THE PRODUCT OF TWO EULERIAN POLYNOMIALS

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The Bernoulli and Euler polynomials can be defined by means of

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}, \qquad \frac{2e^{xt}}{e^t + 1} = \sum_{m=0}^{\infty} E_m(x) \frac{t^m}{m!}.$$

The formula

$$B_m(x)B_n(x) = \sum_r \left\{ \binom{m}{2r} n + \binom{n}{2r} m \right\} \frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} \tag{1}$$

is proved in Nielsen's book [3, p. 75]; a different proof occurs in [2]. Nielsen also obtains similar formulas for

$$E_m(x)E_n(x) \quad \text{and} \quad E_m(x)B_n(x).$$

The Eulerian polynomial $H_m(x|\lambda)$ can be defined by means of

$$\frac{(1-\lambda)e^{xt}}{e^t - \lambda} = \sum_{m=0}^{\infty} H_m(x \mid \lambda) \frac{t^m}{m!}; \tag{2}$$

for properties of $H_m(x|\lambda)$ see for example [1]. Since

$$H_m(x \mid -1) = E_m(x),$$

it may be of interest to get a formula for the product of two Eulerian polynomials.

We assume that $\alpha \neq 1$, $\beta \neq 1$, $\alpha\beta \neq 1$. It follows from (2) that

$$\begin{aligned}
\sum_{m,n=0}^{\infty} H_m(x \mid \alpha) H_n(x \mid \beta) \frac{u^m v^n}{m!n!} &= \frac{(1-\alpha)e^{xu}}{e^u - \alpha} \frac{(1-\beta)e^{xv}}{e^v - \beta} \\
&= \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} \frac{(1-\alpha\beta)e^{x(u+v)}}{e^{u+v} - \alpha\beta} \frac{e^{u+v} - \alpha\beta}{(e^u - \alpha)(e^v - \beta)} \\
&= \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} \frac{(1-\alpha\beta)e^{x(u+v)}}{e^{u+v} - \alpha\beta} \left\{ 1 + \frac{\alpha}{e^u - \alpha} + \frac{\beta}{e^v - \beta} \right\} \\
&= \frac{1}{1-\alpha\beta} \sum_{m,n=0}^{\infty} H_{m+n}(x \mid \alpha\beta) \frac{u^m v^n}{m!n!} \\
&\quad \cdot \left\{ (1-\alpha)(1-\beta) + \alpha(1-\beta) \sum_{r=0}^{\infty} H_r[\alpha] \frac{u^r}{r!} + \beta(1-\alpha) \sum_{s=0}^{\infty} H_s[\beta] \frac{v^s}{s!}, \right.
\end{aligned}$$

where we have put

$$H_r[\alpha] = H_r(0 \mid \alpha) \tag{3}$$

the so-called Eulerian number. Comparison of coefficients evidently yields

$$H_m(x \mid \alpha) H_n(x \mid \beta) = H_{m+n}(x \mid \alpha\beta)$$

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$$(4)\qquad \begin{aligned}&+\frac{\alpha(1-\beta)}{1-\alpha\beta}\sum_{r=1}^{m}\binom{m}{r}H_r[\alpha]H_{m+n-r}(x\mid\alpha\beta)\\&+\frac{\beta(1-\alpha)}{1-\alpha\beta}\sum_{s=1}^{n}\binom{n}{s}H_s[\beta]H_{m+n-s}(x\mid\alpha\beta),\end{aligned}$$

provided $\alpha\neq 1$, $\beta\neq 1$, $\alpha\beta\neq 1$.

In the next place we have

$$\begin{aligned}\sum_{m,n=0}^{\infty}B_m(x)H_n(x\mid\alpha)\frac{u^mv^n}{m!n!}&=\frac{ue^{xu}}{e^u-1}\,\frac{(1-\alpha)e^{xv}}{e^v-\alpha}\\&=u\,\frac{(1-\alpha)e^{x(u+v)}}{e^{u+v}-\alpha}\,\frac{e^{u+v}-\alpha}{(e^u-1)(e^v-\alpha)}\\&=\frac{(1-\alpha)e^{x(u+v)}}{e^{u+v}-\alpha}\left\{u+\frac{u}{e^u-1}+\frac{\alpha u}{e^v-\alpha}\right\}\\&=\sum_{m,n=0}^{\infty}H_{m+n}(x\mid\alpha)\frac{u^mv^n}{m!m!}\cdot\left\{u+\sum_{r=0}^{\infty}B_r\frac{u^r}{r!}+\frac{u}{1-\alpha}\sum_{s=0}^{\infty}H_s[\alpha]\frac{v^s}{s!}\right\}.\end{aligned}$$

It follows that

$$(5)\qquad \begin{aligned}B_m(x)H_n(x\mid\alpha)&=mH_{m+n-1}(x\mid\alpha)+\sum_{r=0}^{m}\binom{m}{r}B_rH_{m+n-r}(x\mid\alpha)\\&\quad+\frac{m\alpha}{1-\alpha}\sum_{s=0}^{n}\binom{n}{s}H_s[\alpha]H_{m+n-s-1}(x\mid\alpha),\end{aligned}$$

provided $\alpha\neq 1$.

