EULERIAN NUMBERS AND POLYNOMIALS

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1. Introduction. Following Euler [6, pp. 487-491], we may put

$$\frac{1-\lambda}{e^x-\lambda} = \sum_{n=0}^{\infty} H_n \frac{x^n}{n!} \qquad (\lambda \neq 1), \tag{1.1}$$

where $H_n = H_n[\lambda]$ is a rational function of λ; indeed

$$R_n = R_n[\lambda] = (\lambda-1)^n H_n[\lambda] \tag{1.2}$$

is a polynomial in λ of degree $n-1$ with integral coefficients. If we put

$$R_n = \sum_{s=1}^{n} A_{ns}\lambda^{s-1} \qquad (n \geqq 1), \tag{1.3}$$

then the first few values of A_{ns} are given by the following table, where n denotes the row and s the column;

(1.4)

1					
1	1				
1	4	1			
1	11	11	1		
1	26	66	26	1	
1	57	302	302	57	1

Alternatively, Worpitzky [15] showed that the A_{ns} may be defined by means of

$$x^n = \sum_{s=1}^{n} A_{ns} \binom{x+s-1}{n}. \tag{1.5}$$

The rational functions H_n were studied in great detail by Frobenius [7], who was particularly interested in their relationship to the Bernoulli numbers. More recently Vandiver [14] has also made use of this relationship to obtain new properties of the Bernoulli numbers. Other recent occurrences are [1], [2], [12]; generalizations occur in [4], [5], [13]. In view of the long history of H_n and A_{ns} it is rather curious that, on the whole, these quantities are not very well known. Indeed an examination of Mathematical Reviews for the past ten years will indicate that they have been frequently rediscovered. Actually there is no detailed discussion of H_n in any book. On the other hand, Riordan, in his recent book [11], does develop a few basic properties and indicates the connection of A_{ns} with certain combinational problems.

The present paper is mainly expository. We include numerous properties of H_n and the related polynomial

$$H_n(u \mid \lambda) = \sum_{r=0}^{n} \binom{n}{r} H_r u^{n-r}, \tag{1.6}$$

indicate the connection with Bernoulli numbers and polynomials and finally obtain some arithmetic properties of H_n. For the combinatorial applications the reader is referred to Riordan's book [11]. The H_n also occur in certain criteria for Fermat's last theorem; this is discussed at length in Bachmann's book [3] and will not be considered in the present paper.

2. The defining relation (1.1) is evidently equivalent to

$$(H+1)^n = \lambda H_n \qquad (n > 0), \quad H_0 = 1, \tag{2.1}$$

where after expansion of the left member, H^n is replaced by H_n; we shall use this convention frequently. If $f(z)$ is an arbitrary polynomial in z, (2.1) implies

$$f(H+1) = \lambda f(H) + (1-\lambda) f(0). \tag{2.2}$$

In particular, for $f(z) = \binom{z}{m}$, we get

$$\binom{H+1}{m} = \lambda \binom{H}{m} \qquad (m \geqq 1), \tag{2.3}$$

which implies

$$(\lambda-1)\binom{H}{m} = \binom{H}{m-1} \qquad (m \geqq 1). \tag{2.4}$$

Repeated application of (2.3) gives

$$(\lambda-1)^r \binom{H}{m} = \binom{H}{m-r} \qquad (m \geqq r);$$

in particular we have

$$(\lambda-1)^m \binom{H}{m} = 1. \tag{2.5}$$

It also follows from (2.2) and (2.3) that

$$\binom{H+r}{m} = \lambda^r\binom{H}{m} \qquad (m \geqq r); \tag{2.6}$$

in particular by (2.5)

$$\binom{H+m}{m} = \frac{\lambda^m}{(1-\lambda)^m}. \tag{2.7}$$

Again, if $f(z)$ is an arbitrary polynomial of degree n, we recall that

$$f(z) = \sum_{r=0}^{n} \binom{z}{r}\Delta^r f(0),$$

where

$$\Delta f(z) = f(z+1) - f(z), \quad \Delta^r f(z) = \Delta^{r-1}f(z+1) - \Delta^{r-1}f(z).$$

Using (2.5) we get

$$f(H) = \sum_{r=0}^{n} (\lambda-1)^{-r}\Delta^r f(0). \tag{2.8}$$

Since

$$\Delta^r f(0) = \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s}f(s),$$

we may write

$$f(H) = \sum_{r=0}^{n} (\lambda-1)^{-r} \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s}f(s). \tag{2.9}$$

