



Note on Norlund's Polynomial $B^{\{z\}}_n$

L. Carlitz

Proceedings of the American Mathematical Society, Vol. 11, No. 3 (Jun., 1960), 452-455.

Stable URL:

<http://links.jstor.org/sici?sici=0002-9939%28196006%2911%3A3%3C452%3ANONP%3E2.0.CO%3B2-T>

Proceedings of the American Mathematical Society is currently published by American Mathematical Society.

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/about/terms.html>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/journals/ams.html>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is an independent not-for-profit organization dedicated to creating and preserving a digital archive of scholarly journals. For more information regarding JSTOR, please contact support@jstor.org.

NOTE ON NÖRLUND'S POLYNOMIAL $B_n^{(z)}$

L. CARLITZ¹

1. Nörlund [2, p. 146] has defined the polynomial $B_n^{(z)}$ by means of

$$(1) \quad \left(\frac{x}{e^x - 1} \right)^z = \sum_{n=0}^{\infty} B_n^{(z)} \frac{x^n}{n!}.$$

Thus $B_n^{(z)}$ is a polynomial in z of degree n with rational coefficients; it should not be confused with the Bernoulli polynomial $B_n(z)$ defined by

$$\frac{xe^{xz}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{x^n}{n!}.$$

The Stirling numbers $S_1(n, k)$ and $S_2(n, k)$ of the first and second kind, respectively, are related to Nörlund's polynomial by means of

$$(2) \quad (-1)^k S_1(n-1, k) = \binom{n-1}{k} B_k^{(n)},$$

$$(3) \quad S_2(n, k) = \binom{n+k}{k} B_k^{(-n)},$$

where, to begin with, n is a positive integer in (2) and (3). The formulas, however, may be used to define $S_1(n, k)$, $S_2(n, k)$ for arbitrary n ; k is restricted to integral values ≥ 0 . In particular (2) and (3) imply the reciprocity relations

$$(4) \quad S_1(-n-1, k) = S_2(n, k), \quad S_2(-n-1, k) = S_1(n, k).$$

Gould [1] has proved the elegant formula

$$(5) \quad B_k^{(s)} = \sum_{j=0}^k (-1)^j \binom{k+1}{j+1} B_k^{(-js)},$$

which, in view of (2) and (3), yields

$$(6) \quad \begin{aligned} & (-1)^k S_1(n-1, k) \\ &= \binom{n-1}{k} \sum_{j=0}^k (-1)^j \binom{k+1}{j+1} \binom{k+jn}{k}^{-1} S_2(jn, k), \end{aligned}$$

Received by the editors August 6, 1959.

¹ Research sponsored by National Science Foundation grant NSF G-9425.

$$(7) \quad (-1)^k S_2(n, k) = \binom{k+n}{k} \sum_{j=0}^k (-1)^j \binom{k+1}{j+1} \binom{jn-1}{k}^{-1} S_1(jn-1, k).$$

He has also proved that

$$(8) \quad (-1)^k \binom{z}{k} B_k^{(k-z)} = \sum_{j=0}^k \binom{k-z}{k+j} \binom{k+z}{k-j} \binom{k+j-1}{k} B_k^{(j+k)},$$

which yields

$$(9) \quad S_1(n-1, k) = \sum_{j=0}^k \binom{k-n}{k+j} \binom{k+n}{k-j} S_2(j, k),$$

$$(10) \quad S_2(n-k, k) = \sum_{j=0}^k \binom{k-n}{k+j} \binom{k+n}{k-j} S_1(k+j-1, k).$$

Of these (9) is due to Schläfli, while (10) is presumably new.

2. It may be of interest to point out that (5) can be proved rapidly as follows. Since, as observed above, $B_n^{(z)}$ is a polynomial in z of degree n , it follows from a familiar formula in finite differences that

$$\sum_{s=0}^{k+1} (-1)^s \binom{k+1}{s} B_k^{(x-sz)} = 0$$

for all x, z . If we take $x=z$, this becomes

$$B_k^{(z)} - \sum_{j=0}^k (-1)^j \binom{k+1}{j+1} B_k^{(-jz)} = 0,$$

which is the same as (5).

