–L), which interprets determinants in terms of weighted path systems in an acyclic directed graph [11,14]. A path system in a directed graph, or an *n-path*, is an n-tuple of paths $(p_1, p_2, \ldots, p_n)$ where p_i is a path from an origin vertex o_i to a destination vertex $d_{\sigma(i)}$ for some permutation σ in S_n. The sign of an n-path P, $\text{sgn}(P)$, is the sign of the permutation by which it is induced.

The G–V–L method relies on the fact that, given an acyclic directed graph $\mathcal{D}$, if $a_{i,j}$ is the number of paths in $\mathcal{D}$ from o_i to d_j and σ is some permutation in S_n then the product $\prod_{i=1}^{n} a_{i\,\sigma(i)}$ is the number of ways to construct an n-path in $\mathcal{D}$ induced by σ. Since $\det(a_{i,j})_{0 \leqslant i,j \leqslant n-1} = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i\,\sigma(i)}$, where $\text{sgn}(\sigma)$ is the sign of the permutation σ, this determinant is the number of n-paths induced by even permutations (called *even n-paths*) minus the number of n-paths induced by odd permutations (called *odd n-paths*).

If an n-path P has two paths p_r, p_s that intersect at a vertex of $\mathcal{D}$, we say that P is *intersecting*. An sign-reversing involution exists between even intersecting n-paths and odd intersecting n-paths. This involution simplifies the evaluation of the determinant further because $\det(a_{i,j})_{0 \leqslant i,j \leqslant n-1}$ reduces to the number of even *nonintersecting* n-paths minus the number of odd *nonintersecting* n-paths. The method also works for graphs with weighted edges. The weight of an n-path is the product of the weights of its paths. If $w_{i,j}$ is the total weight of all paths in $\mathcal{D}$ from origin o_i to destination d_j and $\mathcal{J}$ is the set of all nonintersecting n-paths in $\mathcal{D}$ then

$$
\begin{aligned}
\det(w_{i,j})_{0 \leqslant i,j \leqslant n-1} &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} w_{i\,\sigma(i)} \\
&= \sum_{P \in \mathcal{J}} \text{sgn}(P) wgt(P) \\
&= (\text{the total } \textit{weight} \text{ of even } \textit{nonintersecting } n\text{-paths}) \\
&\quad - (\text{thetotal } \textit{weight} \text{ of odd } \textit{nonintersecting } n\text{-paths}).
\end{aligned}
$$

For the determinant evaluations in this paper, we choose to use the following graph $\mathcal{D}$. See Fig. 1.

Definition 2.1. $\mathcal{D}$ is the directed graph with vertex set $\{(x, y) : x, y \in \mathbb{Z}\}$ and having four types of edges:

1. *up steps*, i.e., edges of weight one of the form $((x, y), (x + 1, y + 1))$, where $y \geqslant 0$,
2. *down steps*, i.e., edges of weight one of the form $((x, y), (x + 1, y - 1))$, where $y \geqslant 0$,
3. *level steps*, i.e., edges of weight t of the form $((x, y), (x + 1, y))$ where $y \geqslant 0$ and
4. *south steps*, i.e., edges of weight one of the form $((x, 0), (x, -1))$, where $x \geqslant 1$.

Notice that a path in this graph from vertex $(i, 0)$ to $(j, 0)$ with $i \leqslant j$ is a t-Motzkin path.

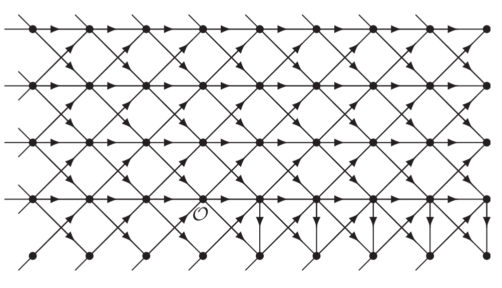

Fig. 1. The directed graph $\mathcal{D}$ used throughout. The origin $(0, 0)$ is designated by $\mathcal{O}$. Horizontal edges have weight t, all other edges have weight one.

