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PRODUCTS OF APPELL POLYNOMIALS

by

L. CARLITZ

1. Let

$$f(t) = \sum_0^\infty f_n \frac{t^n}{n!} \quad (f_0 = 1) \tag{1.1}$$

denote a formal power series and put

$$f(t)\, e^{xt} = \sum_0^\infty f_n(x) \frac{t^n}{n!}, \tag{1.2}$$

so that

$$f_n(x) = \sum_{r=0}^n \binom{n}{r} f_{n-r}\, x^n,\ f_n(0) = f_n. \tag{1.3}$$

Thus with every power series $f(t)$ is associated the set of APPELL plynomials $\{f_n(x)\}$ defined by means of (1.2) or (1.3). The polynomials $f_n(x)$ have the property.

$$f'_n(x) = n f_{n-1}(x)\,; \tag{1.4}$$

conversely if a set of polynomials have the property (1.4) then there exists a power series $f(t)$ such that (1.2) holds.

It is convenient to put

$$(f(t))^{-1} = \sum_0^\infty f'_n \frac{t^n}{n!} \quad (f'_0 = 1),$$

so that

$$x^n = \sum_{r=0}^n \binom{n}{r} f'_{n-r}\, f_r(x). \tag{1.5}$$

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In addition to $f(t)$ we also take

$$g(t) = \sum_0^\infty g_n \frac{t^n}{n!}, \quad h(t) = \sum_0^\infty h_n \frac{t^n}{n!} \quad (g_0 = h_0 = 1)$$

and put

$$g(t)\, e^{xt} = \sum_0^\infty g_n(x) \frac{t^n}{n!}, \quad h(t)\, e^{xt} = \sum_0^\infty h_n(x) \frac{t^n}{n!}.$$

Now consider

$$f_m(ax)\, g_n(bx) = \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} f_{m-r}\, g_{n-s}\, a^r b^s x^{r+s}.$$

As in (1.5)

$$x^n = \sum_{r=0}^{n} \binom{n}{r} h'_{n-r}\, h_r(x),$$

where

$$(1.6) \qquad (h(t))^{-1} = \sum_0^\infty h'_n \frac{t^n}{n!}.$$

Thus

$$f_m(ax)\, g_n(bx) = \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} f_{m-r}\, g_{n-r}\, a^r b^s. \sum_{j=0}^{r+s} \binom{r+s}{j} h'_{r+s-j}\, h_j(x).$$

We put

$$(1.7) \qquad f_m(ax)\, g_n(bx) = \sum_{j=0}^{m+n} A_j^{(m,n)}\, h_j(x),$$

where

$$(1.8) \quad A_j^{(m,n)} = A_j^{(m,n)}(a,b) = \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r}\binom{n}{s}\binom{r+s}{j} f_{m-r}\, g_{n-s}\, h'_{r+s-j}\, a^r b^s.$$

Then

$$\sum_{m,n=0}^\infty A_j^{(m,n)} \frac{u^m v^n}{m!n!} = \sum_{m,n=0}^\infty \frac{u^m v^n}{m!n!} \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r}\binom{n}{s}\binom{r+s}{j} f_{m-r}\, g_{n-s}\, h'_{r+s-j}\, a^r b^s.$$

$$= \sum_{r,s=0}^\infty \frac{(au)^r\, (bv)^s}{r!\, s!} \binom{r+s}{j} h'_{r+s-j} \cdot \sum_{m,n=0}^\infty f_m\, g_n \frac{u^m\, v^n}{m!\, n!}$$

$$= f(u) g(v) \sum_{r,s=0}^\infty \frac{(au)^r\, (bv)^s}{r!\, s!} \binom{r+s}{j} h'_{r+s-j}$$

$$= f(u)\, g(v) \sum_{k=j}^{\infty} \binom{k}{j} h'_{k-j} \sum_{r+s=k} \frac{(au)^r\,(bv)^s}{r!\ s!}$$

$$= f(u)\, g(v) \sum_{k=j}^{\infty} \binom{k}{j} h'_{k-j} \frac{(au+bv)^k}{k!}$$

$$= f(u)\, g(v) \sum_{k=0}^{\infty} h'_k \frac{(au+bv)^{k+j}}{j!\ k!} = \frac{f(u)\, g(v)}{h(au+bv)} \frac{(au+bv)^j}{j!}$$

by (1.6). We have therefore obtained

$$\sum_{m,n=0}^{\infty} A_j^{(m,n)} \frac{u^m\ v^n}{m!\ n!} = \frac{f(u)\, g(v)}{h(au+bv)} \frac{(au+bv)^j}{j!}, \tag{1.9}$$

a generating function for $A_j^{(m,\,n)}$ with j fixed.

