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# Some Properties of the Nörlund Polynomial *B* †

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Article first published online: 12 NOV 2006

DOI: 10.1002/mana.19670330504

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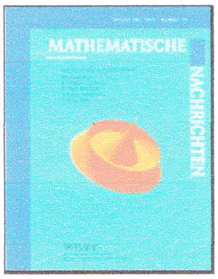
Issue




## Mathematische Nachrichten

Volume 33, Issue 5-6, (/doi/10.1002/mana.v33:5/6/issuetoc) pages 297–311, 1967

Additional Information

### How to Cite

Carlitz, L. (1967), Some Properties of the Nörlund Polynomial *B*. Math. Nachr., 33: 297–311. doi: 10.1002/mana.19670330504

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† Herrn JOSEF NAAS zum 60. Geburtstag am 16. October 1966 gewidmet

### Publication History

1. Issue published online: 12 NOV 2006
2. Article first published online: 12 NOV 2006
3. Manuscript Received: 23 FEB 1966

also

$$1-|x\in_w X_1 - H(x)|\geqq 1-\{1-X_1 =_w X_2 + |x\in_w X_2 - H(x)|\}$$
$$=1-|x\in_w X_2 - H(x)| + X_1 =_w X_2 - 1$$
$$1-|x\in_w X_2 - H(x)|\geqq 1-\{1-X_1 =_w X_2 + |x\in_w X_1 - H(x)|\}$$
$$=1-|x\in_w X_1 - H(x)| + X_1 =_w X_2 - 1,$$

also

$$\inf_{x\in E}(1-|x\in_w X_1 - H(x)|)$$
$$\geqq X_1 =_w X_2 - 1 + \inf_{x\in E}(1-|x\in_w X_2 - H(x)|)$$
$$\inf_{x\in E}(1-|x\in_w X_2 - H(x)|)$$
$$\geqq X_1 =_w X_2 - 1 + \inf_{x\in E}(1-|x\in_w X_1 - H(x)|),$$

also

$$X_1 =_w X_2 - 1 \leqq -|\inf_{x\in E}(1-|x\in_w X_1 - H(x)|)$$
$$-\inf_{x\in E}(1-|x\in_w X_2 - H(x)|)|$$
$$= -|\forall_w x\,(x\in_w X_1 \longleftrightarrow_w H(x)) - \forall_w x\,(x\in_w X_2 \longleftrightarrow_w H(x))|.$$

Man bemerkt beim Beweis von Satz 6b), daß b) für beliebige $X_1, X_2 \in E$ gilt. Satz 6 liefert hinsichtlich der Extensionalität den zum klassischen Fall völlig analogen Sachverhalt für beliebige $H\in w^E$ und $X_1, X_2\in E$:

$$[\mathrm{Me}\,X_1 \wedge_w \mathrm{Me}\,X_2 . \to ._w \forall_w x\,(x\in_w X_1 \longleftrightarrow_w H(x))$$
$$\wedge^w \forall_w x\,(x\in_w X_2 \longleftrightarrow_w H(x)) \to_w X_1 =_w X_2] = 1$$
$$[X_1 =_w X_2 . \to ._w \forall_w x\,(x\in_w X_1 \longleftrightarrow_w H(x)) \longleftrightarrow_w \forall_w x\,(x\in_w X_2 \longleftrightarrow_w H(x))] = 1.$$

Definiert man noch für $H\in w^E$:

$$\exists_w!\,x\,H(x) = \forall_w x_1 \forall_w x_2\,(H(x_1)\wedge^w H(x_2) \to_w x_1 =_w x_2),$$

so erhält man über die Kommutativität und Assoziativität von $\wedge^w$ und wegen

$$t_1 \to_w (t_2 \to_w t_3) = (t_1 \wedge^w t_2) \to_w t_3 \qquad (t_1, t_2, t_3 \in w)$$

sofort für beliebige $H\in w^E$:

$$[\exists_w!\,X\,(\mathrm{Me}\,X \wedge^w_{(w)} \forall_w x\,(x\in_w X \longleftrightarrow_w H(x)))] = 1.$$

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*Institut für Reine Mathematik*

# Some Properties of the Nörlund Polynomial $B_n^{(x)}$

Herrn Josef Naas zum 60. Geburtstag am 16. 10. 1966 gewidmet

By L. Carlitz of Durham

(Eingegangen am 23. 2. 1966)

1. Following Stefani [3] we put

$$C_k^n = \sum_{s=0}^{k-1} (-1)^{k-1-s}\binom{k-1}{s}(s+1)^n. \tag{1.1}$$

Stefani showed that

$$C_x^{x+s-1} = \frac{(x+s)!}{s\cdot s!\,(s+1)!}\,C_s(x) \qquad (s, x = 1, 2, 3, \ldots), \tag{1.2}$$

where $C_s(x)$ is a polynomial in $x$ of degree $s-1$ with rational coefficients. He put

$$C_s(x) = \sum_{i=1}^{s} a_{is}\,x^{s-i} \tag{1.3}$$

and discussed some arithmetic properties of the coefficients $a_{is}$. In particular he proved the following two results.