If $\alpha\neq 1$ but $\alpha\beta=1$ we take

$$\begin{aligned}(u+v)\sum_{m,n=0}^{\infty}H_m(x\mid\alpha)H_n(x\mid\alpha^{-1})\frac{u^mv^n}{m!n!}&=(u+v)\frac{(1-\alpha)e^{xu}}{e^u-\alpha}\,\frac{(1-\alpha^{-1})e^{xv}}{e^v-\alpha^{-1}}\\&=(1-\alpha)(1-\alpha^{-1})\frac{(u+v)e^{x(u+v)}}{e^{u+v}-1}\left\{1+\frac{\alpha}{e^u-\alpha}+\frac{\alpha^{-1}}{e^v-\alpha^{-1}}\right\}.\end{aligned}$$

This implies

$$\begin{aligned}&mH_{m-1}(x\mid\alpha)H_n(x\mid\alpha^{-1})+nH_m(x\mid\alpha)H_{n-1}(x\mid\alpha^{-1})\\&\quad=(1-\alpha)(1-\alpha^{-1})B_{m+n}(x)-(1-\alpha)\sum_{r=0}^{m}\binom{m}{r}H_r[\alpha]B_{m+n-r}(x)\\&\qquad-(1-\alpha^{-1})\sum_{s=0}^{n}\binom{n}{s}H_s[\alpha^{-1}]B_{m+n-s}(x)\\&\quad=-(1-\alpha)\sum_{r=1}^{m}\binom{m}{r}H_r[\alpha]B_{m+n-r}(x)\\&\qquad-(1-\alpha^{-1})\sum_{s=1}^{n}\binom{n}{s}H_s[\alpha^{-1}]B_{m+n-s}(x).\end{aligned}$$

Since

$$\frac{\partial}{\partial x} H_n(x \mid \alpha) = nH_{n-1}(x \mid \alpha),$$

it is clear from (6) that

$$\begin{aligned} H_m(x \mid \alpha)H_n(x \mid \alpha^{-1}) = &- (1-\alpha)\sum_{r=0}^{m-1}\binom{m}{r+1}H_{r+1}[\alpha]\frac{B_{m+n-r}(x)}{m+n-r} \\ &- (1-\alpha^{-1})\sum_{s=0}^{n-1}\binom{n}{s+1}H_{s+1}[\alpha^{-1}]\frac{B_{m+n-s}(x)}{m+n-s} + C_{m,n}, \end{aligned} \tag{7}$$

where $C_{m,n}$ is independent of x. To determine $C_{m,n}$ we notice first that (6) and (7) imply

$$mC_{m-1,n} + nC_{m,n-1} = 0,$$

so that

$$C_{m,n} = -\frac{n}{m+1}C_{m+1,n-1}.$$

Repeated application of this recursion leads to

$$C_{m,n} = (-1)^n\frac{m!n!}{(m+n)!}C_{m+n,0}. \tag{8}$$

Now if we put $n=0$, $x=0$ in (7) we get

$$\begin{aligned} H_m[\alpha] &= -(1-\alpha)\sum_{r=0}^{m-1}\binom{m}{r+1}H_{r+1}[\alpha]\frac{B_{m-r}}{m-r} + C_{m,0} \\ &= -\frac{1-\alpha}{m+1}\sum_{r=1}^{m}\binom{m+1}{r}H_r[\alpha]B_{m-r+1} + C_{m,0}. \end{aligned}$$

Similarly (5) implies

$$B_{m+1} = (m+1)H_m[\alpha] + \sum_{r=0}^{m+1}\binom{m+1}{r}H_r[\alpha]B_{m-r+1} + \frac{(m+1)\alpha}{1-\alpha}H_m[\alpha],$$

so that

$$(m+1)C_{m,0} = -(1-\alpha)H_{m+1}[\alpha].$$

Therefore by (8)

$$C_{m,n} = (-1)^{n+1}\frac{m!n!}{(m+n+1)!}(1-\alpha)H_{m+n+1}[\alpha]. \tag{9}$$

(Since

$$H_n[\alpha^{-1}] = (-1)^n H_n[\alpha],$$

the right member of (9) remains unchanged when we interchange m and n and replace α by α^{-1}).

