Collectanea Mathematica 1973 (vol.24,2): 173-199

EULERIAN NUMBERS AND OPERATORS

by

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1. INTRODUCTION. The Eulerian numbers $A_{n,k}$ are usually introduced by means of [1], [6, Ch. 8]

$$\frac{1-\lambda}{1-\lambda e^{(1-\lambda)x}} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} A_{n,k} x^k. \tag{1.1}$$

It follows from (1.1) that

$$A_{n+1,k} = (n-k+2)\, A_{n,k-1} + k A_{n,k} \tag{1.2}$$

and

$$A_{n,k} = A_{n,n-k+1} \qquad (1 \leq k \leq n). \tag{1.3}$$

It is evident from (1.2) and $A_{1,1} = 1$ that the $A_{n,k}$ are positive integers for $n \geq k \geq 1$.

The symmetry relation (1.3) is by no means obvious from the generating function (1.1). This has motivated the introduction of the symmetric notation [3]

$$A(r, s) = A_{r+s+1,r+1} = A_{r+s+1,s+1} = A(s, r), \tag{1.4}$$

where now $r \geq 0$, $s \geq 0$. It then follows from (1.1) that

$$\sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r+s+1)!} = F(x, y), \tag{1.5}$$

where

$$F(x, y) = \frac{e^x - e^y}{x e^y - y e^x} \tag{1.6}$$

The recurrence (1.2) becomes

$$A(r, s) = (r+1) A(r, s-1) + (s+1) A(r-1, s). \tag{1.7}$$

Moreover in addition to (1.5) there is a second generating function

$$\sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r+s)!} = (1 + xF(x, y))(1 + yF(x, y)) \tag{1.8}$$

with $F(x, y)$ defined by (1.6).

If we put

$$A_n = A_n(x, y) = \sum_{r+s=n} A(r, s) x^r y^s, \tag{1.9}$$

it follows from (1.7) that

$$A_n(x, y) = (x + y + xy(D_x + D_y)) A_{n-1}(x, y), \tag{1.10}$$

where $D_x = \partial/\partial x$, $D_y = \partial/\partial y$. Iteration of (1.10) gives

$$A_n(x, y) = (x + y + xy(D_x + D_y))^n \cdot 1. \tag{1.11}$$

It is accordingly of interest to consider the expansion of the operator

$$\Omega^n \equiv (x + y + xy(D_x + D_y))^n. \tag{1.12}$$

We shall show that

$$\Omega^n = \sum_{k=0}^{n} C_{n,k}(x, y)(xy)^k (D_x + D_y)^k, \tag{1.13}$$

where

$$C_{n,k}(x, y) = \frac{1}{k!(k+1)!} (D_x + D_y)^k A_n(x, y) \qquad (0 \le k \le n). \tag{1.14}$$

The generating function (1.8) suggests the generalization [3]

$$\sum_{r,s=0}^{\infty} A(r, s | \alpha, \beta) \frac{x^r y^s}{(r+s)!} = (1 + xF(x, y))^\alpha (1 + yF(x, y))^\beta \tag{1.15}$$

where again $F(x, y)$ is defined by (1.6). Thus

$$A(r, s) = A(r, s \mid 1,1).$$

It follows from (1.15) that

(1.16) $$A(r, s \mid \alpha, \beta) = (r+\beta) A(r, s-1 \mid \alpha, \beta) + (s+\alpha) A(r-1, s \mid \alpha, \beta).$$

which evidently reduces to (1.7) when $\alpha = \beta = 1$; also

(1.17) $$A(r, s \mid \alpha, \beta) = A(s, r \mid \beta, \alpha).$$

By (1.16), $A(r, s \mid \alpha, \beta)$ is a polynomial in α, β with positive integral coefficients. Combinatorial properties of $A(r, s \mid \alpha, \beta)$ are discussed in [3].

Put

(1.18) $$A_n(x, y \mid \alpha, \beta) = \sum_{r+s=n} A(r, s \mid \alpha, \beta)\, x^r y^s.$$

Then by (1.16)

(1.19) $$A_n(x, y \mid \alpha, \beta) = (\alpha x + \beta y + xy(D_x + D_y))\, A_{n-1}(x, y \mid \alpha, \beta),$$

so that

(1.20) $$A_n(x, y \mid \alpha, \beta) = (\alpha x + \beta y + xy(D_x + D_y))^n \cdot 1.$$

It is therefore of interest to consider the expansion of the operator

(1.21) $$\Omega^n_{\alpha,\beta} \equiv (\alpha x + \beta y + xy(D_x + D_y))^n.$$

We shall show that

(1.22) $$\Omega^n_{\alpha,\beta} = \sum_{k=0}^{n} C^{(\alpha,\beta)}_{n,k}(x, y)\,(xy)^k (D_x + D_y)^k,$$

where

(1.23) $$C^{(\alpha,\beta)}_{n,k}(x, y) = \frac{1}{k!\,(\alpha+\beta)_k}(D_x + D_y)^k A_n(x, y \mid \alpha, \beta) \quad (0 \leq k \leq n),$$

where

$$(\alpha+\beta)_k = (\alpha+\beta)(\alpha+\beta+1)\dots(\alpha+\beta+k-1).$$

The case $\alpha + \beta$ equal to zero or a negative integer requires special treatment.