**Theorem 1.** *The denominators of the coefficients $a_{is}$ contain no prime divisors except* 2.

**Theorem 2.** *If $s = 2^e s'$, where $s'$ is odd, then*

$$a_{is} \equiv 0 \pmod{s'} \qquad (i = 1, 2, \ldots, s).$$

In addition Stefani showed that the $a_{is}$ are closely related to the Bernoulli numbers $B_n$ defined by

$$\frac{y}{e^y - 1} = \sum_{n=0}^{\infty} B_n \frac{y^n}{n!}.$$

Thus for example

$$a_{1s} = (s+1)!\,B_s \qquad (s > 1), \tag{1.4}$$

[1]) Supported in part by NSF grant GP-5174.

while for $s > 1$ and $p$ a prime $> s+1$,

$$B_s \equiv \frac{s \cdot s!}{(p+s)!} C_p^{p+s-1} \pmod p. \tag{1.5}$$

In the present paper we first point out the relationship of the numbers $C_k^n$ to the BERNOULLI numbers of higher order defined by

$$\left(\frac{y}{e^y - 1}\right)^k = \sum_{n=0}^{\infty} B_n^{(k)} \frac{y^n}{n!}$$

and of the polynomial $C_s(x)$ to the polynomial $B_n^{(x)}$ defined by

$$\left(\frac{y}{e^y - 1}\right)^x = \sum_{n=0}^{\infty} B_n^{(x)} \frac{y^n}{n!}.$$

The precise relationship is in fact

$$C_s(x) = \frac{s(s+1)!}{x} B_s^{(-x)} \qquad (s \geqq 1). \tag{1.6}$$

By means of this representation we are able to prove Theorems 1 and 2 rather rapidly. Moreover we obtain some additional properties of $C_s(x)$. Put

$$\frac{1}{s} C_s(x) = \frac{\bar{C}_s(x)}{d_s}, \tag{1.7}$$

where $\bar{C}_s(x)$ is a *primitive* polynomial with integral coefficients, that is, if

$$\bar{C}_s(x) = \sum_{i=1}^{s} \bar{a}_{is} x^{s-i},$$

then

$$(\bar{a}_{1s}, \bar{a}_{2s}, \ldots, \bar{a}_{ss}) = 1.$$

By Theorems 1 and 2 the denominator $d_s$ is a power of 2. The exact power is determined as follows.

**Theorem 3.** *Let $2^{\nu(s)}$ denote the highest power of 2 that divides $s!$ and let $2^t$ denote the highest power of 2 that divides $s(s+1)$. Then*

$$d_s = 2^{s-\nu(s)-t}.$$

As a refinement of Theorem 2 we have

**Theorem 4.** *Let $p$ be a prime $> 3$. Then the coefficients of*

$$\frac{1}{s(s+1)} C_s(x) = \frac{s!}{x} B_s^{(-x)}$$

*are integral* (mod $p$) *provided $s \neq p-1$. When $s = p-1$, however, we have*

$$C_{p-1}(x) \equiv 1 \pmod p.$$



Indeed the first part of the theorem holds for $p = 3$ provided $s$ is sufficiently large.

As for the polynomial $B_n^{(-x)}$ we may state the following result which contains Theorem 4.

**Theorem 5.** *Let $p^{\nu(n)}$ denote the highest power of $p$ that divides $n!$ Then the coefficients of*

$$p^{\nu(n+m)-\nu(n)} B_n^{(-x)},$$

*where $m = [n/(p-1)]$, are integral* (mod $p$).

The writer has proved [1] that if

$$k = a_1 p^{k_1} + \cdots + a_r p^{k_r} \qquad (0 \leqq k_1 < k_2 < \cdots < k_r;\ 0 < a_j < p)$$

then $p^r B_n^{(k)}$ is integral (mod $p$) for all $n$; moreover for certain $n$ the exponent cannot be reduced. It is of some interest to contrast this result with Theorem 5.