Combining (7) and (9) we get

$$
\begin{aligned}
H_m(x \mid \alpha)H_n(x \mid \alpha^{-1}) = &-(1-\alpha)\sum_{r=1}^{m}\binom{m}{r}H_r[\alpha]\frac{B_{m+n-r+1}(x)}{m+n-r+1} \\
&-(1-\alpha^{-1})\sum_{s=1}^{n}\binom{n}{s}H_s[\alpha^{-1}]\frac{B_{m+n-s+1}(x)}{m+n-s+1} \\
&+(-1)^{n+1}\frac{m!n!}{(m+n+1)!}(1-\alpha)H_{m+n+1}[\alpha],
\end{aligned} \tag{10}
$$

where of course $\alpha \neq 1$.

In particular if we take $\alpha=-1$, (5) and (10) reduce to

$$
\begin{aligned}
B_m(x)E_n(x) = &\ E_{m+n}(x) + \sum_{r=2}^{m}\binom{m}{r}B_rE_{m+n-r}(x) \\
&-\frac{m}{2}\sum_{s=1}^{n}\binom{n}{s}2^{-s}C_sE_{m+n-s-1}(x),
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
E_m(x)E_n(x) = &-2\sum_{r=1}^{m}\binom{m}{r}2^{-r}C_r\frac{B_{m+n-r+1}(x)}{m+n-r+1} \\
&-2\sum_{s=1}^{n}\binom{n}{s}2^{-s}C_s\frac{B_{m+n-s+1}(x)}{m+n-s+1} \\
&+(-1)^{n+1}2^{-m-n}\frac{m!n!}{(m+n+1)!}C_{m+n+1},
\end{aligned} \tag{12}
$$

where [4, p. 28]

$$C_n = 2^nE_n(0) = (2-2^{-n})\frac{B_{n+1}}{n+1}\,.$$

The formulas (11) and (12) may be compared with [3, p. 77, formulas (12), (16)].

We note also that since

$$\int_0^1 B_m(x)dx = \frac{B_{m+1}(1)-B_{m+1}(0)}{m+1} = 0 \qquad (m \geqq 1),$$

(10) yields

$$\int_0^1 H_m(x \mid \alpha)H_n(x \mid \alpha^{-1})dx = (-1)^{n+1}\frac{m!n!}{(m+n+1)!}(1-\alpha)H_{m+n+1}[\alpha] \qquad (m \geqq 1,\ n \geqq 1). \tag{13}$$

Finally we remark that (4), (5) and (10) imply the following special formulas:

$$(14) \qquad H_m(x \mid \alpha) = H_m(x \mid \beta) + \frac{\alpha - \beta}{1 - \beta} \sum_{r=1}^{m} \binom{m}{r} H_r[\alpha] H_{m-r}(x \mid \beta) \qquad (\alpha \neq 1, \beta \neq 1),$$

$$(15) \qquad B_m(x) = m H_{m-1}(x \mid \alpha) + \sum_{r=0}^{m} \binom{m}{r} B_r H_{m-r}(x \mid \alpha) \qquad (\alpha \neq 1),$$

$$(16) \qquad H_m(x \mid \alpha) = -\frac{1 - \alpha}{m + 1} \sum_{r=1}^{m+1} \binom{m+1}{r} H_r[\alpha] B_{m-r+1}(x) \qquad (\alpha \neq 1).$$

It is not difficult to prove these formulas directly. For example (14) follows easily from the identity

$$\frac{(1 - \alpha)e^{xu}}{e^u - \alpha} = \frac{1}{1 - \beta} \left\{ 1 - \alpha + (\alpha - \beta) \frac{1 - \alpha}{e^u - \alpha} \right\} \frac{(1 - \beta)e^{xu}}{e^x - \beta} \cdot$$

References

1. L. Carlitz, Eulerian numbers and polynomials, Mathematics Magazine, **32** (1959) 247–260.
2. L. Carlitz, Note on the integral of the product of several Bernoulli polynomials, Journal of the London Mathematical Society, **34** (1959) 361–363.
3. N. Nielsen, Traité élémentaire des nombres de Bernoulli, Paris, 1923.
4. N. E. Norlund, Vorlesungen über Differenzenrechnung, Berlin, 1924.

NUMBER THEORY

A pump's a composite of handle and spout
That has to be primed, or nothing comes out.
A gun's a composite of barrel and butt
That has to be primed, or nothing will sput.
In the arts, composition is carefully timed
And one doesn't begin till the surface is primed.
You will find composition is easy to do
When you start with a primer and carry it thru.

Marlow Sholander