We consider also the inverse of (1.22), that is,

$$(1.24) \qquad (xy)^n (D_x + D_y)^n = \sum_{k=0}^{n} B_{n,k}^{(\alpha,\beta)} \Omega_{\alpha,\beta}^k.$$

We show that

$$(1.25) \quad (D_x + D_y) B_{n,k}^{(\alpha,\beta)}(x, y) = n(\alpha + \beta + n - 1) B_{n-1,k}^{(\alpha,\beta)}(x, y)$$

and

(1.26)

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{k=0}^{n} B_{n,k}^{(\alpha,\beta)}(x, y)(x - y)^k v^k = \left(\frac{1 - xu}{1 - yu}\right)^{-v} (1 - xu)^{-\alpha} (1 - yu)^{-\beta}.$$

Additional properties of $B_{n,k}^{(\alpha,\beta)}(x, y)$ are given in §§ 8-10.

In recent years the Eulerian numbers and certain generalizations have been encountered in a number of combinatorial problems [2], [3], [4], [5], [6], [7]. The study of Eulerian operators is of intrinsic interest and may be useful for applications.

2. It is convenient to first discuss (1.13), that is,

$$(2.1) \quad (x + y + xy(D_x + D_y))^n = \sum_{k=0}^{n} C_{n,k}(x, y)(xy)^k (D_x + D_y)^k.$$

We shall require the following operational formulas:

$$(2.2) \qquad (D_x + D_y)^k (x + y) = 2k(D_x + D_y)^{k-1} + (x + y)(D_x + D_y)^k,$$

$$(2.3) \qquad (D_x + D_y)^k xy = k(k - 1)(D_x + D_y)^{k-2} + k(x + y)(D_x + D_y)^{k-1} + xy(D_x + D_y)^k.$$

The proof is by induction on k. For (2.2) we have

$$\begin{aligned}(D_x + D_y)^k (x + y) &= 2k(D_x + D_y)^k + (D_x + D_y)(x + y)(D_x + D_y)^k \\ &= 2k(D_x + D_y)^k + [2 + (x + y)(D_x + D_y)](D_x + D_y)^k \\ &= 2(k + 1)(D_x + D_y)^k + (x + y)(D_x + D_y)^{k+1}.\end{aligned}$$

As for (2.3), we have

$$(D_x + D_y)^{k+1} xy = k(k-1)(D_x + D_y)^{k-1} +$$
$$+ k(D_x + D_y)(x + y)(D_x + D_y)^{k-1} + (D_x + D_y)\, xy\, (D_x + D_y)^k$$
$$= k(k-1)(D_x + D_y)^{k-1} + k[2 + (x + y)(D_x + D_y)](D_x + D_y)^{k-1}$$
$$+ [x + y + xy(D_x + D_y)](D_x + D_y)^k$$
$$= k(k+1)(D_x + D_y)^{k-1} + (k+1)\, xy\,(D_x + D_y)^k + xy\,(D_x + D_y)^{k+1}.$$

Incidentally, the special case $k = 1$ of (2.3) may be noted:

(2.4) $$(D_x + D_y)\, xy = x + y + xy(D_x + D_y) \equiv \Omega$$

Thus

(2.5) $$\Omega^n = [(D_x + D_y)\, xy]^n.$$

We now apply Ω to both sides of (2.1). Then

$$\Omega^{n+1} = \sum_{k=0}^{n} \Omega \{C_{n,k}(x, y)(xy)^k\}(D_x + D_y)^k.$$

Since

$$(D_x + D_y)\{C_{n,k}(x, y)(xy)^k\}$$
$$= k(xy)^{k-1}(x + y)\, C_{n,k}(x, y)$$
$$+ (xy)^k (D_x + D_y)\, C_{n,k}(x, y) + (xy)^k C_{n,k}(x, y)(D_y + D_x),$$

it follows that

$$\Omega^{n+1} = (x + y)\sum_{k=0}^{\infty} C_{n,k}(x, y)(xy)^k (D_x + D_y)^k$$
$$+ xy \sum_{k=0}^{n} \{k(xy)^{k-1}(x + y)\, C_{n,k}(x, y) + (xy)^k (D_x + D_y)\, C_{n,k}(x, y)$$
$$+ (xy)^k C_{n,k}(x, y)(D_x + D_t)\}(D_x + D_y)^k$$
$$= \sum_k (xy)^k \{[(k+1)(x + y) + xy(D_x + D_y)]\, C_{n,k}(x, y)$$
$$+ C_{n,k-1}(x, y)\}(D_x + D_y)^k$$

We therefore have the recurrence

(2.6)
$$C_{n+1,k}(x, y) = [(k+1)(x+y) + xy(D_x + D_y)]\, C_{n,k}(x, y) + C_{n,k-1}(x, y).$$

This establishes the existence of the expansion (2.1) and indeed shows that $C_{n,k}(x, y)$ is a homogeneous polynomial in x, y of degree $n - k$.