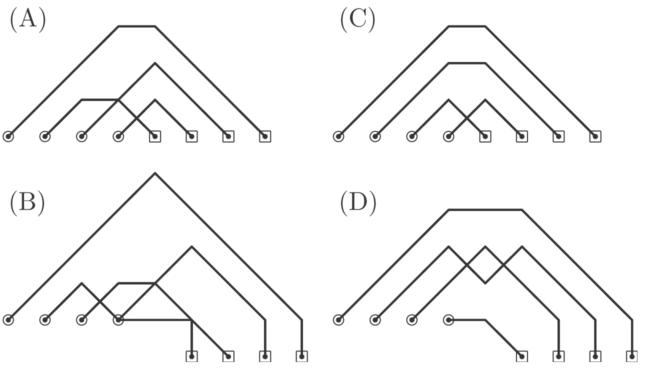

Fig. 2. (A) and (B) are examples of *intersecting* 4-paths in $\mathcal{D}$, while (C) and (D) are *nonintersecting*. (Origin vertices are encircled, destination vertices are boxed.) The 4-paths on the right are nonintersecting since no two paths intersect at a vertex of $\mathcal{D}$.

See Fig. 2 for examples of n-paths (both intersecting and nonintersecting) in $\mathcal{D}$ with different sets of origins and destinations.

In each subsequent section of this paper, we evaluate the Hankel determinants for sums of consecutive t-Motzkin numbers $\{m^t_{k+r} + m^t_{k+r+1}\}_{k\geqslant 0}$ when $r = 0$ and $r = 1$. To evaluate these determinants, it is helpful to consider first Hankel determinants for the original t-Motzkin sequence $\{m^t_{k+r+1}\}_{k\geqslant 0}$, that is,

$$\det\left(m^t_{i+j+r+1}\right)_{0\leqslant i,j\leqslant n-1}, \tag{2}$$

(in particular for the case when $r = 0, 1$) and subsequently their relationship to the sums of consecutive t-Motzkin numbers counterpart in (1). The determinant in (2), for the case when $r = 0, 1$, has been treated in [21,12]. We follow the same treatment of this determinant here using Viennot's combinatorial model, considering only nonintersecting n-paths in $\mathcal{D}$ with particular sets of origins and destinations. For the sake of simplicity later, these particular n-paths are described in the following definitions.

Definition 2.2. $\mathcal{G}^{t,r}_n$ is the set of all *nonintersecting* n-paths in $\mathcal{D}$ with designated origins $(-n+1, 0)$, $(-n+2, 0), \ldots, (0, 0)$ and destinations $(r+1, 0), (r+2, 0), \ldots, (r+n, 0)$.

Notice that the n-path in Fig. 2(C) is an element of $\mathcal{G}^{t,0}_4$. Now since, for $i, j \geqslant 0$, the total weight of all paths from origin $(-i, 0)$ to destination $(r+1+j, 0)$ in $\mathcal{D}$ is $m^t_{i+j+r+1}$, we will use $\mathcal{G}^{t,r}_n$ to evaluate (2). Now, we introduce another set of nonintersecting paths.

Definition 2.3. $\mathcal{H}^{t,r}_n$ is the set of all nonintersecting n-paths in $\mathcal{D}$ with designated origins $(-n+1, 0), (-n+2, 0), \ldots, (0, 0)$ and destinations $(r+1, -1), (r+2, -1), \ldots, (r+n, -1)$.

For example, the n-path in Fig. 2(D) is an element of $\mathcal{H}^{t,1}_4$. By contrast, $\mathcal{H}^{t,r}_n$ will be used to evaluate (1), since the total weight of all paths in $\mathcal{D}$ from origin $(-i, 0)$ to destination $(r+1+j, -1)$ is $m^t_{i+j+r} + m^t_{i+j+r+1}$.

All n-paths in $\mathcal{G}^{t,r}_n$ and $\mathcal{H}^{t,r}_n$ are nonintersecting, so for any n-path contained in these sets, no two paths may share a common vertex. However, it is possible for edges of two different paths to "cross" in the plane without creating a point of intersection. This is made precise in the following definition.

Definition 2.4. A *crossing* in an n-path P in $\mathcal{D}$ is the existence of an up step and a down step whose endpoints occupy all four vertices of a 1×1 square of the form $\{(a, b), (a+1, b), (a+1, b+1),$

$(a, b + 1) : a, b \in \mathbb{Z}\}$. (The up step and down step in this case are necessarily from two different paths in P.) The *(vertical) position* of this crossing is the value b corresponding to its height above the x-axis.