We remark that there is no difficulty in proving the corresponding formula for the product of an arbitrary number of APPELL polynomials. If

$$f^{(i)}(t) = \sum_{n=0}^{\infty} f_n^{(i)} \frac{t^n}{n!},\ f_0^{(i)} = 1 \qquad (i = 1,\ \ldots,\ k),$$

$$f^{(i)}(t)\, e^{xt} = \sum_{n=0}^{\infty} f_n^{(i)}(x) \frac{t^n}{n!}$$

and we put

$$f_{m_1}^{(1)}(a_1 x) \ \ldots\ f_{m_k}^{(k)}(a_k x) = \sum_{j=0}^{m_1+\ldots+m_k} A_j^{(m_1,\ldots,m_k)} h_j(x), \tag{1.10}$$

then we find that

$$\sum_{m_1,\ldots,m_k=0}^{\infty} A_j^{(m_1,\ldots,m_k)} \frac{u_1^{m_1}\ \ldots\ u_k^{m_k}}{m_1!\ldots m_k!} \tag{1.11}$$

$$= \frac{f^{(1)}(u_1)\ \ldots\ f^{(k)}(u_k)}{h(a_1 u_1 + \ldots + a_k u_k)} \frac{(a_1 u_1 + \ldots + a_k u_k)^j}{j!}.$$

2. In place of (1.7) we may write

$$f_m(ax)\, g_n(bx) = \sum_{j=0}^{m+n} A_{m+n-j}^{(m,\,n)} h_{m+n-j}(x). \tag{2.1}$$

This suggests evaluating

$$\sum_{m,\,n=0}^{\infty} A_{m+n-j}^{(m,\,n)} \frac{u^m\ v^n}{m!\ n!}.$$

By (1.8) this sum is equal to

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!\, n!} \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} \binom{r+s}{m+n-j} f_{m-r}\, g_{n-s} \,.\, h'_{r+s-m-n+j}\, a^r b^s$$

$$= \sum_{m,n=0}^{\infty} \frac{u\, v^n}{m!\, n!} \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} \binom{m+n-r-s}{j-r-s} f_r\, g_s\, h'_{j-r-s} \,.\, a^{m-r}\, b^{n-s}$$

$$= \sum_{r+s\le j} f_r\, g_s\, h'_{j-r-s} \frac{u\, v^s}{r!\, s!} \cdot \sum_{m,n=0}^{\infty} \binom{m+n}{j-r-s} \frac{(au)^m\, (bv)^n}{m!\; n!}$$

$$= \sum_{r+s\le j} f_r\, g_s\, h'_{j-r-s} \frac{u\, v^s}{r!\, s!} \cdot \sum_{m=0}^{\infty} \binom{m}{j-r-s} \frac{(au+bv)^m}{m!}$$

$$= \sum_{r+s\le j} f_r\, g_s \frac{h'_{j-r-s}}{(j-r-s)!} \frac{u^r\, v^s}{r!\, s!} \sum_{m=0}^{\infty} \frac{(au+bv)^{m+j-r-s}}{m!} .$$

We have therefore proved

$$\text{(2.2)} \qquad \sum_{m,n=0}^{\infty} A_{m+n-j}^{(m,n)} \frac{u^m\, v^n}{m!\, n!} = e^{au+bv} \sum_{r+s\le j} f_r\, g_s\, h'_{j-r-s} \frac{u^r\, v^s\, (au+bv)^{j-r-s}}{r!\, s!\, (j-r-s)!} .$$

This suggests that we multiply by t^j and sum over j. Then

$$\sum_{j=0}^{\infty} t^j \sum_{r+s\le j} f_r\, g_s\, h'_{j-r-s} \frac{u^r\, v^s}{r!\, s!} \frac{z^{j-r-s}}{(j-r-s)!}$$

$$= \sum_{r,s=0}^{\infty} f_r\, g_s \frac{(tu)^r\, (tv)^s}{r!\, s!} \sum_{j=r+s}^{\infty} h'_{j-r-s} \frac{(tz)^{j-r-s}}{(j-r-s)!}$$

$$= \frac{1}{h\,(tz)} \sum_{r,s=0}^{\infty} f_r\, g_s \frac{(tu)^r\, (tv)^s}{r!\, s!} = \frac{f\,(tu)\; g\,(tv)}{h\,(tz)} .$$

We have therefore

$$\text{(2.3)} \qquad \sum_{m,n,j=0}^{\infty} A_{m+n-j}^{(m,n)} \frac{u^m\, v^n}{m!\; n!} t^j = e^{au+bv} \frac{f\,(tu)\; g\,(tv)}{h\,(t\,(au+bv))} .$$

The generalization corresponding to (1.11) will be omitted.