2. By a HURWITZ series (briefly $H$-series) we understand a (formal) power series of the type

$$H(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \tag{2.1}$$

where the $a_n$ are rational integers. We recall a few simple properties of such series. In the first place sum, difference or product of two $H$-series is again an $H$-series; if $a_0 = \pm 1$ the reciprocal of $H(x)$ is also an $H$-series. Let

$$H_1(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n!} \tag{2.2}$$

denote an $H$-series without constant term. Then we have the useful result that

$$\frac{1}{k!} (H_1(x))^k \tag{2.3}$$

is an $H$-series for $k = 0, 1, 2, \ldots$ It follows from this result that if $H(x)$ is an arbitrary $H$-series and $H_1(x)$ is of the form (2.2) then $H(H_1(x))$ is also an $H$-series.

By the statement

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \equiv \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} \pmod m$$

is understood the system of ordinary congruences

$$a_n \equiv b_n \pmod m \qquad (n = 0, 1, 2, \ldots).$$

We have the following

**Lemma.** *If $H_1(x)$, $H_2(x)$ are H-series without constant terms such that*

(2.4) $$H_1(x) \equiv H_2(x) \pmod{m}$$

*and $H(x)$ is an arbitrary H-series, then*

(2.5) $$H(H_1(x)) \equiv H(H_2(x)) \pmod{m}.$$

Indeed it follows from (2.4) that

$$H_2(x) = H_1(x) + m\,H_0(x),$$

where $H_0(x)$ is an $H$-series without constant term. Then if $H(x)$ is defined by (2.1) we have

$$\begin{aligned} H(H_2(x)) &= H(H_1(x) + m\,H_0(x)) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!} (H_1(x) + m\,H_0(x))^n \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!} \sum_{j=0}^{n} \binom{n}{j} (H_1(x))^j m^{n-j} (H_0(x))^{n-j} \\ &= \sum_{j,k=0}^{\infty} a_{j+k}\, m^k \frac{(H_1(x))^j}{j!} \frac{(H_0(x))^k}{k!} \\ &\equiv \sum_{j=0}^{\infty} a_j \frac{(H_1(x))^j}{j!} \equiv H(H_1(x)) \pmod{m}. \end{aligned}$$

This evidently proves (2.5).

**3.** We now return to the definition of the Nörlund polynomial $B_n^{(x)}$ [2, Ch. 6]:

(3.1) $$\left(\frac{y}{e^y - 1}\right)^x = \sum_{n=0}^{\infty} B_n^{(x)} \frac{y^n}{n!}.$$

It is clear from (3.1) that

(3.2) $$x \mid B_n^{(x)} \quad (n \geqq 1).$$

Also by comparison with

(3.3) $$\frac{y}{e^y - 1} = \sum_{n=0}^{\infty} B_n \frac{y^n}{n!}$$

we have $B_n^{(1)} = B_n$.

It is easily verified, using (3.3), that

(3.4) $$\sum_{n=1}^{\infty} (-1)^n \frac{B_n}{n} \frac{y^n}{n!} = \log \frac{e^y - 1}{y}.$$



Thus (3.1) becomes

(3.5) $$\exp\left\{x \sum_{n=1}^{\infty} (-1)^n \frac{B_n}{n} \frac{y^n}{n!}\right\} = \sum_{n=0}^{\infty} B_n^{(-x)} \frac{y^n}{n!}.$$

If we put

(3.6) $$B_n^{(x)} = \sum_{i=1}^{n}{}' \, b_{in} x^{n-i+1} \qquad (n \geqq 1)$$

it follows from (3.5) that

(3.7) $$b_{nn} = (-1)^n \frac{B_n}{n} \qquad (n \geqq 1)$$

and $\left(\text{since } B_1 = -\frac{1}{2}\right)$

(3.8) $$b_{1n} = \frac{1}{2^n} \qquad (n \geqq 0).$$

In the next place, if we differentiate (3.1) with respect to $y$ we get

$$\sum_{n=0}^{\infty} B_{n+1}^{(-x)} \frac{y^n}{n!} = x \left(\frac{y}{e^y - 1}\right)^{-x} \left(\frac{e^y}{e^y - 1} - \frac{1}{y}\right),$$

so that

$$\begin{aligned} \sum_{n=0}^{\infty} n\,B_n^{(-x)} \frac{y^n}{n!} &= x \left(\frac{y}{e^y - 1}\right)^{-x} \left(\frac{y\,e^y}{e^y - 1} - 1\right) \\ &= x \sum_{n=0}^{\infty} B_n^{(-x)} \frac{y^n}{n!} \sum_{j=1}^{\infty} (-1)^j B_j \frac{y^j}{j!}. \end{aligned}$$

Equating coefficients of $y^n$ we get [2, p. 146]

(3.9) $$n\,B_n^{(-x)} = x \sum_{j=1}^{n} (-1)^j \binom{n}{j} B_j B_{n-j}^{(-x)}.$$

It is evident that (3.9) contains both (3.7) and (3.8).