For example, Fig. 2(C) has a crossing at position zero and Fig. 2(D) has two crossings at position one.

3. The case when $r = 0$

Our goal in this section is to evaluate the Hankel determinant for sums of consecutive t-Motzkin numbers, that is,

$$\det\left(m^t_{i+j} + m^t_{i+j+1}\right)_{0 \leqslant i,j \leqslant n-1}. \tag{3}$$

It will turn out that this determinant is a Chebychev polynomial of the second kind. This fact will be established in two ways: (1) by establishing a recurrence relation for the determinant and (2) by a combinatorial proof involving tilings. Recall that the Chebychev polynomials of the second kind are defined by $U_0(x) = 1, U_1(x) = 2x$ and $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$, for $n \geqslant 2$. For convenience and simplicity later, we use $S_n(x)$ to denote the polynomial $U_n(x/2)$, which will be noted in the following definition.

Definition 3.1. $S_0(x) = 1,\ S_1(x) = x,\ S_n(x) = xS_{n-1}(x) - S_{n-2}(x),$ for $n \geqslant 2$.

Here is the main result of this section.

Theorem 3.2 (Recursive version)**.** *Let* $H^t_n = \left(m^t_{i+j} + m^t_{i+j+1}\right)_{0 \leqslant i,j \leqslant n-1}$. *Then for* $n \geqslant 2$

$$\left|H^t_n\right| = (t+1)\left|H^t_{n-1}\right| - \left|H^t_{n-2}\right|, \text{ with } \left|H^t_0\right| = 1 \text{ and } \left|H^t_1\right| = t+1.$$

Observe that the determinant we wish to evaluate in (3) is equal to the total weight of even nonintersecting n-paths in $\mathcal{H}^{t,0}_n$ minus the total weight of odd nonintersecting n-paths in $\mathcal{H}^{t,0}_n$. (To simplify notation in this section, we will let $\mathcal{H}^{t,0}_n = \mathcal{H}^t_n$ and $\mathcal{G}^{t,0}_n = \mathcal{G}^t_n$.) However, it is instructive to consider the analogous quantity in $\mathcal{G}^t_n$, in other words

$$\det\left(m^t_{i+j+1}\right)_{0 \leqslant i,j \leqslant n-1},$$

as $\mathcal{H}^t_n$ and $\mathcal{G}^t_n$ share obvious graphic similarities and the latter determinant is known to bear a recursion similar to the one in Theorem 3.2. In [12, Theorem 29] and [20, Proposition 2.2], one can find proofs of the following result.

Theorem 3.3. *If* $G^t_n = \left(m^t_{i+j+1}\right)_{0 \leqslant i,j \leqslant n-1}$ *then for* $n \geqslant 2$

$$\left|G^t_n\right| = t\left|G^t_{n-1}\right| - \left|G^t_{n-2}\right|, \text{ with } \left|G^t_0\right| = 1 \text{ and } \left|G^t_1\right| = t. \tag{4}$$

Our approach then to proving Theorem 3.2 is to show that $|H^t_n| = |G^{t+1}_n|$ by establishing a correspondence between $\mathcal{G}^{t+1}_n$ and $\mathcal{H}^t_n$. The correspondence will be based on an equivalence relation on $\mathcal{H}^t_n$, which we establish in Lemma 3.4.

Lemma 3.4. *Let $\sim$ be the relation on $\mathcal{H}^t_n$ defined as follows: $Q \sim Q'$ if and only if the number and position of all crossings in Q equals the number and position of all crossings in Q'. Then $\sim$ is an equivalence relation.*

Proof of Theorem 3.2. Given $P \in \mathcal{G}^{t+1}_n$, we define the map $C : \mathcal{G}^{t+1}_n \to \mathcal{H}^t_n$ by letting $C(P)$ be the n-path in $\mathcal{H}^t_n$ obtained by adding a south step $(0, -1)$ at each of the destinations in P. See Fig. 3.

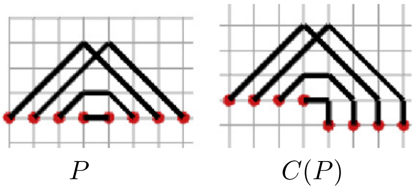

Fig. 3. An example of an 4-path $P \in \mathcal{G}_4^{t+1}$ and its image $C(P) \in \mathcal{H}_4^t$ under the map C. The level steps in P have weight $t+1$ while the level steps in $C(P)$ have weight t.