Another variant may be mentioned. Rewrite (2.1) in the form

$$\text{(2.4)} \qquad f_m\,(ax)\; g_n\,(bx) = \sum_{j=0}^{m+n} B_j^{(m,n)} \frac{h_{m+n-j}\,(x)}{(m+n-j)\,!} ,$$

so that

$$B_j^{(m,n)} = (m+n-j)\,!\; A_{m+n-j}^{(m,n)} .$$

Making use of (1.8) we get

$$B_j^{(m,n)} = \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} \binom{m+n-r-s}{j-r-s} (m+n-j)! \, . \, f_r \, g_s \, h'_{j-r-s} \, a^{m-r} \, b^{n-s}$$

$$= \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} \frac{(m+n-r-s)!}{(j-r-s)!} f_r \, g_s \, h'_{j-r-s} \, . \, a^{m-r} \, b^{n-s}.$$

This evidently defines $B_j^{(m,n)}$ for all m, n, j; the condition $m+n \geq j$ is no longer needed. We now find that

$$\text{(2.5)} \qquad \sum_{m,n=0}^{\infty} B_j^{(m,n)} \frac{u^m v^n}{m! \, n!} = \frac{1}{1-au-bv} \sum_{r+s \leq j} f_r \, g_s \frac{h'_{j-r-s}}{(j-r-s)!} \frac{u^r v^s}{r! \, s!}$$

and

$$\text{(2.6)} \qquad \sum_{m,n,j=0}^{\infty} B_j^{(m,n)} \frac{u^m v^n}{m! \, n!} t^j = \frac{1}{1-au-bv} \frac{f(tu) \, g(tv)}{h(t)} .$$

This formula admits of an obvious generalization analogous to (1.11).

3. There are many well known instances of APPELL polynomials. The BERNOULLI and EULER polynomials defined by means of

$$\text{(3.1)} \qquad \frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

and

$$\text{(3.2)} \qquad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} ,$$

respectively, furnish examples; more generally this is also true of the BERNOULLI and EULER polynomials of higher order [4, Ch. 5] which may be defined by

$$\text{(3.3)} \qquad \left(\frac{t}{e^t-1} \right)^z e^{xt} = \sum_{n=0}^{\infty} B_n^{(z)} (x) \frac{t^n}{n!} ,$$

$$\text{(3.4)} \qquad \left(\frac{2}{e^t-1} \right)^z e^{xt} = \sum_{n=0}^{\infty} E_n^{(z)} (x) \frac{t^n}{n!} ,$$

where z is an arbitrary complex number. We may also cite NÖRLUND's polynomials of order k defined by

$$(3.5)\qquad \frac{\omega_1 \ldots \omega_k\, t^k\, e^{xt}}{(e^{\omega_1 t}-1)\ldots(e^{\omega_k t}-1)} = \sum_{n=0}^{\infty} B_n^{(k)}(x \mid \omega_1, \ldots, \omega_k)\frac{t^n}{n!},$$

$$(3.6)\qquad \frac{2^k\, e^{xt}}{(e^{\omega_1 t}+1)\ldots(e^{\omega_k t}+1)} = \sum_{n=0}^{\infty} E_n^{(k)}(x \mid \omega_1, \ldots, \omega_k)\frac{t^n}{n!}.$$

A less familiar example is that of the «EULERIAN» polynomial [1]

$$\phi_n(x) = \phi_n(x \mid \lambda)$$

defined by

$$(3.7)\qquad \frac{1-\lambda}{e^x-\lambda}\, e^{xt} = \sum_{n=0}^{\infty} \phi_n(x \mid \lambda)\frac{t^n}{n!}\quad (\lambda \neq 1).$$

This polynomial can be generalized along the lines of (3.4) and (3.6).