**4.** By (1.1) we have, for $k \geqq 1$,

$$\begin{aligned} C_k^n &= \sum_{s=0}^{k-1} (-1)^{k-1-s} \binom{k-1}{s} (s+1)^n \\ &= \frac{1}{k} \sum_{s=0}^{k-1} (-1)^{k-1-s} \binom{k}{s+1} (s+1)^{n+1} \\ &= \frac{1}{k} \sum_{s=1}^{k} (-1)^{k-s} \binom{k}{s} s^{n+1}, \end{aligned}$$

so that

$$\sum_{n=0}^{\infty} C_k^n \frac{y^{n+1}}{(n+1)!} = \frac{1}{k} \sum_{s=1}^{k} (-1)^{k-s} \binom{k}{s} \sum_{n=1}^{\infty} \frac{s^n y^n}{n!}$$
$$= \frac{1}{k} \sum_{s=1}^{k} (-1)^{k-s} \binom{k}{s} (e^{sy} - 1)$$
$$= \frac{1}{k} (e^y - 1)^k$$
$$= \frac{y^k}{k} \left(\frac{e^y - 1}{y}\right)^k$$
$$= \frac{y^k}{k} \sum_{n=0}^{\infty} B_n^{(-k)} \frac{y^n}{n!}.$$

Equating coefficients of $y^{n+1}$ we get

$$C_k^n = \frac{(n+1)!}{k(n-k+1)!} B_{n-k+1}^{(-k)};$$

replacing $n$ by $n + k - 1$ this becomes

(4.1) $$C_k^{n+k-1} = \frac{(n+k)!}{k \cdot n!} B_n^{(-k)}.$$

Since, by (1.2),

$$C_k^{n+k-1} = \frac{(n+k)!}{n \cdot n!\,(n+1)!} C_n(k),$$

(4.1) becomes

(4.2) $$C_n(k) = \frac{n(n+1)!}{k} B_n^{(-k)}.$$

Since (4.2) holds for $k = 1, 2, 3, \ldots$ we have the polynomial identity

(4.3) $$C_n(x) = \frac{n(n+1)!}{x} B_n^{(-x)}.$$

Thus with $a_{in}$ defined by (1.3) and $b_{in}$ defined by (3.6) we see that

(4.4) $$a_{in} = n(n+1)!\, b_{in}.$$

In particular it follows from (3.7) and (3.8) that

(4.5) $$a_{nn} = (-1)^n (n+1)!\, B_n \qquad (n \geqq 1)$$

and

(4.6) $$a_{1n} = \frac{n(n+1)!}{2^n} \qquad (n \geqq 1).$$



5. It is convenient to put

(5.1) $$P_n(x) = n!\, B_n^{(-x)},$$

so that (4.3) becomes

(5.2) $$C_n(x) = \frac{n(n+1)}{x} P_n(x)$$

and (3.9) implies

(5.3) $$P_n(x) = x \sum_{j=1}^{n} (-1)^j \binom{n}{j} \frac{(n-1)!}{(n-j)!} B_j P_{n-j}(x).$$

Multiplying both sides of (5.3) by $n + 1$ we get

(5.4) $$(n+1) P_n(x) = x \sum_{j=1}^{n} (-1)^j \binom{n}{j} \frac{(n+1)!}{n(n-j+1)!} B_j \times (n-j+1) P_{n-j}(x).$$

We shall prove that the denominators of the coefficients of

(5.5) $$(n+1) P_n(x) = \frac{x\, C_n(x)}{n}$$

contain no primes except possibly 2. This will be proved by induction on $n$. For $n = 1$ we have $2 P_1(x) = x$. Assuming that the above assertion holds up to and including the value $n - 1$ we make use of (5.4). We shall also require the Staudt-Clausen theorem:

(5.6) $$B_{2n} = G_{2n} - \sum_{p-1 \mid 2n} \frac{1}{p} \qquad (n > 0),$$

where $G_{2n}$ is an integer and the summation on the right is over all primes $p$ (including 2) such that $p - 1 \mid 2n$. Now let $p$ denote an arbitrary odd prime. Then it is only necessary to consider those values of $j$ in the right member of (5.4) such that $p - 1 \mid j$. The quotient

$$\frac{(n+1)!}{n(n-j+1)!} = (n+1) \prod_{s=1}^{j-2} (n-s) = \frac{1}{n} \prod_{s=0}^{j-1} (n-s+1).$$

For $j = p - 1$ the binomial coefficient

$$\binom{n}{p-1} \equiv 0 \qquad (\mathrm{mod}\ p)$$

unless $n \equiv p - 1 \pmod{p}$; in this case the presence of the factor $n + 1$ neutralizes the prime $p$ in the denominator of $B_{p-1}$. For $j = t(p-1)$, $t > 1$, $p > 3$, the product of $j - 2$ consecutive integers

$$\prod_{s=1}^{j-2} (n-s)$$

is divisible by $(j-2)!$ Since $j-2 \geqq 2p-4 \geqq p$, the product is divisible by $p$. Finally for $p=3, j=2t, t>2$, we have $j-2=2t-2 \geqq 4$, which implies divisibility by 3 so that we need only consider the case $p=3, j=4$. The coefficient to examine is

$$\binom{n}{4}(n+1)(n-1)(n-2) = \binom{n+1}{4}(n-1)(n-2)(n-3)$$
$$\equiv 0 \pmod{3}.$$

This completes the proof of the assertion about the coefficients of $P_n(x)$ and therefore proves Theorems 1 and 2. It also gives some information about the possible odd prime divisors of the denominators of the coefficients of $P_n(x)$, namely a necessary condition that an odd prime $p$ have this property is that $p \mid n+1$. We shall however obtain a more precise result below.

6. We now determine the power of 2 occurring in the denominator of $P_n(x)$. More precisely let $e(n)$ denote the smallest integer such that the coefficients of the polynomial

$$\bar{P}_n(x) = 2^{e(n)} P_n(x) \tag{6.1}$$

have odd denominators. We shall make use of (3.5), which we rewrite in the form

$$\exp\left\{-x \sum_{n=1}^{\infty} (-1)^n \frac{B_n}{n} \frac{2^n y^n}{n!}\right\} = \sum_{n=0}^{\infty} B_n^{(x)} \frac{2^n y^n}{n!}. \tag{6.2}$$

Now

$$\sum_{n=1}^{\infty} (-1)^n \frac{B_n}{n} \frac{2^n y^n}{n!} = y + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n} \frac{2^{2n} y^{2n}}{(2n)!}.$$

For $n=1$ we get $2B_2 = \frac{1}{3}$; for $n>1$ we have

$$\frac{2^{2n} B_{2n}}{2n} \equiv 0 \pmod{2}.$$

Thus

$$\sum_{n=1}^{\infty} (-1)^n \frac{B_n}{n} \frac{2^n y^n}{n!} \equiv y + \frac{y^2}{2} \pmod{2}.$$

Applying the Lemma of § 2 it therefore follows from (3.5) that

$$\sum_{n=0}^{\infty} B_n^{(x)} \frac{2^n y^n}{n!} \equiv \sum_{k=0}^{\infty} \frac{x^k}{k!}\left(y + \frac{y^2}{2}\right)^k$$
$$\equiv \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{y^{k+j}}{2^j}$$
$$\equiv \sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{2j \leqq n} \frac{n!}{2^j j!\,(n-2j)!} x^{n-j} \pmod{2},$$



which yields

$$2^n B_n(x) \equiv \sum_{2j \leqq n} \binom{n}{2j} 1.3 \ldots (2j-1)\, x^{n-j} \pmod{2}. \tag{6.3}$$

It is now easy to evaluate $e(n)$ as defined by (6.1). Put

$$\nu(n) = \nu(n, 2) = \left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right] + \cdots,$$

so that $2^{\nu(n)} \mid n!$, $2^{\nu(n)+1} \nmid n!$. Then comparing (5.1) with (6.1) and (6.3) we get

$$e(n) = n - \nu(n). \tag{6.4}$$

In the next place if

$$2^t \mid n(n+1), \quad 2^{t+1} \nmid n(n+1) \tag{6.5}$$

and we define

$$f(n) = e(n) - t = n - \nu(n) - t, \tag{6.6}$$

then it follows from (5.2) and Theorems 1 and 2 that the coefficients of

$$2^{f(n)} C_n(x) \tag{6.7}$$

are integers and that $f(n)$ is the minimum exponent that has this property. For example, we have

$$n=5, \quad \nu(5)=3, \quad t=1, \quad f(5)=1,$$
$$n=6, \quad \nu(6)=4, \quad t=1, \quad f(6)=1,$$
$$n=7, \quad \nu(7)=4, \quad t=3, \quad f(7)=0,$$

in agreement with Stefani [3, p. 135].