In the next place we apply Ω to both sides of (6.1) but now on the right. Then

$$\Omega^{n+1} = \sum_{k=0}^{n} C_{n,k}(x, y)\,(xy)^k\,(D_x + D_y)^k\,[x + y + xy\,(D_x + D_y)].$$

Applying (2.2) and (2.3), we get

$$\Omega^{n+1} = \sum_{k=0}^{n} C_{n,k}(x, y)\,(xy)^k\,\{2k\,(D_x + D_y)^{k-1} + (x + y)\,(D_x + D_y)^k\}$$

$$+ \sum_{k=0}^{n} C_{n,k}(x, y)\,(xy)^k\,\{k\,(k-1)\,(D_x + D_y)^{k-2} + k\,(x + y)\,(D_x + D_y)^{k-1} + xy\,(D_x + D_y)^k\}\,(D_x + D_y).$$

It follows that

$$C_{n+1,k}(x, y) = (k + 1)\,(x + y)\,C_{n,k}(x, y) + (k + 1)\,(k + 2)\,xy\,C_{n,k+1}(x, y) + C_{n,k-1}(x, y). \tag{2.7}$$

Comparing (2.7) with (2.6), we get

$$(D_x + D_y)\,C_{n,k}(x, y) = (k + 1)\,(k + 2)\,C_{n,k+1}(x, y). \tag{2.8}$$

It is clear from (2.8) that

$$C_{n,k}(x, y) = \frac{1}{k!\,(k + 1)!}\,(D_x + D_y)^k\,C_{n,0}(x, y). \tag{2.9}$$

Since, by (1.1),

$$C_{n,0}(x, y) = A_n(x, y),$$

(2.9) becomes

$$C_{n,k}(x, y) = \frac{1}{k!\,(k + 1)!}\,(D_x + D_y)^k\,A_n(x, y) \qquad (0 \le k \le n). \tag{2.10}$$

so that we have proved (1.14).

3. Put

$$f_n(x, y, z) = \sum_{k=0}^{n} (k + 1)!\,C_{n,k}(x, y)\,z^k. \tag{3.1}$$

Then $f_n(x, y, z)$ is homogeneous in x, y, z of degree n. We also define

$$g_k(x, y) = \sum_{n=k}^{\infty} \frac{1}{(n+1)!} C_{n,k}(x, y). \tag{3.2}$$

Since, by (1.5) and (1.9),

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} A_n(x, y) = F(x, y),$$

it follows that

$$g_k(x, y) = \frac{1}{k!\,(k+1)!} (D_x + D_y)^k F(x, y). \tag{3.3}$$

It is easily verified that

$$(D_x + D_y) F = F^2$$

and therefore

$$(D_x + D_y)^k F = k!\, F^{k+1}. \tag{3.4}$$

Thus (3.3) becomes

$$g_k(x, y) = \frac{1}{(k+1)!} F^{k+1}(x, y). \tag{3.5}$$

Therefore

$$G(x, y, z) \equiv \sum_{k=0}^{\infty} (k+1)!\, g_k(x, y) z = \frac{F(x, y)}{1 - zF(x, y)}. \tag{3.6}$$

Also, since

$$G(x, y, z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} f_n(x, y, z),$$

we get

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} f_n(x, y, z) = \frac{F(x, y)}{1 - zF(x, y)}. \tag{3.7}$$

By (1.6),

$$\begin{aligned} \frac{F(x, y)}{1 - zF(x, y)} &= \frac{e^x - e^y}{(xe^y - ye^x) - z(e^x - e^x)} \\ &= \frac{e^x - e^y}{(x+z)e^x - (y+z)e^x} \\ &= \frac{e^{x+z} - e^{y+z}}{(x+z)e^{y+z} - (y+z)e^{x+z}} = F(x+z, y+z). \end{aligned}$$

Thus (3.7) becomes

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} f_n(x, y, z) = F(x+z, y+z). \tag{3.8}$$

Since

$$F(x+z, y+z) = \sum_{n=0}^{\infty} A_n(x+z, y+z),$$

it follows that

$$f_n(x, y, z) = A_n(x+z, y+z). \tag{3.9}$$

This formula can also be proved without the use of generating functions.