In the next place the HERMITE polynomial $H_n(x)$ may be mentioned. We have the familiar generating function

$$(3.8)\qquad e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!};$$

thus the polynomials $\{H_n(x/2)\}$ form an APPELL set. In this case we evidently have

$$f(t) = e^{-t^2},$$

so that

$$f_n = \begin{cases} (-1)^r \dfrac{(2r)!}{r!} & (n = 2r) \\ & (n \ odd). \end{cases}$$

If we take

$$(3.9)\qquad f(t) = g(t) = h(t) = e^{-t^2}$$

we find that (2.2) and (2.5) can be simplified. We may assume j even. Then

$$\sum_{r+s \leq j} f_{2r}\, g_{2s}\, h'_{2j-2r-2s}\, \frac{u^{2r}\, v^{2s}}{(2r)!\,(2s)!}\, \frac{(au+bv)^{2j-2r-2s}}{(2j-2r-2s)!}$$

$$= \sum_{r+s \leq j} (-1)^{r+s}\, \frac{u^{2r}\, v^{2s}}{r!\, s!}\, \frac{(au+bv)^{2j-2r-2j}}{(j-r-s)!} = \frac{1}{j!}(1 - a^2 u^2 - b^2 v^2)^j.$$

Thus (2.2) becomes

$$(3.10)\qquad \sum_{m,\,n=0}^{\infty} A_{m+n-2j}^{(m,\,n)} \frac{u^m\, v^n}{m\,!\, n\,!} = \frac{1}{j\,!}\,(1 - a^2\, u^2 - b^2\, v^2)^j\, e^{au+bv}\,.$$

Similarly we find that, when (3.9) is satisfied (2.5) reduces to

$$(3.11)\qquad \sum_{m,\,n=0}^{\infty} B_{2j}^{(m,\,n)} \frac{u^m\, v^n}{m\,!\, n\,!} = \frac{1}{j\,!}\,\frac{(1 - a^2\, u^2 - b^2\, v^2)^j}{1 - au - bv}\,.$$

We remark that (3.11) was derived in [3] for the product of an arbitrary number of HERMITE polynomials.

4. For the LAGUERRE polynomial $L_n^{(\alpha)}(x)$ we have the two generating functions

$$(4.1)\qquad \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n = (1-t)^{-\alpha-1}\, e^{-xt/(1-t)}\,,$$

$$(4.2)\qquad \sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)\, t^n = (1+t)^{\alpha}\, e^{-xt}\,.$$

Thus the polynomials $\{L_n^{(\alpha-n)}(-x)\}$ constitute an APPELL set and therefore the results of §§ 1,2 apply. This is however not true of the $\{L_n^{(\alpha)}(x)\}$. We therefore consider the following situation.

Let $f(t)$ be of the form (1.1) and let

$$F(t) = \sum_{n=1}^{\infty} F_r\, t^r$$

be a formal power series without constant term. Put

$$(4.3)\qquad f(t)\, e^{xF(t)} = \sum_{n=0}^{\infty} f_n(x)\, \frac{t^n}{n\,!}\,.$$

We also put

$$g(t)\, e^{xG(t)} = \sum_{n=0}^{\infty} g_n(x)\, \frac{t^n}{n\,!}\,,\quad h(t)\, e^{xH(t)} = \sum_{n=0}^{\infty} h_n(x)\, \frac{t^n}{n\,!}\,,$$

where $G(t)$, $H(t)$ are series similar to $F(t)$.

Let

$$f_n(x) = \sum_{r=0}^{n} \binom{n}{r} f_{n,\,r}\, x^r,\ g_n(x) = \sum_{r=0}^{n} \binom{n}{r} g_{n,\,r}\, x^r,\ h_n(x) = \sum_{r=0}^{\infty} \binom{n}{r} h_{n,\,r}\, x^r\,.$$

Then we have

$$f_m(ax)\, g_n(bx) = \sum_{j=0}^{m+n} C_j^{(m,\,n)}\, h_j(x)\,, \tag{4.4}$$

where

$$C_j^{(m,\,n)} = \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r}\binom{n}{s}\binom{r+s}{j} a^r b^s .\, f_{m,r}\, g_{n,s}\, h'_{r+s,j} \tag{4.5}$$

and

$$x^n = \sum_{j=0}^{n} \binom{n}{j} h'_{n,j}\, h_j(x)\,. \tag{4.6}$$

It follows that

$$\sum_{m,\,n=0}^{\infty} C_j^{(m,\,n)} \frac{u^m v^n}{m!\,n!} = \sum_{r,\,s=0}^{\infty} \frac{(au)^r\,(bv)^s}{r!\,s!} \binom{r+s}{j} h'_{r+s,j} \cdot \sum_{m,\,n=0}^{\infty} f_{m+r,\,r}\, g_{n+s,\,s} \frac{u^m v^n}{m!\,n!}\,. \tag{4.7}$$

