NOTE ON NÖRLUND'S POLYNOMIAL $B_n^{(z)}$

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1. Nörlund [2, p. 146] has defined the polynomial $B_n^{(z)}$ by means of

$$(1) \qquad \left(\frac{x}{e^x - 1}\right)^z = \sum_{n=0}^{\infty} B_n^{(z)} \frac{x^n}{n!}.$$

Thus $B_n^{(z)}$ is a polynomial in z of degree n with rational coefficients; it should not be confused with the Bernoulli polynomial $B_n(z)$ defined by

$$\frac{xe^{xz}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{x^n}{n!}.$$

The Stirling numbers $S_1(n, k)$ and $S_2(n, k)$ of the first and second kind, respectively, are related to Nörlund's polynomial by means of

$$(2) \qquad (-1)^k S_1(n-1, k) = \binom{n-1}{k} B_k^{(n)},$$

$$(3) \qquad S_2(n, k) = \binom{n+k}{k} B_k^{(-n)},$$

where, to begin with, n is a positive integer in (2) and (3). The formulas, however, may be used to define $S_1(n, k)$, $S_2(n, k)$ for arbitrary n; k is restricted to integral values $\geqq 0$. In particular (2) and (3) imply the reciprocity relations

$$(4) \qquad S_1(-n-1, k) = S_2(n, k), \qquad S_2(-n-1, k) = S_1(n, k).$$

Gould [1] has proved the elegant formula

$$(5) \qquad B_k^{(z)} = \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} B_k^{(-jz)},$$

which, in view of (2) and (3), yields

$$(6) \qquad \begin{aligned} &(-1)^k S_1(n-1, k) \\ &\quad = \binom{n-1}{k} \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} \binom{k+jn}{k}^{-1} S_2(jn, k), \end{aligned}$$

Received by the editors August 6, 1959.

[1] Research sponsored by National Science Foundation grant NSF G-9425.

$$(-1)^k S_2(n, k) = \binom{k+n}{k} \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} \binom{jn-1}{k}^{-1} S_1(jn-1, k). \tag{7}$$

He has also proved that

$$(-1)^k \binom{z}{k} B_k^{(k-z)} = \sum_{j=0}^{k} \binom{k-z}{k+j} \binom{k+z}{k-j} \binom{k+j-1}{k} B_k^{(j+k)}, \tag{8}$$

which yields

$$S_1(n-1, k) = \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} S_2(j, k), \tag{9}$$

$$S_2(n-k, k) = \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} S_1(k+j-1, k). \tag{10}$$

Of these (9) is due to Schläfli, while (10) is presumably new.

2. It may be of interest to point out that (5) can be proved rapidly as follows. Since, as observed above, $B_n^{(z)}$ is a polynomial in z of degree n, it follows from a familiar formula in finite differences that

$$\sum_{s=0}^{k+1} (-1)^s \binom{k+1}{s} B_k^{(x-sz)} = 0$$

for all x, z. If we take $x=z$, this becomes

$$B_k^{(z)} - \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} B_k^{(-jz)} = 0,$$

which is the same as (5).

As for (8), if we put

$$g(z) = (-1)^k \binom{z}{k} B_k^{(k-z)},$$

then $g(z)$ is a polynomial in z of degree $2k$. Consequently it will suffice to show that (8) holds for $2k+1$ distinct values of z. For $z=0, 1, \cdots, k-1$, it is evident that $g(z)=0$; since also

$$\binom{k-z}{k+j} \binom{k+j-1}{k} = 0 \qquad (0 \leqq z < k;\ 0 \leqq j \leqq k),$$

it follows that (8) holds for these values of z. For $z=k$, we get

$$(-1)^k B_k^{(0)} = \sum_{j=0}^{k} \binom{k}{k+j}\binom{k}{k-j}\binom{k+j-1}{k} B_k^{(j+k)},$$

which is correct in view of

$$B_0^{(0)} = 1, \qquad B_k^{(0)} = 0 \qquad (k \geqq 1). \tag{11}$$

Finally for $z=-s$, where $s=1, 2, \cdots, k$ we remark that the right member of (8) reduces to a single term, namely

$$\binom{k+s}{k+s}\binom{k-s}{k-s}\binom{k+s-1}{k} B_k^{(s+k)} = (-1)^k \binom{-s}{k} B_k^{(s+k)},$$

so that (8) holds in this case also. We have therefore verified that (8) is satisfied for the $2k+1$ values $0, \pm 1, \cdots, \pm k$.

3. Examination of the above proofs reveals the somewhat surprising fact that the only property of $B_k^{(z)}$ that we have made use of is that $B_k^{(z)}$ is a polynomial in z of degree k which satisfies (11). We have therefore the following generalization. Let $f_k(z)$ denote an arbitrary polynomial in z of degree k. Then it follows that

$$f_k(z) = \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} f_k(-jz). \tag{12}$$

If moreover

$$f_k(0) = 0 \qquad (k \geqq 1), \tag{13}$$

then we have also

$$\begin{aligned}(-1)^k \binom{z}{k} f_k(k-z) \\ = \sum_{j=0}^{k} \binom{k-z}{k+j}\binom{k+z}{k-j}\binom{k+j-1}{k} f_k(j+k).\end{aligned} \tag{14}$$

In addition if we define

$$(-1)^k F_1(n-1, k) = \binom{n-1}{k} f_k(n), \tag{15}$$

$$F_2(n, k) = \binom{n+k}{k} f_k(-n), \tag{16}$$

then (12) and (14) yield

$$(17)\quad (-1)^k F_1(n-1,k) = \binom{n-1}{k}\sum_{j=0}^{k}(-1)^j\binom{k+1}{j+1}\binom{k+jn}{k}^{-1}F_2(jn,k),$$

$$(18)\quad (-1)^k F_2(n,k) = \binom{k+n}{k}\sum_{j=0}^{k}(-1)^j\binom{k+1}{j+1}\binom{jn-1}{k}^{-1}F_1(jn-1,k),$$

$$(19)\quad F_1(n-1,k) = \sum_{j=0}^{k}\binom{k-n}{k+j}\binom{k+n}{k-j}F_2(j,k),$$

$$(20)\quad F_2(n-k,k) = \sum_{j=0}^{k}\binom{k-n}{k+j}\binom{k+n}{k-j}F_1(k+j-1,k).$$

Note also that (15) and (16) imply

$$(21)\quad F_1(-n-1,k) = F_2(n,k), \qquad F_2(-n-1,k) = F_1(n,k).$$

References

1. H. W. Gould, *Stirling number representation problems*, Proc. Amer. Math. Soc. vol. 11 (1960) pp. 447–451.

2. N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Berlin, 1924.

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