4. We now consider the general case:

$$\Omega^n_{\alpha,\beta} = \sum_{k=0}^{n} C^{(\alpha,\beta)}_{n,k}(x, y)\,(xy)^k (D_x + D_y)^k, \tag{4.1}$$

where

$$\Omega_{\alpha,\beta} \equiv \alpha x + \beta y + xy(D_x + D_y). \tag{4.2}$$

We apply $\Omega_{\alpha,\beta}$ on the left of each side of (4.1). Since

$$\Omega_{\alpha,\beta}\left\{C^{(\alpha,\beta)}_{n,k}(x, y)\,(xy)^k (D_y + D_y^k)\right\}$$
$$= (\alpha x + \beta y)\, C^{(\alpha,\beta)}_{n,k}(x, y)\,(xy)^k + k(x+y)\, C^{(\alpha,\beta)}_{n,k}(x, y)\,(xy)^k$$
$$+ (xy)^{k+1}(D_x + D_y)\, C^{(\alpha,\beta)}_{n,k}(x, y) + (xy)^{k+1} C^{(\alpha',\beta)}_{n,k}(x, y)\,(D_x + D_y),$$

we get the recurrence

$$C^{(\alpha,\beta)}_{n+1,k}(x, y) = (\alpha x + \beta y)\, C^{(\alpha,\beta)}_{n,k}(x, y)$$
$$+ [k(x+y) + xy(D_x + D_y)]\, C^{(\alpha,\beta)}_{n,k}(x, y) + C^{(\alpha,\beta)}_{n,k-1}(x, y). \tag{4.3}$$

Next, apply $\Omega_{\alpha,\beta}$ on the right. Since

$$(D_x + D_y)^k (\alpha x + \beta y) = \sum_{j=0}^{k} \binom{k}{j} D_x^j D_y^{k-j} (\alpha x + \beta y)$$
$$= \sum_{j=0}^{k} \binom{k}{j} \left\{\alpha (x D_x^j + j D_x^{j-1}) D_y^{k-j} + \beta D_x^j (y D_j^{k-j} + (k-j) D_y^{k-j-1})\right\}$$

$$= \alpha x \sum_{j=0}^{k} \binom{k}{j} D_x^j D_y^{k-j} + k\alpha \sum_{j=1}^{k} \binom{k-1}{j-1} D_x^{j-1} D_y^{k-j} + \beta y \sum_{j=0}^{k} \binom{k}{j} D_x^j D_y^{k-j}$$

$$+ k\beta \sum_{j=0}^{k-1} \binom{k-1}{j} D_x^j D_y^{k-j}$$

$$= (\alpha x + \beta y)(D_x + D_y)^k + k(\alpha + \beta)(D_x + D_y)^{k-1}$$

and, by (2.3),

$$(D_x + D_y)^k xy + k(k-1)(D_x + D_y)^{k-2}$$
$$+ k(x+y)(D_x + D_y)^{k-1} + xy(D_x + D_y)^k,$$

we get

$$\Omega_{\alpha,\beta}^{n+1} = \sum_{k=0}^{n} C_{n,k}^{(\alpha,\beta)}(x,y)(xy)^k \{(\alpha x + \beta y)(D_x + D_y)^k$$
$$+ k(\alpha+\beta) + (D_x + D_y)^{k-1}$$
$$+ k(k-1)(D_x + D_y)^{k-1} + k(x+y)(D_x + D_y)^k + xy(D_x + D_y)^{k+1}\}.$$

It follows that

$$C_{n+1,k}^{(\alpha,\beta)}(x,y) = ((k+\alpha)x + (k+\beta)y)\, C_{n,k}^{(\alpha,\beta)}(x,y) \tag{4.4}$$
$$+ (k+1)(k+\alpha+\beta)\, xy\, C_{n,k+1}^{(\alpha,\beta)}(x,y) + C_{n,k-1}^{(\alpha,\beta)}(x,y).$$

Comparison of (4.4) with (4.3) gives

$$(D_x + D_y)\, C_{n,k}^{(\alpha,\beta)}(x,y) = (k+1)(k+\alpha=\beta)\, C_{n,k-1}^{(\alpha,\beta)}(x,y). \tag{4.5}$$

It follows that

$$C_{n.k}^{(\alpha,\beta)}(x,y) = \frac{1}{k!\,(\alpha+\beta)_k} (D_n + D_y)^k\, C_{n,0}^{(\alpha,\beta)}(x,y), \tag{4.6}$$

provided $\alpha + \beta$ is not equal to zero or a negative integer. Moreover, by (1.20),

$$C_{n,0}^{(\alpha,\beta)}(x,y) = A_n(x, y \mid \alpha, \beta),$$

so that (4.6) becomes

$$C_{n,k}^{(\alpha,\beta)}(x,y) = \frac{1}{k!\,(\alpha+\beta)_k} (D_x + D_y)^k A_n(x, y \mid \alpha, \beta). \tag{4.7}$$