On the other hand, we have

$$\sum_{m=0}^{\infty} f_m(x) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{r=0}^{m} \binom{m}{r} f_{m,r}\, x^r = \sum_{r=0}^{\infty} \frac{(xt)^r}{r!} \sum_{m=0}^{\infty} f_{m+r,\,r} \frac{t^m}{m!}$$

$$= \sum_{r=0}^{\infty} \frac{(xt)^r}{r!} f_r(t)\,,$$

say. But since by (4.3)

$$\sum_{m=0}^{\infty} f_m(x) \frac{t^m}{m!} = f(t)\, e^{xF(t)} = \sum_{r=0}^{\infty} f(t)\, F^r(t) \frac{x^r}{r!}\,,$$

it is clear that

$$t^r f_r(t) = f(t)\, F^r(t)\,. \tag{4.8}$$

Thus (4.7) becomes

$$\sum_{,\,n=0}^{\infty} C_j^{(m,\,n)} \frac{u^m v^n}{m!\,n!} = \sum_{r,\,s=0}^{\infty} \frac{a^r b^s}{r!\,s!} \binom{r+s}{j} h'_{r+s,j} \cdot f(u) F^r(u) \,.\, g(v) G^s(v) \tag{4.9}$$

$$= f(u)\, g(v) \sum_{k=j}^{\infty} \binom{k}{j} h'_{k,j} \sum_{r+s=k} \frac{a^r b^s\, F^r(u)\, G^s(v)}{r!\,s!}$$

$$= f(u)\, g(v) \sum_{k=j}^{\infty} \binom{k}{j} h'_{k,j} \frac{(a\, F(u) + b\, G(v))^k}{k\,!}$$

$$= f(u)\, g(v) \frac{(a\, F(u) + bG(v))^j}{j\,!} \cdot \sum_{k=0}^{\infty} h'_{k+j,j} \frac{(a\, F(u) + bG(v))^k}{k\,!}.$$

We now make an additional assumption concerning

$$H(t) = \sum_{n=1}^{\infty} H_n\, t^n,$$

namely $H_1 \neq 0$. This is equivalent to assuming the existence of a series $I(t)$ such that

$$H(I(t)) = I(H(t)) = t. \tag{4.10}$$

Using (4.6) we get

$$e^{xH(t)} = \sum_{j=0}^{\infty} h_j(x) \frac{H^j(t)}{j\,!} \sum_{n=0}^{\infty} h'_{n+j,j} \frac{H^n(t)}{n\,!}.$$

Since

$$e^{xH(t)} = \frac{1}{h(t)} \sum_{j=0}^{\infty} h_j(x) \frac{t^j}{j\,!}.$$

it follows that

$$H^j(t) \sum_{n=0}^{\infty} h'_{n+j,j} \frac{H^n(t)}{n\,!} = \frac{t^j}{h(t)}.$$

Now using (4.10) it follows that

$$t^j \sum_{n=0}^{\infty} h'_{n+i,j} \frac{t^n}{n\,!} = \frac{I^j(t)}{h(I(t))}.$$

Thus (4.9) becomes

$$\sum_{m,n=0}^{\infty} C_j^{(m,n)} \frac{u^m v^n}{m!\, n!} = \frac{f(u)\, g(v)}{h\{I(aF(u) + bG(v))\}} \frac{I^j(a\,F(u) + bG(v))}{j\,!}. \tag{4.11}$$

The generalization corresponding to a product of k polynomials is immediate.

We now examine the special case suggested by (4.1), that is

$$H(t) = \frac{t}{1-t}, \quad I(t) = \frac{t}{1+t}.$$

If in addition we take

$$h(t) = (1 - t)^{-\alpha-1},$$

we find that the right member of (4.11) reduces to

$$\frac{f(u)\,g(v)}{j\,!}\;\frac{(a\,F(u) + bG(v))^j}{(1 + a\,F(u) + bG(v))^{\alpha+j+1}}, \tag{4.12}$$

which is in agreement with [3, (6)].