It is perhaps of interest to mention that (2.9) can also be obtained as follows from (1.1). For $|x|$ sufficiently small we have

$$\frac{1-\lambda}{e^x-\lambda} = \left(1 - \frac{e^x-1}{\lambda-1}\right)^{-1} = \sum_{r=0}^{\infty} (\lambda-1)^{-r}(e^x-1)^r$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{r=0}^{n} (\lambda-1)^{-r} \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s}s^n,$$

so that

$$H_n = \sum_{r=0}^{n} (\lambda-1)^{-r} \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s} s^n \tag{2.10}$$

and (2.9) follows at once.

Returning again to (1.1) it is easily verified that

$$R_n[\lambda] = \lambda^{n-1} R_n[\lambda^{-1}]; \tag{2.11}$$

also differentiation yields

$$R_{n+1} = (n+1)\lambda R_n + (1-\lambda)\frac{d}{d\lambda}(\lambda R_n), \tag{2.12}$$

where R_n is defined by (1.2). From (2.11) and (2.12) we immediately obtain

$$A_{ns} = A_{n,n-s+1} \tag{2.13}$$

and

$$A_{n+1,s} = sA_{n,s-1} + (n-s+2)A_{ns}. \tag{2.14}$$

By means of (2.14) one can easily extend the table (1.4). A convenient check is furnished by the formula

$$R_n[1] = \sum_{s=1}^{n} A_{ns} = n! \qquad (n \geqq 1).$$

We also note that $R_{2n}[-1] = 0$ for $n \geqq 1$.

Frobenius remarks that it follows from (2.12) that the $n-1$ roots of $R_n[\lambda] = 0$ are real, negative and distinct; also for each root λ_0 the reciprocal λ_0 is also a root. Moreover the roots of $R_{n+1}[\lambda] = 0$ are separated by the roots of $R_n[\lambda] = 0$. In the next place, by (2.10)

$$R_n = \sum_{r=0}^{n} (\lambda-1)^{n-r} \sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} j^n$$

$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} (-1)^{n-r-s}\binom{n-r}{s}\lambda^s \sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} j^n$$

$$= \sum_{s=0}^{n} \lambda^s \sum_{j=0}^{n-s} (-1)^{n-s-j} j^n \sum_{r=j}^{n-s} \binom{n-r}{s}\binom{r}{j}$$

$$= \sum_{s=0}^{n} \lambda^s \sum_{j=0}^{n-s} (-1)^{n-s-j} \binom{n+1}{s+j+1} j^n.$$

Hence, by (1.3) and (2.13), we get

$$A_{ns} = \sum_{j=0}^{s} (-1)^j \binom{n+1}{j} (s-j)^n, \tag{2.15}$$

a formula due to Euler.

Since

$$x^n = \sum_{r=0}^{n} \binom{x}{r} \Delta^r f(0) = \sum_{r=0}^{n} \Delta^{n-r} \binom{x}{n} \Delta^r f(0),$$

it follows from (2.10) and (2.15)

$$x^n = R_n(1+\Delta)\binom{x}{n} = \sum_{s=1}^{n} A_{ns}(1+\Delta)^{s-1}\binom{x}{n} = \sum_{s=1}^{n} A_{ns}\binom{x+s-1}{n},$$

which establishes (1.5). This proof is taken from Frobenius.

3. We now consider the polynomial

$$H_n(u) = H_n(u \mid \lambda) = (u+H)^n \tag{3.1}$$

defined by (1.6). We evidently have the generating function

$$\frac{1-\lambda}{e^x-\lambda} e^{xu} = \sum_{n=0}^{\infty} \frac{x^n}{n!} H_n(u). \tag{3.2}$$