It follows from (1.19) and (4.7) that

(4.8)
$$A_{m+n}(x, y \mid \alpha, \beta)$$
$$= \sum_{k=0}^{\min(m,n)} \frac{1}{k!\,(\alpha+\beta)_k} (xy)^k (D_x + D_y)^k A_m(x, y \mid \alpha, \beta)\,(D_x + D_y)^k A_n(x, y \mid \alpha, \beta).$$

5. Put

$$f_n(x, y, z \mid \alpha, \beta) = \sum_{k=0}^{n} (\alpha+\beta)_k C_{n,k}^{(\alpha,\beta)}(x, y)\, z^k, \tag{5.1}$$

$$g_k(x;\, y \mid \alpha, \beta) = \sum_{n=k}^{\infty} \frac{1}{n!} C_{n,k}^{(\alpha,\beta)}(x, y), \tag{5.2}$$

$$\Phi_{\alpha,\beta}(x, y) = (1 + xF(x, y))^{\alpha}\,(1 + yF(x, y))^{\beta}. \tag{5.3}$$

Since

$$\sum_{n=0}^{\infty} \frac{1}{n!} A_n(x, y \mid \alpha, \beta) = \Phi_{\alpha,\beta}(x, y),$$

it follows from (4.7) and (5.2) that

$$g_k(x, y) = \frac{1}{k!\,(\alpha+\beta)_k} (D_x + D_y)^k \Phi_{\alpha,\beta}(x, y). \tag{5.4}$$

But

$$(D_x + D_y)^k \Phi_{\alpha,\beta}(x, y) = (\alpha+\beta)_k F^k(x, y)\, \Phi_{\alpha,\beta}(x, y), \tag{5.5}$$

so that (5.4) becomes

$$g_k(x, y \mid \alpha, \beta) = \frac{1}{k!} F^k(x, y)\, \Phi_{\alpha,\beta}(x, y). \tag{5.6}$$

Now put

$$G(x, y, z \mid \alpha, \beta) = \sum_{k=0}^{\infty} (\alpha+\beta)_k\, g_k(x, y \mid \alpha, \beta)\, z^k.$$

Then, by (5.6),

$$G(x, y, z \mid \alpha, \beta) = \frac{\Phi_{\alpha,\beta}(x, y)}{(1 - zF(x, y, y))^{\alpha+\beta}}. \tag{5.7}$$

Since

$$\Phi_{\alpha,\beta}(x, y) = \frac{(x-y)^{\alpha+\beta} e^{x+y}}{(xe^x - ye^y)^{\alpha+\beta}}$$

and

$$\frac{\Phi_{\alpha,\beta}(x, y)}{(1 - zF(x, y))^{\alpha+\beta}} = \frac{(x-y)^{\alpha+\beta} e^{\alpha x+y\beta}}{[(xe^y - ye^x) - z(e^x - e^y)]}$$

$$= \frac{(x-y)^{\alpha+\beta} e^{\alpha x+\beta y}}{[(x+z)e^y - (y+z)e^x]}$$

$$= \frac{(x-y)^{\alpha+\beta} e^{\alpha(x+z)+\beta(x+z)}}{[(x+z)e^{y+z} - (y+z)e^{x+z}]^{\alpha+\beta}},$$

(5.7) becomes

(5.8) $$G(x, y, z \mid \alpha, \beta) = \Phi_{\alpha,\beta}(x+z, y+z).$$

On the other hand, by (5.1) and (5.2),

$$G(x, y, z \mid \alpha, \beta) = \sum_{k=0}^{\infty} (\alpha+\beta)_k z^k \sum_{n=k}^{\infty} \frac{1}{n!} C_{n,k}^{(\alpha,\beta)}(x, y)$$

$$= \sum_{k=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} (\alpha+\beta)_k C_{n,k}^{(\alpha,\beta)}(x, y) z^k$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f_n(x, y, z \mid \alpha, \beta),$$

so that, by (5.8),

(5.9) $$\sum_{n=0}^{\infty} \frac{1}{n!} f_n(x, y, z \mid \alpha, \beta) = \Phi_{\alpha,\beta}(x+z,\ y+z).$$

Therefore, by (1.15) and (1.18), we get

(5.10) $$f_n(x, y, z \mid \alpha, \beta) = A_n(x+z,\ y+z \mid \alpha, \beta).$$

This identity implies

(5.11) $$A_n(x+z,\ y+z \mid \alpha, \beta) = \frac{z^k}{k!}(D_x + D_y)^k A_n(x, y \mid \alpha, \beta),$$

which can also be obtained by applying Taulor's theorem to $A_n(x, y \mid \alpha, \beta)$.