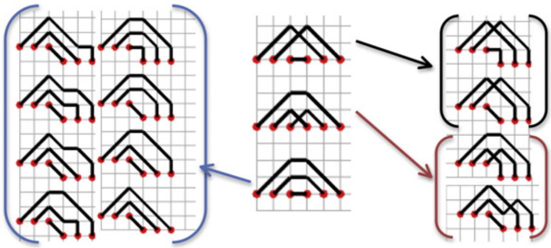

Fig. 4. The three 3-paths in the center are elements of $\mathcal{G}_3^2$ and have weights -2, -2 and 8, respectively. Each 3-path in the center is mapped to a class of 3-paths in $\mathcal{H}_3^1$ with equal weight.

Clearly, C is an injection and $\text{sgn}(P) = \text{sgn}(C(P))$. Let $\overline{\mathcal{H}_n^t}$ denote the set of all equivalence classes in $\mathcal{H}_n^t$, that is, $\overline{\mathcal{H}_n^t} = \{[Q] : Q \in \mathcal{H}_n^t\}$. Define the map $\overline{C} : \mathcal{G}_n^{t+1} \to \overline{\mathcal{H}_n^t}$ by $\overline{C}(P) = [C(P)]$.

Claim 1: $\overline{C}$ is a bijection.

Proof of Claim 1. To see that $\overline{C}$ is injective, assume that $P \neq P'$. Then P and P' must differ in either the number or vertical positions of their crossings, and hence the same will be true of $C(P)$ and $C(P')$. Therefore, $C(P) \nsim C(P')$ and $[C(P)] \neq [C(P')]$. To see that $\overline{C}$ is onto, consider $[Q]$ in $\overline{\mathcal{H}_n^t}$. There is a unique n-path Q^* in $[Q]$ such that all edges incident to destinations are south steps. Since all edges to destinations in Q^* are south steps, all steps in Q^* occurring to the right of the line $x = 1$ must be down steps, and all steps to the left of the line $x = 0$ must be up steps. Therefore, any crossing in Q^* is forced to occur in the central column between the lines $x = 0$ and $x = 1$. Let $P^* \in \mathcal{G}_n^{t+1}$ be the n-path obtained from Q^* by deleting the n south steps incident to its destinations and increasing by one the weight of each level step. Then we have $[Q] = [Q^*] = [C(P^*)] = \overline{C}(P^*)$, so $\overline{C}$ is onto.

Claim 2: If $P \in \mathcal{G}_n^{t+1}$, then $wgt(P) = \sum_{Q \in [C(P)]} wgt(Q)$.

This claim is illustrated in Fig. 4 for the case where $n = 3$ and $t = 1$.

Proof of Claim 2. Suppose P has exactly k level steps, where $0 \leqslant k \leqslant n$. On one hand, $wgt(P) = (t+1)^k$. On the other, we consider the n-paths in $[C(P)]$. Notice $C(P)$ must also have k level steps located between the lines $x = 0$ and $x = 1$, and the number of crossings in $C(P)$ is $\frac{n-k}{2}$, an integer. If $Q \sim C(P)$, then Q must have the same number of crossings as $C(P)$, which means that Q has at most k level steps. (If not and $Q \in [C(P)]$ has x level steps, where $x > k$, then the number of paths in $C(P)$ would be $x + 2\left(\frac{n-k}{2}\right) = x + n - k > n$.) Therefore, we can say that each Q in $[C(P)]$ has i level steps, where $0 \leqslant i \leqslant k$. If Q has exactly i level steps, then its weight is t^i. There are $\binom{k}{i}$ n-paths in $[C(P)]$ having exactly i level steps. Hence,

$$\sum_{Q \in [C(P)]} wgt(Q) = \sum_{i=0}^{k} \binom{k}{i} t^i = (t+1)^k.$$

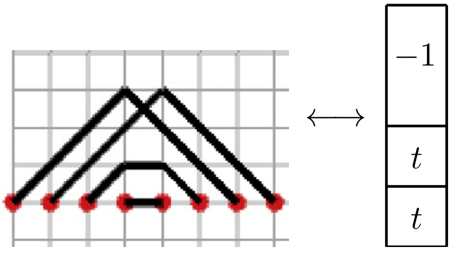

Fig. 5. A 4-path in $\mathcal{G}_4^t$ and its associated tiling of a 4×1 board with squares of weight t and dominoes of weight -1.