It follows from (3.2) that

$$H_n(u+1) - \lambda H_n(u) = (1-\lambda)u^n \tag{3.3}$$

Moreover (3.3) uniquely determines the polynomial $H_n(u)$. If $f(z)$ is an arbitrary polynomial in z then by (3.3)

$$f(u+1+H) - \lambda f(u+H) = (1-\lambda) f(u),$$

from which it follows that for given $f(z)$ the difference equation

(3.4) $$g(u+1) - \lambda g(u) = (1-\lambda)f(u)$$

has the unique solution

$$g(u) = g(u \mid \lambda) = f(u+H).$$

It follows at once from (3.2) that

(3.5) $$H_n'(u) = nH_{n-1}(u)$$

and generally

$$H_n^{(r)}(u) = \frac{n!}{(n-r)!}H_{n-r}(u) \qquad (n \geqq r),$$

which implies

(3.6) $$H_n(u+v) = \sum_{r=0}^{n} \binom{n}{r} u^{n-r} H_n(v).$$

If we differentiate (3.2) with respect to λ we get

(3.7) $$H_{n+1}(u \mid \lambda) + \lambda \frac{\partial}{\partial\lambda} H_n(u \mid \lambda) = \left(u - \frac{1}{1-\lambda}\right) H_n(u \mid \lambda),$$

which reduces to (2.12) for $u = 0$. We also note that (3.2) implies

(3.8) $$H_n(1-u \mid \lambda^{-1}) = (-1)^n H_n(u, \lambda).$$

We remark that by (2.10) and (3.1) we have

(3.9) $$H_n(u) = \sum_{r=0}^{n} (\lambda-1)^{-r} \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} (u+s)^n.$$

Again, if we put

(3.10) $$f_n(u, m) = \sum_{j=0}^{m-1} (u+j)^n \lambda^{m-1-j}$$

then it follows from (3.3) that

(3.11) $$f_n(u, m) = \frac{H_n(u+m) - \lambda^m H_n(u)}{1-\lambda}.$$

The polynomial $f_n(0, m)$ is usually called a Mirimonoff polynomial; a more general polynomial is discussed by Vandiver [14], see also Bachmann [1, p. 117].

To get a *multiplication* theorem for $H_n(u)$ we consider

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{r=0}^{m-1}\lambda^{m-1-r}H_n\left(u+\frac{r}{m}\mid\lambda^m\right)=\frac{1-\lambda^m}{e^x-\lambda^m}e^{xu}\sum_{r=0}^{m-1}\lambda^{m-1-r}e^{rx/m}$$

$$=\frac{1-\lambda^m}{e^{x/m}-\lambda}e^{xu}$$

$$=\frac{1-\lambda^m}{1-\lambda}\sum_{n=0}^{\infty}\frac{x^n m^{-n}}{n!}H_n(mu\mid\lambda),$$

which yields

$$m^n\sum_{r=0}^{m-1}\lambda^{m-1-r}H_n\left(u+\frac{r}{m}\mid\lambda^m\right)=\frac{1-\lambda^m}{1-\lambda}H_n(mu\mid\lambda). \tag{3.12}$$

An interesting special case of (3.12) is obtained by taking $\lambda=\zeta$, where $\zeta^{m-1}=1$, $\zeta\neq 1$. Then (3.12) reduces to

$$m^n\sum_{r=0}^{m-1}\zeta^{-r}H_n\left(u+\frac{r}{m}\mid\zeta\right)=H_n(u\mid\zeta). \tag{3.13}$$

Nielsen [9, p. 54] has proved that the multiplication theorems for the Bernoulli and Euler polynomials characterize those polynomials. Suppose now that $f(u\mid\lambda)$ is a polynomial in u of degree n that satisfies the equation

$$m^n\sum_{r=0}^{m-1}\lambda^{m-1-r}f\left(u+\frac{r}{m}\mid\lambda^m\right)=\frac{1-\lambda^m}{1-\lambda}f(mu\mid\lambda) \tag{3.14}$$

for some value of $m>1$. Put

$$f(u\mid\lambda)=\sum_{s=0}^{n}A_s[\lambda]H_s(u\mid\lambda).$$

Then by (3.12) and (3.14)

$$\sum_{s=0}^{n} A_s[\lambda^m]m^{n-s}H_s(mu \mid \lambda) = \sum_{s=0}^{n} A_s[\lambda]H_s(mu \mid \lambda).$$

This requires that

$$m^{n-s}A_s[\lambda^m] = A_s[\lambda] \qquad (0 \leqq s \leqq n). \tag{3.15}$$

Now assume that (3.14) holds for *two* values of $m>1$, say m_1 and m_2. Then it is clear that (3.15) becomes

$$A_n[\lambda^m] = A_n[\lambda], \quad A_s[\lambda] = 0 \qquad (0 \leqq s < n).$$