5. Finally we discuss briefly a q-analog of (1.9). Put

$$e(t) = \prod_0^\infty (1 - q^n t)^{-1} = \sum_0^\infty \frac{t^n}{(q)_n}, \tag{5.1}$$

where

$$(q)_n = (1 - q)(1 - q^2)\ldots(1 - q^n).$$

Let

$$f(t)\,e(xt) = \sum_0^\infty f_n(x)\frac{t^n}{(q)_n}, \tag{5.2}$$

where

$$f(t) = \sum_0^\infty f_n \frac{t^n}{(q)_n} \qquad (f_0 = 1);$$

then

$$f_n(x) = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} f_{n-r}\, x^r, \tag{5.3}$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(q)_n}{(q)_r\,(q)_{n-r}}.$$

We also put

$$\frac{1}{f(t)} = \sum_0^\infty f'_n \frac{t^n}{(q)_n} \qquad (f'_0 = 1), \tag{5.4}$$

so that

$$x^n = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} f'_{n-r}\, f_n(x). \tag{5.5}$$

Now let $g(t)$, $h(t)$ be two other series of the same nature as $f(t)$ and define $g_n(x)$, $h_n(x)$, g'_n, h'_n by formulas like (5.4) and (5.5).

Then if we put

$$f_m(ax)\, g_n(bx) = \sum_{j=0}^{m+n} A_j^{(m,n)}\, h_j(x)\,, \tag{5.6}$$

we find that

$$A_j^{(m,n)} = \sum_{r=0}^{m} \sum_{s=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} r+s \\ j \end{bmatrix} \cdot f_{m-r}\, g_{n-s}\, h'_{r+s-j}\, a^r b^s\,. \tag{5.7}$$

Thus

$$\sum_{m,n=0}^{\infty} A_j^{(m,n)} \frac{u^m v^n}{(q)_m\,(q)_n} = \sum_{r,s=0}^{\infty} \frac{(au)^r\,(bv)^s}{(q)_r\,(q)_s} \begin{bmatrix} r+s \\ j \end{bmatrix} h'_{r+s-j} \cdot \sum_{m,n=0}^{\infty} f_m\, g_n \frac{u^m v^n}{(q)_m\,(q)_n}\,.$$

We have therefore

$$\sum_{m,n=0}^{\infty} A_j^{(m,n)} \frac{u^m v^n}{(q)_m\,(q)_n} = f(u)\, g(v) \sum_{r,s=0}^{\infty} \frac{(au)^r\,(bv)^s}{(q)_r\,(q)_s} \begin{bmatrix} r+s \\ j \end{bmatrix} h'_{r+s-j}. \tag{5.8}$$

If we put

$$H_k(a, b) = \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix} a^r b^{k-r}\,, \tag{5.9}$$

the sum on the right of (5.8) becomes

$$\frac{1}{(q)_j} \sum_{k=0}^{\infty} \frac{h'_k}{(q)_k} H_{k+j}(au, bv)\,. \tag{5.10}$$

This does not seem to admit of further simplification. If we multiply both sides of (5.8) by t^j and sum over j, we find similarly that

$$\sum_{m,n,j=0}^{\infty} A_j^{(m,n)} \frac{u^m v^n}{(q)_m\,(q)_n}\, t = f(u)\, g(v) \sum_{n=0}^{\infty} \frac{1}{(q)_n} h'_n(t)\, H_n(au, bv)\,, \tag{5.11}$$

where

$$\frac{e(xt)}{h(t)} = \sum_{0}^{\infty} h'_n(x) \frac{t^n}{(q)_n}\,. \tag{5.12}$$

If we take $h(t) = (e(t))^{-1}$, then since

$$\sum_{k=0}^{\infty} \frac{1}{(q)_k} H_k(a, b) = e(a)\, e(b),$$

it follows that

$$h'_n(x) = H_n(x, 1). \tag{5.13}$$

Incidentally, since

$$\frac{e(xt)}{e(t)} = \sum_{n=0}^{\infty} \frac{(x-1)(x-q)\ldots(x-q^{n-1})}{(q)_n} t^n,$$

it is clear that in the present case

$$h_n(x) = (x-1)(x-q)\ldots(x-q^{n-1}).$$

In view of (5.13), the right member of (5.11) is equal to

$$f(u)\, g(v) \sum_{n=0}^{\infty} \frac{(aut)^n}{(q)_n} H_n(t, 1)\, H_n\left(\frac{bv}{an}, 1\right).$$