Suppose $\overline{\mathcal{H}_n^t} = \{[Q_1], [Q_2], \ldots, [Q_s]\}$ so that $s = |\overline{\mathcal{H}_n^t}| = |\mathcal{G}_n^{t+1}|$.

$$\begin{aligned}\left|H_n^t\right| &= \sum_{Q\in\mathcal{H}_n^t} \operatorname{sgn}(Q)wgt(Q)\\ &= \sum_{i=1}^{s}\sum_{Q\in[Q_i]} \operatorname{sgn}(Q)wgt(Q)\\ &= \sum_{i=1}^{s}\sum_{Q\in[C(P_i)]} \operatorname{sgn}(C(P_i))wgt(Q)\\ &= \sum_{i=1}^{s}\operatorname{sgn}(C(P_i))\sum_{Q\in[C(P_i)]} wgt(Q)\\ &= \sum_{i=1}^{s}\operatorname{sgn}(P_i)wgt(P_i)\\ &= \left|G_n^{t+1}\right|. \qquad \square\end{aligned}$$

Remark. At this point, we will detail a useful combinatorial interpretation of (4). While this interpretation is already known and documented in the literature, for example in [21, Chapter 1–2] and more recently in [5, Theorem 3], we describe it here because it will be useful in the following section when we consider what happens as the parameter r increases.

The recurrence in (4) describes the total weight of all the tilings of a $n \times 1$ board using dominoes of weight -1 and squares of weight t. (The weight of a tiling is the product of the weights of all its tiles.) We can draw a one-to-one correspondence between these tilings and the signed n-paths in $\mathcal{G}_n^t$. Since an n-path in $\mathcal{G}_n^t$ is nonintersecting, it is completely determined by the steps that occur in the $n \times 1$ rectangle between the lines $x = 0, x = 1, y = 0$ and $y = n$. All steps that occur to left of this rectangle are up steps and all those to the right are down steps (all of which have weight 1). If we divide this rectangle into n 1×1 blank spaces, we find that there are only two ways that the n-path may occupy these spaces: (1) with a crossing which requires one blank space above it in order to avoid an intersection, and (2) with a level step. Since every crossing in the n-path effectively uses two blank spaces and contributes a factor of -1 to the signed weight of the n-path, we can associate each crossing with a 2×1 domino of weight -1. Similarly, each level step in the rectangle is associated with a 1×1 square of weight t. See Fig. 5.

The sum of the weights of the tilings of a $n \times 1$ board using dominoes of weight -1 and squares of weight t is given by $S_n(t)$. This implies the following alternate form of (4), which appears as a special case of Theorem 29 in [12]

$$\left|G_n^t\right| = S_n(t), \tag{5}$$

Consequently, we have an alternate, nonrecursive version of the Theorem 3.2.

Theorem 3.2 (Nonrecursive version)**.** $\det\left(m^t_{i+j} + m^t_{i+j+1}\right)_{0 \leqslant i,j \leqslant n-1} = S_n(t+1)$.

Let us summarize the approach used to establish Theorem 3.2, the main result so far. In order to evaluate the Hankel determinant associated with the sequence of consecutive sums of t-Motzkin numbers with parameter $r = 0$, we first considered the Hankel determinant associated with the original t-Motzkin sequence, with the expectation that the set of n-paths associated with each have a quantifiable relationship. The latter determinant happened to have a combinatorial interpretation in terms of tilings which made its evaluation straightforward. The work then (i.e., the proof of Theorem 3.2) lies in qualifying the essential difference between the sets of n-paths associated with t-Motzkin numbers and consecutive sums of t-Motzkin numbers, that is, between $\mathcal{G}^t_n$ and $\mathcal{H}^t_n$. In the next section, a similar approach is taken as we increase the parameter r by one and see how the situation changes.