Therefore

$$f(u \mid \lambda) = A_n[\lambda]H_n(u \mid \lambda).$$

For (3.13) the situation is somewhat simpler. If $g(u \mid \zeta)$ is a polynomial in u satisfying the equation

$$m^n \sum_{r=0}^{m-1} \zeta^{-r} g\left(u+\frac{r}{m} \mid \zeta\right) = g(mu \mid \zeta), \tag{3.16}$$

where $\zeta^{m-1} = 1$, $\zeta \neq 1$, then if we put

$$g(u \mid \zeta) = \sum_{s=0}^{n} A_s(\zeta)H_n(u \mid \zeta)$$

and assume that (3.16) holds for *one* value of $m>1$, then it follows readily that

$$g(u \mid \zeta) = A_n(\zeta)H_n(u \mid \zeta),$$

where $A_n(\zeta)$ is arbitrary.

It may be of interest to mention also an *addition* theorem satisfied by $H_n(u \mid \lambda)$. Since

$$\frac{1-\lambda}{e^x-\lambda}e^{xu}\frac{1+\lambda}{e^x+\lambda}e^{xv} = \frac{1-\lambda^2}{e^{2x}-\lambda^2}e^{x(u+v)},$$

it follows at once that

$$\sum_{r=0}^{n} \binom{n}{r} H_r(u \mid \lambda)H_{n-r}(v \mid -\lambda) = H_n(u+v \mid \lambda^2). \tag{3.17}$$

We note also that from the identity

$$\frac{(1-\lambda)(1+\lambda)}{e^x-\lambda}-\frac{(1-\lambda)(1+\lambda)}{e^x+\lambda}=\frac{2\lambda(1-\lambda^2)}{e^{2x}-\lambda^2}$$

follows

$$(3.18)\qquad (1+\lambda)H_n(u\mid\lambda)-(1-\lambda)H_n(u\mid-\lambda)=2^{n+1}\lambda H_n\left(\frac{u}{2},\lambda^2\right),$$

while from

$$\frac{2\lambda-1}{(e^x-\lambda)(e^x-1+\lambda)}=\frac{1}{e^x-\lambda}-\frac{1}{e^x-1+\lambda}$$

follows

$$(3.19)\qquad (2\lambda-1)\sum_{r=0}^{n}\binom{n}{r}H_r(u\mid\lambda)H_{n-r}(v\mid 1-\lambda)=\lambda H_n(u+v\mid\lambda)-(1-\lambda)H_n(u+v\mid 1-\lambda).$$

4. It is familiar that the Bernoulli polynomial $B_n(u)$ may be defined by [10, Chapter 2]

$$(4.1)\qquad \frac{xe^{xu}}{e^x-1}=\sum_{n=0}^{\infty}B_n(u)\frac{x^n}{n!}\,.$$

then

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{s=0}^{m-1}\zeta^{-rs}B_n\left(u+\frac{s}{m}\right)=\frac{xe^{xu}}{e^x-1}\sum_{s=0}^{m-1}\zeta^{-rs}e^{sx/m}$$

$$=\frac{xe^{xu}}{\zeta^{-r}e^{x/m}-1}=\frac{\zeta^r xe^{xu}}{e^{x/m}-\zeta^r},$$

where ζ is a primitive m-th root of unity and $m\nmid r$. This evidently implies

$$(4.2)\qquad m^{n-1}\sum_{s=0}^{m-1}\zeta^{-rs}B_n\left(u+\frac{s}{m}\right)=\frac{n\zeta^r}{1-\zeta^r}H_{n-1}(mu\mid\zeta^r).$$

For $m\mid r$, on the other hand, we have the multiplication theorem for $B_n(x)$:

$$(4.3)\qquad m^{n-1}\sum_{s=0}^{m-1}B_n\left(u+\frac{s}{m}\right)=B_n(mu)$$

Multiplying both sides of (4.2) by ζ^{rt}, summing over r and using (4.3), we get

$$m^n B_n\left(u+\frac{t}{m}\right) = B_n(mu) + n \sum_{r=1}^{m-1} \zeta^{rt} \frac{H_{n-1}(mu \mid \zeta^r)}{\zeta^{-r}-1}, \tag{4.4}$$

where $0 \leqq t < m$.