6. As noted above, (4.7) is not valid when $\alpha + \beta$ is zero or a negative integer We shall now consider the excluded values It is

convenient to begin with the special case $\alpha = \beta = 0$ In place of (4.1) we now have

$$(6.1) \qquad (xy(D_x + D_y))^n = \sum_{k=1}^{n} C_{n,k}^{(0,0)}(x, y)\,(xy)^k (D_x + D_y) \qquad (n \geq 1)$$

The recurrence (4.3) reduces to

$$(6.2) \quad C_{n+1,k}^{(0,0(}(x, y) = [k(x+y) + xy(D_x + D)_y] C_{n.k}^{(0,0)}(x, y) + C_{n,k-1}^{(0,0)}(x, y),$$

while (4.4) becomes

$$(6.3) \qquad C_{n+1,k}^{(0,0)}(x, y) = k(x + y)\, C_{n,k}^{(0,0)}(x, y) + k(k + 1)\, xy\, C_{n,k+1}^{(0,0)}(x, y) + C_{n,k-1}^{(0,0)}(x, y)$$

Hence

$$(6.4) \qquad (D_x + D_y)\, C_{n,k}^{(0,0)}(x, y) = k(k + 1)\, C_{n,k+k}^{(0,0)}(x, y),$$

so that

$$(6.5) \qquad C_{n,k}^{(0,0)}(x, y) = \frac{1}{k!\,(k-1)!}(D_x + D_y)^{k-1} C_{n,1}^{(0,0)}(x, y) \qquad (k \geq 1)$$

For $k = 1$, (6.2) reduces to

$$C_{n+1,1}^{(0,0)}(x, y) = [x + y + xy(D_x + D_y)]\, C_{n,1}^{(0,0)}(x, y),$$

which yields

$$C_{n,1}^{(0,0)}(x, y) = [x + y + xy(D_x + D_y)]^{n-1} \cdot C_{1,1}^{(0,0)}(x, y)$$

It is clear from (6.1) that $C_{1,1}^{(0,0)}(x, y) = 1$ and therefore

$$(6.6) \qquad C_{n,1}^{(0,0)}(x, y) = A_{n-1}(x, y) \equiv A_{n-1}(x, y \mid 1,1)$$

Thus (6.5) becomes

$$(6.7) \qquad C_{n,k}^{(0,0)}(x, y) = \frac{1}{k!\,(k-1)!}(D_x + D_y)^{k-1} A_{n-1}(x, y)$$

Before discussing the general case $\alpha + \beta$ equal to zero or a negative integer, we consider the expansion

$$\Omega^n_{\alpha,\beta} = \sum_{k=0}^{n} Q^{(\alpha,\beta)}_{n,k}(x, y)\,(xy\,(D_x + D_y))^k, \tag{6.8}$$

where the $Q^{(\alpha,\beta)}_{n,k}(x, y)$ are to be determined Clearly

$$\begin{aligned}\Omega^{n+1}_{\alpha,\beta} &= (\alpha x + \beta y) \sum_{k=0}^{n} Q^{(\alpha,\beta)}_{n,k}(x, y)\,(xy\,(D_x + D_y))^k \\ &+ xy \sum_{k=0}^{n} (D_x + D_y)\,Q^{(\alpha,\beta)}_{n,k}(x, y)\cdot(xy\,(D_x + D_y))^k \\ &+ \sum_{k=0}^{n} Q^{(\alpha,\beta)}_{n,k}(x, y)\,(xy\,(D_x + D_y))^k,\end{aligned}$$

so that

$$Q^{(\alpha,\beta)}_{n+1,k}(x, y) = \Omega_{\alpha,\beta}\,Q^{(\alpha,\beta)}_{n,k}(x, y) + Q^{(\alpha,\beta)}_{n,k-1}(x, y) \tag{6.9}$$

For $k = 0$, (6.9) reduces to

$$Q^{(\alpha,\beta)}_{n+1,0}(x, y) = \Omega_{n,\beta}\,Q^{(\alpha,\beta)}_{n,0}(x, y)$$

Since, by (6.8),

$$Q^{(\alpha,\beta)}_{1,0}(x, y) = \alpha x + \beta y = A_1(x, y \mid x, y),$$

it is clear that

$$Q^{(\alpha,\beta)}_{n,0}(x, y) = A_n(x, y \mid \alpha, \beta) \tag{6.10}$$

We shall now show that

$$Q^{(\alpha,\beta)}_{n,0}(x, y) = \binom{n}{k} A_{n-k}(x, y \mid \alpha, \beta) \qquad (0 \le k \le n) \tag{6.11}$$

Clearly (6.11) holds for $n = 0$ Assuming that it holds up to and including the value n, we have by (6.9),

$$Q_{n+1,k}^{(\alpha,\beta)}(x, y) = \binom{n}{k} \Omega_{\alpha,\beta} A_{n-k}(x, y \mid \alpha, \beta) + \binom{n}{k-1} A_{n-k+1}(x, y \mid \alpha, \beta)$$

$$= \binom{n}{k} A_{n-k+1}(x, y \mid \alpha, \beta) + \binom{n}{k-1} A_{n-k+1}(x, y \mid \alpha, \beta)$$

$$= \binom{n}{k} A_{n-k+1}(x, y \mid \alpha, \beta)$$

We have therefore proved

(6.12) $$\Omega_{\alpha,\beta}^n = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}(x, y \mid \alpha, \beta)\,(xy\,(D_x + D_y))^k$$