It should be noted that $f(n)$ may be negative. For example

$$f(4) = 4 - 3 - 2 = -1.$$

Indeed, for $t \geqq 0$,

$$f(2^t) = 2^t - (2^{t-1} + \cdots + 1) - t = 1 - t,$$
$$f(2^{t+1} + 2^t) = 2^{t+1} + 2^t - (2^{t+1} + 2^t - 2) - t = 2 - t,$$
$$f(2^{t+2} + 2^t) = 2^{t+2} + 2^t - (2^{t+2} + 2^t - 2) - t = 2 - t,$$

and so on. It would be of some interest to determine all $n$ such that $f(n) < 0$. In this connection we remark that if

$$n = 2^t m, \quad 2 \nmid m, \quad t > 0,$$

then we find that

$$f(n) = m - \nu(m) - t = f(m) - t + t',$$

where $2^{t'} \mid m+1$, $2^{t'+1} \nmid m+1$. Thus for fixed odd $m$, $f(2^t m) < 0$ provided $t$ is sufficiently large.

7. Let $p$ denote a fixed odd prime and let $e(n) = e(n, p)$ denote the smallest integer such that the coefficients of the polynomial

$$\bar{P}_n(x) = p^{e(p)} P_n(x) \tag{7.1}$$

are integral (mod $p$).

To begin with we shall examine several special cases. First, for $n = p - 1$, (5.3) becomes

$$P_{p-1}(x) = x \sum_{j=1}^{p-1} (-1)^j \binom{p-1}{j} \frac{(p-2)!}{(p-1-j)!} B_j P_{n-1-j}(x).$$

Applying the STAUDT-CLAUSEN theorem we have at once

$$p P_{p-1}(x) \equiv -x \pmod{p}. \tag{7.2}$$

This evidently implies

$$e(p-1) = 1. \tag{7.3}$$

For $n = p$ we have

$$P_p(x) = x \sum_{j=1}^{p} (-1)^j \binom{p}{j} \frac{(p-1)!}{(p-j)!} B_j P_{p-j}(x). \tag{7.4}$$

The terms for which $j = 1$ or $p - 1$ require examination. For $j = 1$ we have

$$-\binom{p}{1} B_1 P_{p-1}(x) = \frac{1}{2} p P_{p-1}(x) \equiv -\frac{1}{2} x \pmod{p}$$

by (7.2). For $j = p - 1$ we have

$$\binom{p}{p-1} (p-1)!\, B_{p-1}\, P_1(x) \equiv P_1(x) \equiv \frac{1}{2} x \pmod{p}$$

by the STAUDT-CLAUSEN theorem. We have therefore

$$P_p(x) \equiv 0 \pmod{p}. \tag{7.5}$$

This result shows only that $e(p) \leqq -1$. However by (3.8) and (5.1) the coefficient of $x^p$ in $P_p(x)$ is equal to $-p!/2^p$ and therefore

$$e(p) = -1. \tag{7.6}$$

For $n = p + 1$ we have

$$P_{p+1}(x) = x \sum_{j=1}^{p+1} (-1)^j \binom{p+1}{j} \frac{p!}{(p-j+1)!} B_j P_{p-j+1}(x).$$

The terms for which $j = 1, 2, p - 1, p + 1$ require examination. We have, for $p > 3$,



$$-\binom{p+1}{1} B_1 P_p \equiv 0,$$

$$\binom{p+1}{2} p B_2 P_{p-1}(x) \equiv 0,$$

$$\binom{p+1}{p-1} \frac{p!}{2!} B_{p-1} P_2(x) \equiv 0,$$

$$\binom{p+1}{p+1} p!\, B_{p+1} P_0(x) \equiv 0,$$

so that

$$P_{p+1}(x) \equiv 0 \pmod{p} \qquad (p > 3). \tag{7.7}$$

Thus $e(p + 1) \leqq -1$. Since as above the coefficient of $x^{p+1}$ in $P_{p+1}(x)$ is equal to $(p + 1)!/2^p$ it follows that

$$e(p+1) = -1 \qquad (p > 3). \tag{7.8}$$

For $p = 3$, however, we have $e(4) = 0$.