We recall that the Bernoulli function $\bar{B}_n(u)$ is defined by

$$\bar{B}_n(u) = B_n(u) \quad (0 \leqq u < 1), \qquad \bar{B}_n(u+1) = \bar{B}_n(u).$$

Similarly we define

$$\bar{H}_n(u \mid \zeta) = H_n(u \mid \zeta) \quad (0 \leqq u < 1), \qquad \bar{H}_n(u+1 \mid \zeta) = \zeta \bar{H}_n(u \mid \zeta), \tag{4.5}$$

where ζ is some root of unity $\neq 1$. With these definitions of $\bar{H}_n(u \mid \zeta)$ and $\bar{B}_n(x)$ it is easily verified that the formulas (3.8), (3.12), (3.13), (4.2), (4.3), (4.4) hold for the barred functions; in particular in (4.4) the restriction $0 \leqq t < m$ is no longer necessary.

We remark that for m even and $\zeta^r = -1$, (4.2) reduces to the known formula

$$m^n \sum_{s=0}^{m-1} (-1)^s B_n\left(u+\frac{s}{m}\right) = -\frac{n}{2} E_{n-1}(mu),$$

where $E_{n-1}(u)$ is the Euler polynomial of degree $n-1$.

For $\zeta^r = \omega$, where $\omega^2 + \omega + 1 = 0$, $m = 3$, $u = 0$, (4.2) becomes

$$\frac{n\omega}{1-\omega} H_{n-1}[\omega] = 3^n \left\{ B_n + \omega^{-1} B_n(\tfrac{1}{3}) + \omega B_n(\tfrac{2}{3}) \right\}.$$

For n even, it is known that [10, p. 22]

$$B_n(\tfrac{1}{3}) = B_n(\tfrac{2}{3}) = \tfrac{1}{2}(3^{1-n}-1) B_n,$$

from which it follows that

$$\frac{2n\omega}{1-\omega} H_{n-1}[\omega] = (3^n-1) B_n \qquad (n \text{ even}). \tag{4.6}$$

On the other hand for n odd > 1 we get

$$n H_{n-1}[\omega] = 3^n \omega B_n(\tfrac{1}{3}) \tag{4.7}$$

Again for $\zeta^r = i$, $m = 4$, we get

$$\frac{ni}{1-i} H_{n-1}[i] = 4^{n-1}\left\{B_n - iB_n(\tfrac{1}{4}) - B_n(\tfrac{1}{2}) - iB_n(\tfrac{3}{4})\right\}.$$

It follows that

$$n(i-1)H_{n-1}[i] = 2^n(2^n-1)B_n \qquad (n \text{ even}), \tag{4.8}$$

$$n(i+1)H_{n-1}[i] = -2^{2n}B_n(\tfrac{1}{4}) \qquad (n \text{ odd} > 1). \tag{4.9}$$

For $\lambda = -1$ we have [10, p. 28]

$$H_{n-1}[-1] = 2^{1-n}C_{n-1} = 2(1-2^n)\frac{B_n}{n}. \tag{4.10}$$

5. We now obtain some congruences satisfied by H_n. If in (2.8) we take $f(z) = z^n(1-z^w)^r$, we get

$$H^n(1-H^w)^k = \sum_r (\lambda-1)^{-r} \sum_{s=0}^{n} (-1)^{r-s}\binom{r}{s} s^n(1-s^w)^k.$$

Assume that p is a prime such that

$$(p-1)p^{e-1} \mid w; \tag{5.1}$$

then by Fermat's theorem

$$s^n(1-s^w)^k \equiv 0 \left(\text{mod } (p^n, p^{kw})\right).$$

We thus obtain (Frobenius)

$$\sum_{s=0}^{k} (-1)^{k-s}\binom{k}{s} H_{n+sw} \equiv 0 \left(\text{mod } (p^n, p^{kw})\right) \tag{5.2}$$

valid for $k \geqq 0$, $n \geqq 0$, provided w satisfies (5.1). This result is referred to as Kummer's congruence for H_n. In (5.2) λ may be an indeterminate or an algebraic number such that $(p, 1-\lambda) = (1)$. In particular λ may be an l-th root of unity, where $l \neq p^f$.

By means of (5.2) it is proved in [5] that the coefficient A_{ns} of (1.3) satisfies the congruence

$$A_{n+b,\, s} \equiv A_{n,\, s} \pmod{p^e}, \tag{5.3}$$

where $p^{j-1} < s \leqq p^j$, $n \geqq e$, $b = p^{j+s-1}(p-1)$.