4. The case when $r = 1$

In this section, our goal is to evaluate the following determinant:

$$\det\left(m^t_{i+j+1} + m^t_{i+j+2}\right)_{0 \leqslant i,j \leqslant n-1}. \tag{6}$$

But first, as the approach in the previous section suggests, we consider a simpler, related problem, which is the following determinant:

$$\det\left(m^t_{i+j+2}\right)_{0 \leqslant i,j \leqslant n-1}.$$

This determinant evaluation is already known. The evaluation of an even more general form of this determinant is presented in [12]. While the next result is not new, in order to help motivate the evaluation of the desired determinant in (6), we include here a formal proof of this special case using the same approach described by Krattenthaler in the remarks following [12, Theorem 29] and the square–domino tiling interpretation referenced in the previous section. So to this end, let us consider $\mathcal{G}^{t,1}_n$, the set of all nonintersecting t-Motzkin n-paths, with origins $(-n+1, 0), (-n+2, 0), \ldots, (0, 0)$ and destinations $(2, 0), (3, 0), \ldots, (n+1, 0)$. We know that $\det(m^t_{i+j+2})_{0 \leqslant i,j \leqslant n-1}$ is given by $\sum_{P \in \mathcal{G}^{t,1}_n} \operatorname{sgn}(P) wgt(P)$. Using the square–domino interpretation of n-paths described in the previous section, we confirm with a formal proof, as indicated in [12, Theorem 29], that the Hankel determinant associated with $\{m^t_{k+2}\}_{k \geqslant 0}$ is a sum of squares of Chebychev polynomials.

Theorem 4.1. $\det\left(m^t_{i+j+2}\right)_{0 \leqslant i,j \leqslant n-1} = \sum_{k=0}^{n} [S_k(t)]^2$.

Proof. We assert that any member P of the set $\mathcal{G}^{t,1}_n$, along with its sign, can be associated with a pair of weighted tilings of a $n \times 1$ board with squares of weight t and dominoes of weight -1. Given an n-path P in $\mathcal{G}^{t,1}_n$, observe that P is completely determined by the steps of the n-path used in the $n \times 2$ rectangle between $x = 0, x = 2, y = 0$ and $y = n$. (All steps to the left of this rectangle are up steps and those to the right are down steps.) Now, consider all paths in P from origin $(-i, 0)$, where $0 \leqslant i \leqslant n-1$, which have the form $U^{i+1}D^{i+1}$ and the property that its neighboring path from $(-i-1)$ has the form $U^{i+2}D^{i+2}$. Define the *index* of P to be the number of such paths that P contains. (So the index ranges between 0 and n.) See Fig. 6. Observe that if P has index k, then all steps in P in the $(n-k) \times 2$ rectangle between the lines $x = 0, x = 2, y = 0$ and $y = n-k$ are configured as crosses or level steps. Using the square–domino tiling interpretation described in the remarks preceding equation (5), we can associate P uniquely to a pair of weighted tilings of a $(n-k) \times 1$ board with squares of weight t and dominoes of weight -1. Hence, the total signed weight of all n-paths with index k is the total weight of all pairs of tilings of a $(n-k) \times 1$ board, which equals $S_{n-k}(t)^2$. Summing over all values of the index k produces the result. □

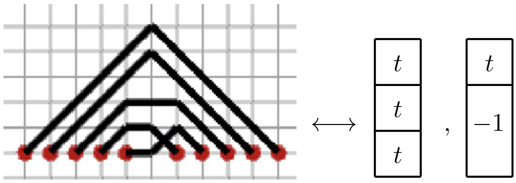

Fig. 6. An 5-path in $\mathcal{G}_5^{t,1}$ with index 2 and its corresponding pair of tilings of a 3×1 board with squares of weight t and dominoes of weight -1.

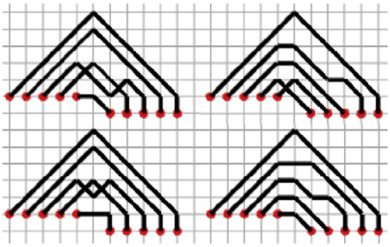

Fig. 7. Four 5-paths in $\mathcal{H}_5^{t,1}$. The two on the left have 2 outer paths, while the two on the right have 1 outer path.

Notice that when $t = 1$, Theorem 4.1 implies the following result.