This suggests the following more general result:

(6.13) $$\Omega_{\alpha+\gamma,\beta+\delta}^n = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}(x, y \mid \alpha, \beta)\,\Omega_{\gamma,\delta}^k$$

To prove (6.13), consider

$$\Omega_{\alpha+\gamma,\beta+\delta}^n = \sum_{k=0}^{n} R_{n,k}\,\Omega_{\gamma,\delta}^k,$$

where the $R_{n,k}$ are functions of x, y, α, β, γ, δ. Then

$$\Omega_{\alpha+\gamma,\beta+\delta}^{n+1} = [(\alpha + \gamma)\,x + (\beta + \delta)\,y + xy\,(D_x + D_y)] \sum_{k=0}^{n} R_{n,k}\,\Omega_{\gamma,\delta}^k$$

$$= [(\alpha + \gamma)\,x + (\beta + \delta)] \sum_{k=0}^{n} R_{n,k}\,\Omega_{\gamma,\delta}^k$$

$$+ xy \sum_{k=0}^{n} (D_x + D_y)\,R_{n,k} \cdot \Omega_{\gamma,\delta}^k + \sum_{k=0}^{n} R_{n,k} \cdot xy\,(D_x + D_y)\,\Omega_{\gamma,\delta}^k$$

$$= \sum_{k=0}^{n} \Omega_{\alpha,\beta}\,R_{n,k} \cdot \Omega_{\gamma,\delta}^k + \sum_{k=0}^{n} R_{n,k}\,\Omega_{\gamma,\delta}^k$$

This evidently implies

(6.14) $$R_{n+1,k} = \Omega_{\alpha,\beta}\,R_{n,k} + R_{n,k-1}.$$

Then, exactly as above, we show first that

$$R_{n,0} = A_n(x, y \mid \alpha, \beta)$$

and generally

$$R_{n,k} = \binom{n}{k} A_{n-k}(x, y \mid \alpha, \beta) \qquad (0 \le k \le n)$$

This completes the proof of (6.13)

As a special case of (6.13), we note

$$(6.15) \qquad (xy(D_x + D_y))^n = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}(x, y \mid -\alpha, -\beta)\, \Omega^k_{\alpha,\beta}.$$

7 We now treat the general excluded case in (4.7), $\alpha + \beta$ equal to zero or a negative integer. We shall require the following formulas:

$$(7.1) \qquad \Phi_{\alpha,\beta}(x, y) = \sum_0^\infty \frac{1}{n!} A_n(x, y \mid \alpha, \beta)$$

$$= (1 + xF(x, y))^\alpha (1 + yF(x, y))^\beta,$$

$$(7.2) \qquad (D_x + D_y)^k F(x, y) = k!\, F^{k+1}(x, y),$$

$$(7.3) \qquad (D_x + D_y)^k \Phi_{\alpha,\beta}(x, y) = (\alpha + \beta)_k F^k(x, y)\, \Phi_{\alpha,\beta}(x, y).$$

If $\alpha + \beta$ is not equal to zero or a negative integer, we have seen that

$$C^{(\alpha,\beta)}_{n,k}(x, y) = \frac{1}{k!\,(\alpha + \beta)_k} (D_x + D_y)^k A_n(x, y \mid \alpha, \beta).$$

It follows that

$$\sum_{n=k}^\infty \frac{1}{n!} C^{(\alpha,\beta)}_{n,k}(x, y) = \frac{1}{k!\,(\alpha + \beta)_k} (D_x + D_y)^k \Phi_{\alpha,\beta}(x, y)$$

and therefore, by (7.3),

$$(7.4) \qquad \sum_{n=k}^\infty \frac{1}{n!} C^{(\alpha,\beta)}_{n,k}(x, y) = \frac{1}{k!} F^k(x, y)\, \Phi_{\alpha,\beta}(x, y).$$

Put

$$(7.5) \qquad F^k(x, y) = \sum_{n=0}^\infty \frac{1}{(n + k)!} A^{(k)}_n(x, y) \qquad (k = 1, 2, 3, \ldots),$$

where $A_n^{(k)}(x, y)$ is homogeneous of degree n in x, y. For $k = 1$ we have

$$A_n^{(1)}(x, y) = A_n(x, y) = A_n(x, y \mid 1, 1). \tag{7.6}$$

It follows from (7.5) and (7.6) that

$$A_n^{(k+1)}(x, y) = \sum_{r=0}^{n} \binom{n+k+1}{r+1} A_r(x, y) A_{n-r}^{(k)}(x, y) \tag{7.7}$$

and therefore the coefficients in $A_n^{(k)}(x, y)$ are positive integers

By (7.4) and (7.5) we get, since $C_{n,k}^{(\alpha,\beta)}(x, y)$ is of degree $n - k$,

$$C_{n,k}^{(\alpha,\beta)}(x, y) = \frac{1}{k!} \sum_{r=k}^{n} \binom{n}{r} A_{r-k}^{(k)}(x, y) A_{n-r}(x, y \mid \alpha, \beta). \tag{7.8}$$