Another interesting result, also due to Frobenius, is

(5.4) $$H_w \equiv \frac{\lambda^{p-1}-1}{\lambda^p-1} \pmod{p^e},$$

where again w satisfies (5.1). Indeed we can prove a slightly more general result by using (3.11) with $n = w$, $u = r$, $m \equiv 0 \pmod{p^e}$. We get

$$\frac{1-\lambda^m}{1-\lambda} H_w(r) \equiv \sum_{\substack{j=0 \\ p \nmid (r+j)}}^{m-1} \lambda^{m-1-j} \equiv \sum_{j=0}^{m-1} \lambda^{m-1-j} - \sum_{i=0}^{\frac{m}{p}-1} \lambda^{m-1-r_0-ip}$$

$$\equiv \frac{\lambda^m-1}{\lambda-1} - \lambda^{p-1-r_0} \frac{\lambda^m-1}{\lambda^p-1} \pmod{p^e},$$

where the integer r_0 is defined by

(5.5) $$r \equiv -r_0 \pmod{p} \qquad (0 \leqq r_0 < p).$$

We have finally

(5.6) $$H_w(r) \equiv 1 - \lambda^{p-1-r_0} \frac{\lambda-1}{\lambda^p-1} \pmod{p^e},$$

which reduces to (5.4) for $r = 0$.

The interesting congruence

(5.7) $$R_{p-2}[1-\lambda] \equiv R_{p-2}[\lambda] \pmod{p}$$

is due to Mirimanoff [8]. It can be proved rapidly as follows. From (3.11) we get

$$(1-\lambda)R_{p-2}[\lambda] \equiv \sum_{j=0}^{p-2} (j+1)^{-1}\lambda^j,$$

so that

(5.8) $$\frac{d}{d\lambda}\{\lambda(1-\lambda)R_{p-2}[\lambda]\} \equiv \frac{\lambda-\lambda^p}{(1-\lambda)}.$$

Replacing λ by $1-\lambda$, we get also

$$\frac{d}{d\lambda}(\lambda(1-\lambda)R_{p-2}[1-\lambda]) \equiv \frac{\lambda-\lambda^p}{(1-\lambda)}.$$

Consequently

$$\lambda(1-\lambda)R_{p-2}[1-\lambda] \equiv \lambda(1-\lambda)R_{p-2}[\lambda] + C,$$

where C is independent of λ. Clearly $C \equiv 0$ and (5.7) follows at once.

Again by (3.11)

$$(1-\lambda)^2 R_{p-3}[\lambda] \equiv \sum_{j=0}^{p-2} (j+1)^{-2}\lambda^j,$$

so that

$$\frac{d}{d\lambda}\{\lambda(1-\lambda)^2 R_{p-3}[\lambda]\} \equiv (1-\lambda)R_{p-2}[\lambda]\,. \tag{5.9}$$

Then by (5.7)

$$(\lambda^p-\lambda)R_{p-2}[\lambda] \equiv \lambda^p(1-\lambda)R_{p-2}[\lambda] - \lambda(1-\lambda)^p R_{p-2}[1-\lambda]\,;$$

now using (5.8) and (5.9), we get

$$-\frac{1}{2}\lambda^2(1-\lambda)^2 R_{p-2}^2[\lambda] \equiv \lambda^{p+1}(1-\lambda)^2 R_{p-3}[\lambda] + \lambda^2(1-\lambda)^{p+1}R_{p-3}[1-\lambda]\,,$$

so that

$$-\frac{1}{2}R_{p-2}^2[\lambda] \equiv \lambda^{p-1}R_{p-3}[\lambda] + (1-\lambda)^{p-1}R_{p-3}[1-\lambda] \pmod{p}, \tag{5.10}$$

where of course $p > 2$. This result also is due to Mirimanoff.

We note that from (2.1) and

$$\binom{p-1}{r} \equiv (-1)^r \pmod{p}$$

it follows that

$$\sum_{r=0}^{p-1} (-1)^r H_r \equiv \lambda H_{p-1} \pmod{p}, \tag{5.11}$$

$$p\sum_{r=1}^{p-1} \frac{(-1)^{r-1}}{r} H_r \equiv (\lambda-1)H_p - 1 \pmod{p^2}. \tag{5.12}$$

Finally from (3.11) follows

$$\sum_{r=1}^{p-1} \left(\frac{r}{p}\right)\lambda^{r-1} \equiv (1-\lambda)^k R_k \pmod{p}, \tag{5.13}$$

where $p = 2k+1$ and (r/p) is the Legendre symbol.

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