The writer has proved the identity [2, (3.9)]

$$\sum_{0}^{\infty} \frac{t^n}{(q)_n} H_n(a, 1)\, H_n(b, 1) = \frac{e(t)\, e(at)\, e(bt)\, e(abt)}{e(abt^2)} \tag{5.14}$$

Thus (5.11) becomes

$$\sum_{m,n,j=0}^{\infty} A_j^{(m,n)} \frac{u^m v^n}{(q)_m (q)_n} t^j = f(u)\, g(v) \frac{e(au)\, e(bv)\, e(aut)\, e(bvt)}{e(abuvt)}. \tag{5.15}$$

While (5.8) and (5.11) can be generalized for the product of an arbitrary number of polynomials, it does not seem possible to do the same for (5.15) unless a suitable generalization of (5.14) can be obtained.

Interesting variants of the above results can be obtained if we make a slight change in the definition of $g_n(x)$ but leave all other quantities unchanged. Put

$$\frac{g(t)}{e(xt)} = \sum_{0}^{\infty} \bar{g}_n(x) \frac{t^n}{(q)_n}. \tag{5.16}$$

Since

$$\frac{1}{e(t)} = \prod_{0}^{\infty} (1 - q^n t) = \sum_{0}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} \frac{t^n}{(q)_n},$$

it follows that

$$\bar{g}_n(x) = \sum_{s=0}^{n} (-1)^s \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}s(s-1)} g_{n-s} x^s .$$

If we now put

$$f_m(ax)\,\bar{g}_n(bx) = \sum_{j=0}^{m+n} B_j^{(m,n)} h_j(x) ,$$

we find

$$B_j^{(m,n)} = \sum_{r=0}^{m}\sum_{s=0}^{n} (-1)^s \begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix}\begin{bmatrix} r+s \\ j \end{bmatrix} q^{\frac{1}{2}s(s-1)} \cdot f_{m-r}\, g_{n-s}\, h'_{r+s-j}\, a^r b^s .$$

It follows that

$$\sum_{m,n=0}^{\infty} B_j^{(m,n)} \frac{u^m v^n}{(q)_m (q)_n} = \frac{f(u)g(v)}{(q)_j} \sum_{k=0}^{\infty} \frac{h'_k}{(q)_k} \sum_{s=0}^{k+j} (-1)^s q^{\frac{1}{2}s(s-1)} \begin{bmatrix} k+j \\ s \end{bmatrix} (au)^{k+j-s} (bv)^s$$

$$(5.17) \qquad = \frac{f(u)\,g(v)}{(q)_j} \sum_{k=0}^{\infty} \frac{h'_k}{(q)_k} (au)^{k+j} \left(\frac{bv}{au}\right)_{k+j}$$

$$= \frac{f(u)\,g(v)}{(q)_j} (au)^j \left(\frac{bv}{au}\right)_j \sum_{k=0}^{\infty} \frac{h'_k}{(q)_k} (au)^k \left(\frac{q^j bv}{au}\right)_k ,$$

where

$$(a)_k = (1-a)(1-qa)\ldots(1-q^{k-1}a) .$$

We again take $h(t) = (e(t))^{-1}$, so that $h'_k = 1$. Then the sum in (5.17) reduces to

$$\sum_{k=0}^{\infty} \frac{(au)^k}{(q)_k} \left(\frac{q^j bv}{au}\right)_k = \frac{e(au)}{e(q^j bv)} = \frac{e(au)}{(bv)_j\, e(bv)} .$$

We have therefore

$$(5.18) \qquad \sum_{m,n=0}^{\infty} B_j^{(m,n)} \frac{u^m v^n}{(q)_m (q)_n} = \frac{f(u)\,g(v)\,e(au)}{e(bv)} \frac{(au)^j}{(q)_j\ (bv)_j} \left(\frac{bv}{au}\right)_j .$$

Finally, if we multiply by $(bv)_j\, t^j$ and sum, we get

$$(5.19) \qquad \sum_{m,n=0}^{\infty} (bv)_j\, B_j^{(m,n)} \frac{u^m\ v^n}{(q)_m\ (q)_n}\, t^j = \frac{f(u)\,g(v)\,e(au)}{(e(bv))^2} ,$$

where in both (5.18) and (5.19) $\bar{g}_n(x)$ is defined by (5.16) and $h(t) = (e(t))^{-1}$.

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