Continuing in this way we show that

$$P_n(x) \equiv 0 \pmod{p} \qquad (p \leqq n < 2p - 2) \tag{7.9}$$

and indeed

$$e(n) = -1 \qquad (p \leqq n < 2p - 2). \tag{7.10}$$

For $n = 2p - 2$ it follows from

$$P_{2p-2}(x) = x \sum_{j=1}^{2p-2} (-1)^j \binom{2p-2}{j} \frac{(2p-3)!}{(2p-j-2)!} B_j P_{2p-j-2}(x)$$

that

$$P_{2p-2}(x) \equiv \frac{1}{2} x \qquad (p > 3), \tag{7.11}$$

so that

$$e(2p-2) = 0 \qquad (p > 3). \tag{7.12}$$

Indeed (7.11) and (7.12) hold also for $p = 3$.

For $n = 2p - 1$ we find that

$$P_{2p-1}(x) \equiv \binom{2p-1}{p-1} \frac{(2p-2)!}{p!} B_{p-1} P_p(x) \pmod{p}.$$

In view of (7.5) and (7.6) it follows that

$$e(2p-1) = 0. \tag{7.13}$$

8. We now consider the general case. As we have seen in § 5, the coefficients of $(n + 1) P_n(x)$ are integral (mod $p$), where $p$ is any odd prime. Thus

if $p^t$ is the highest power of $p$ dividing $n+1$, it follows that the coefficients of $p^t P_p(x)$ are integral (mod $p$). Thus we have

(8.1) $$e(n) \leq t.$$

Returning to (3.5) we write

(8.2) $$\sum_{n=0}^{\infty} B_n^{(-x)} \frac{y^n}{n!} = \exp\{x F(y) + x G(y)\},$$

where

$$F(y) = \sum_{n=1}^{\infty} \frac{B_{n(p-1)}}{n(p-1)} \frac{y^{n(p-1)}}{(n(p-1))!},$$

$$G(y) = \sum_{n=1}^{\infty} (-1)^n \frac{B_n}{n} \frac{y^n}{n!}.$$

Put $y^{p-1} = z$,

$$F(y) = f(z) = \sum_{n=1}^{\infty} \frac{B_{n(p-1)}}{n(p-1)} \frac{z^n}{(n(p-1))!}.$$

Then

$$f(p\,z) = \sum_{n=1}^{\infty} \frac{B_{n(p-1)}}{n(p-1)} \frac{p^n n!}{(n(p-1))!} \frac{z^n}{n!}.$$

Since

$$\frac{(n\,p)!}{(n(p-1))!} = \frac{p^n n!}{(n(p-1))!} \prod_{\substack{j=1 \\ p \nmid n}}^{n-1} (p\,n - j),$$

it follows that the coefficients

$$\frac{B_{n(p-1)}}{n(p-1)} \frac{p^n n!}{(n(p-1))!} \qquad (n = 1, 2, 3, \ldots)$$

are integral (mod $p$), so that $f(p\,z)$ is an $H$-series; moreover the first coefficient

$$\frac{B_{p-1}}{p-1} \frac{p}{(p-1)!} \equiv -1 \qquad (\text{mod } p).$$

If we put

(8.3) $$\exp\{x F(y)\} = \sum_{n=0}^{\infty} U_{n(p-1)}(x) \frac{y^{n(p-1)}}{(n(p-1))!}$$

then

$$\exp\{x f(p\,z)\} = \sum_{n=0}^{\infty} U_{n(p-1)}(x) \frac{p^n n!}{(n(p-1))!} \frac{z^n}{n!},$$

where

(8.4) $$V_{n(p-1)}(x) = U_{n(p-1)}(x) \frac{p^n n!}{(n(p-1))!}$$



is primitive. (A polynomial is primitive if all its coefficients are integral (mod $p$) but not all are divisible by $p$.)

Put

(8.5) $$\exp\{x G(y)\} = \sum_{n=0}^{\infty} W_n(x) \frac{y^n}{n!}.$$

We recall that if $p^r \mid n$ but $p-1 \nmid n$, then, $p^{-r} B_n$ is integral (mod $p$). Consequently $G(y)$ is an $H$-series and therefore the coefficients of $W_n(x)$ are integral (mod $p$). Indeed $W_n(x)$ is primitive since its highest coefficient is $2^{-n}$.