As for (8), if we put

$$g(z) = (-1)^k \binom{z}{k} B_k^{(k-z)},$$

then $g(z)$ is a polynomial in z of degree $2k$. Consequently it will suffice to show that (8) holds for $2k+1$ distinct values of z . For $z=0, 1, \dots, k-1$, it is evident that $g(z)=0$; since also

$$\binom{k-z}{k+j} \binom{k+j-1}{k} = 0 \quad (0 \leq z < k; 0 \leq j \leq k),$$

it follows that (8) holds for these values of z . For $z=k$, we get

$$(-1)^k B_k^{(0)} = \sum_{j=0}^k \binom{k}{k+j} \binom{k}{k-j} \binom{k+j-1}{k} B_k^{(j+k)},$$

which is correct in view of

$$(11) \quad B_0^{(0)} = 1, \quad B_k^{(0)} = 0 \quad (k \geq 1).$$

Finally for $z = -s$, where $s = 1, 2, \dots, k$ we remark that the right member of (8) reduces to a single term, namely

$$\binom{k+s}{k+s} \binom{k-s}{k-s} \binom{k+s-1}{k} B_k^{(s+k)} = (-1)^k \binom{-s}{k} B_k^{(s+k)},$$

so that (8) holds in this case also. We have therefore verified that (8) is satisfied for the $2k+1$ values $0, \pm 1, \dots, \pm k$.

3. Examination of the above proofs reveals the somewhat surprising fact that the only property of $B_k^{(z)}$ that we have made use of is that $B_k^{(z)}$ is a polynomial in z of degree k which satisfies (11). We have therefore the following generalization. Let $f_k(z)$ denote an arbitrary polynomial in z of degree k . Then it follows that

$$(12) \quad f_k(z) = \sum_{j=0}^k (-1)^j \binom{k+1}{j+1} f_k(-jz).$$

If moreover

$$(13) \quad f_k(0) = 0 \quad (k \geq 1),$$

then we have also

$$(14) \quad \begin{aligned} & (-1)^k \binom{z}{k} f_k(k-z) \\ &= \sum_{j=0}^k \binom{k-z}{k+j} \binom{k+z}{k-j} \binom{k+j-1}{k} f_k(j+k). \end{aligned}$$

In addition if we define

$$(15) \quad (-1)^k F_1(n-1, k) = \binom{n-1}{k} f_k(n),$$

$$(16) \quad F_2(n, k) = \binom{n+k}{k} f_k(-n),$$

then (12) and (14) yield

$$(17) \quad \begin{aligned} & (-1)^k F_1(n-1, k) \\ &= \binom{n-1}{k} \sum_{j=0}^k (-1)^j \binom{k+1}{j+1} \binom{k+jn}{k}^{-1} F_2(jn, k), \end{aligned}$$

$$(18) \quad \begin{aligned} & (-1)^k F_2(n, k) \\ &= \binom{k+n}{k} \sum_{j=0}^k (-1)^j \binom{k+1}{j+1} \binom{jn-1}{k}^{-1} F_1(jn-1, k), \end{aligned}$$

$$(19) \quad F_1(n-1, k) = \sum_{j=0}^k \binom{k-n}{k+j} \binom{k+n}{k-j} F_2(j, k),$$

$$(20) \quad F_2(n-k, k) = \sum_{j=0}^k \binom{k-n}{k+j} \binom{k+n}{k-j} F_1(k+j-1, k).$$

Note also that (15) and (16) imply

$$(21) \quad F_1(-n-1, k) = F_2(n, k), \quad F_2(-n-1, k) = F_1(n, k).$$

REFERENCES

1. H. W. Gould, *Stirling number representation problems*, Proc. Amer. Math. Soc. vol. 11 (1960) pp. 447-451.
2. N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Berlin, 1924.

DUKE UNIVERSITY