Corollary 4.2. $\det(m_{i+j+2})_{0 \leqslant i,j \leqslant n-1} = \begin{cases} n-j+1, & \text{if } n = 3j \\ n-j+1, & \text{if } n = 3j+1 \\ n-j, & \text{if } n = 3j+2. \end{cases}$

To evaluate the desired determinant in (6), we now consider $\mathcal{H}_n^{t,1}$ with origin vertices $(-n+1, 0), (-n+2, 0), \dots, (0, 0)$ and destination vertices $(2, -1), (3, -1), \dots, (n+1, -1)$. The determinant we seek is $\sum_{P \in \mathcal{H}_n^{t,1}} \operatorname{sgn}(P) wgt(P)$. As a way of organizing the elements of $\mathcal{H}_n^{t,1}$, we want to extend the notion of index, as described in the proof of the previous theorem, to $\mathcal{H}_n^{t,1}$. We make this precise in the following definition.

Definition 4.3. Given an n-path $P \in \mathcal{H}_n^{t,1}$, we say that the path $p \in P$ leaving origin $(-i, 0)$, where $0 \leqslant i \leqslant n-1$, is an *outer path* if:

1. p goes from $(-i, 0)$ to $(i+2, -1)$,
2. p has the form $U^{i+1}D^{i+1}S$, where U represents an up step, D represents a down step and S represents a south step, and
3. if $0 \leqslant i < n-1$, then the path from $(-i-1, 0)$ to $(i+3, -1)$ is also an outer path.

In other words, any path located directly "above" an outer path is also an outer path. See Fig. 7. We now present the main and final result of this section.

Theorem 4.4. $\det\left(m_{i+j+1}^t + m_{i+j+2}^t\right)_{0 \leqslant i,j \leqslant n-1} = \sum_{k=0}^{n} S_k(t) S_k(t+1).$

Proof. Consider the set $\mathcal{H}_n^{t,1}$. Any n-path P in $\mathcal{H}_n^{t,1}$ can be categorized according to the number of *outer* paths P contains. Given P with k outer paths, we ignore the outer paths and consider just the remaining $(n-k)$-path in P. We split this $(n-k)$-path into two sections X and Y at the line $x = 1$. To the left of $x = 1$ is the section X of the $(n-k)$-path which corresponds to a unique element of $\mathcal{G}_{n-k}^t$. See Fig. 8. Similarly, the section Y located to the right of $x = 1$ corresponds to a unique

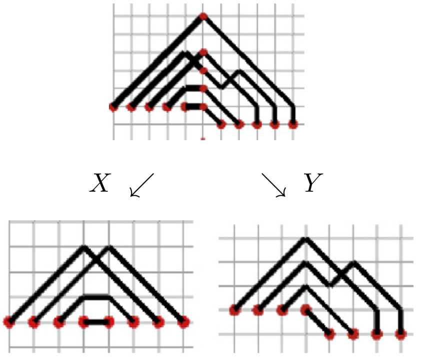

Fig. 8. Top: A 5-path in $\mathcal{H}_5^{t,1}$ with 1 outer path. The darkened section to the left of the line $x = 1$ is X and Y is the lighter section to the right. Bottom: After removing the outer path, X and Y correspond to these unique 4-paths in $\mathcal{G}_4^t$ and $\mathcal{H}_4^t$, respectively.

$(n-k)$-path in $\mathcal{H}_{n-k}^t$. So, by (5) and Theorem 3.2, the signed total weight of all n-paths with k outer paths is $|G_{n-k}^t||H_{n-k}^t| = S_{n-k}(t)S_{n-k}(t+1)$. Hence, the signed total weight of all n-paths in $\mathcal{H}_n^{t,1}$ is

$$\sum_{k=0}^{n} S_{n-k}(t)S_{n-k}(t+1) = \sum_{k=0}^{n} S_k(t)S_k(t+1). \qquad \square$$

4.1. Conclusion

Having evaluated

$$\det\left(m_{i+j+r}^t + m_{i+j+r+1}^t\right)_{0 \leqslant i,j \leqslant n-1} \tag{7}$$

for the case when $r = 0, 1$, we leave open the evaluation of this determinant for all $r > 0$. While the methods presented here offer promising approaches to the cases where $r \geqslant 2$, it is not completely obvious how they generalize. In particular, for the case where $r = 2$, it is unclear how to partition $H_n^{t,2}$ or, furthermore, how to interpret an n-path in $H_n^{t,2}$ as a string of tilings of weighted dominos and squares. Nonetheless, we offer the approaches used here as a potential basis for the successful evaluation of (7) and other related determinants.

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