By (8.2), (8.3), (8.5) we have

(8.6) $$B_n^{(-x)} = \sum_j \binom{n}{j(p-1)} U_{j(p-1)}(x) W_{n-j(p-1)}(x).$$

As we have seen above the coefficients of $V_{n(p-1)}(x)$ in (8.4) are integral (mod $p$). It follows that the coefficients of $p^k B_n^{(-x)}$ are integral (mod $p$), where $p^k$ is the highest power of $p$ dividing

$$\binom{n}{j(p-1)}^{-1} \frac{(j(p-1))!}{(j\,p)!} = \binom{n+j}{j\,p}^{-1} \frac{(n+j)!}{n!}.$$

Thus if $p^{v(n)}$ is the highest power of $p$ that divides $n!$ then certainly the coefficients of

(8.7) $$p^{\nu(n+m)-\nu(n)} B_n^{(-x)},$$

where $m = [n/(p-1)]$, are integral (mod $p$). This evidently completes the proof of Theorem 5.

Comparing (8.7) with (7.1) we get

(8.8) $$e(n) \leq \nu(n+m) - 2\nu(n).$$

To illustrate (8.7) take $n = p^k - 1$, so that $m = (p^k - 1)\cdot(p-1)$ and

$$\nu(n) = \frac{p^k - 1}{p-1} - k = m - k,$$

$$\nu(n+m) = p^{k-1} + 2p^{k-2} + \cdots + k = \frac{p\,m - k}{p-1},$$

$$\nu(n+m) - v(n) = \frac{m + (p-2)k}{p-1},$$

so that

$$v(n+m) - 2v(n) < 0 \qquad (p > 3).$$

We shall now show that, for $p > 3$,

(8.9) $$\nu(n+m) \leq 2\nu(n),$$

where

$$n = m(p-1) + a \qquad (0 \leqq a < p-1),$$

except when $m = 1$, $a = 0$.

For $m = 1$, $1 \leqq a < p - 1$, we have

$$\nu(n+m) = \nu(p+a) = 1, \quad \nu(n) = \nu(p+a-1) = 1.$$

We may accordingly assume that $m > 1$. Next for $1 < m < p$ we have

$$\nu(n+m) = \nu(pm+a) = m, \quad \nu(n) = \nu((p-1)m+a) \geqq m-1,$$

so that $\nu(n+m) = m \leqq 2(m-1) \leqq 2\nu(n)$.

We now assume that $m \geqq p$. Put

$$m = a_0 + a_1 p + \cdots + a_r p^r \qquad (0 \leqq a_j < p;\ 0 < a_r).$$

If we write (8.9) in the form

$$\nu(pm+a) \leqq 2\nu((p-1)m+a),$$

it is clear that it suffices to take $a = 0$.

To evaluate $\nu((p-1)m)$ we write

$$\nu((p-1)m) = \sum_{j=1}^{r+1} \left[\frac{(p-1)m}{p^j}\right] = \sum_{j=1}^{r+1} \left[\frac{m}{p^{j-1}} - \frac{m}{p^j}\right].$$

Since

$$\left[\frac{m}{p^{j-1}} - \frac{m}{p^j}\right] \geqq a_{j-1} + a_j p + \cdots + a_r p^{r-j+1}$$
$$- (a_j + a_{j+1} p + \cdots + a_r p^{r-j}) - 1,$$

it follows that

$$\nu((p-1)m) \geqq m - r - 1.$$

On the other hand

$$\nu(n+m) = \nu(pm) = m + \nu(m) = m + \frac{m - S(m)}{p-1},$$

where $S(m) = a_0 + a_1 + \cdots + a_r$. Thus it suffices to show that

$$\frac{pm - S(m)}{p-1} \leqq 2(m-r-1);$$

that is, since $S(m) \geqq 1$, to

(8.10) $\qquad (p-2)m \geqq 2(p-1)(r+1) - 1.$

Since $m \geqq p^r$, $r \geqq 1$, $p \geqq 5$, we have

$$(p-2)m - 2(p-1)(r+1) + 1 \geqq (p-2)(4r+1)$$
$$- 2(p-1)(r+1) + 1 = 2(p-3)r - (p-1) \geqq 2(p-3)$$
$$- (p-1) = p - 5 \geqq 0.$$



This evidently completes the proof of (8.9). Also it is clear from the above proof that (8.9) holds when $p = 3$ provided $r \geqq 2$.

## References

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