Since both $C_{n,k}^{(\alpha,\beta)}(x, y)$ and $A_n(x, y \mid \alpha, \beta)$ are polynomials in α, β (as well as in x, y), it follows that (7.8) is valid for all α, β. The numerical coefficients in $C_{n,k}^{(\alpha,\beta)}(x, y)$ are integers; however that is not obvious from (7.8)

Since

$$A_n(x, y \mid 0, 0) = 0 \qquad (n > 0),$$

it is evident that (7.8) implies

$$C_{n,k}^{(0,0)}(x, y) = \frac{1}{k!} A_{r-k}^{(k)}(x, y). \tag{7.9}$$

By (7.2) and (7.5) we have

$$(k-1)! \sum_{n=0}^{\infty} \frac{1}{(n+k)!} A_u^{(k)}(x, y) = (D_x + D_y)^{k-1} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} A_n(x, y),$$

so that

$$A_n^{(k)}(x, y) = \frac{1}{(k-1)!} (D_x + D_y)^{k-1} A_{n+k-1}(x, y) \quad (1 \leq k \leq n). \tag{7.10}$$

Thus (7.8) becomes

$$C_{n,k}^{(\alpha,\beta)}(x, y) = \frac{1}{k!\,(k-1)!} \sum_{r=k}^{n} \binom{n}{r} (D_x + D_v)^{k-1} A_{r-1}(x, y) \cdot A_{n-r}(x, y \mid \alpha, \beta) \quad (1 \leq k \leq n). \tag{7.11}$$

For $\alpha = \beta = 0$, (7.11) reduces to

$$C_{n,k}^{(0,0)}(x, y) = \frac{1}{k!\,(k-1)!}(D_x + D_y)^{k-1} A_{n-1}(x, y) \tag{7.12}$$

in agreement with (6.7)

Both (7.8) and (7.11) are valid for all α, β.

8. We now consider the inverse of (4.1), that is,

$$(xy)^n (D_x + D_y)^n = \sum_{k=0}^{n} (-1)^{n-k} B_{n,k}^{(\alpha,\beta)}(x, y)\, \Omega_{\alpha,\beta}^k, \tag{8.1}$$

where, as will appear presently, $B_{n,k}^{(\alpha,\beta)}(x, y)$ is a homogeneous polynomial in x, y of degree $n - k$. The existense of a formula of this kind is evidently implied by (4.1). For example

$$xy\,(D_x + D_y) = -(\alpha x + \beta y) + \Omega_{\alpha,\beta},$$

$$(xy)^2 (D_x + D_y)^2 = (\alpha x + \beta y)^2 + \alpha x^2 + \beta y^2$$

$$- [(2\alpha + 1)x + (2\beta + 1)y]\,\Omega_{\alpha,\beta} + \Omega_{\alpha,\beta}^2.$$

To get a recurrence for the coefficients $B_{n,k}^{(\alpha,\beta)}$ we apply the operator $xy\,(D_x + D_y)$ to both sides of (8.1) — on the left. This gives

$$n\,(x + y)(xy)^n (D_x + D_y) + (xy)^{n+1} (D_x + D_y)^{n+1}$$

$$= xy \sum_{k=0}^{n} (-1)^{n-k} \left\{ (D_x + D_y)\, B_{n,k}^{(\alpha,\beta)}(x, y) + B_{n,k}^{(\alpha,\beta)}(x, y)\,(D_x + D_y) \right\} \Omega_{\alpha,\beta}^k$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \left\{ xy\,(D_x + D_y)\, B_{n,k}^{(\alpha,\beta)}(x, y) - (\alpha x + \beta y)\, B_{n,k}^{(\alpha,\beta)}(x\,y), \right.$$

$$\left. + B_{n,k}^{(\alpha,\beta)}(x, y)\,\Omega_{\alpha,\beta} \right\} \Omega_{\alpha,\beta}^k.$$

It follows that

$$(xy)^{n+1} (D_x + D_y)^{n+1} = -n\,(x + y) \sum_{k=0}^{n} (-1)^{n-k} B_{n,k}^{(\alpha,\beta)}(x, y)\, \Omega_{\alpha,\beta}^k$$

$$+ \sum_{k=0}^{n} (-1)^{n-k} \left\{ xy\,(D_x + D_y)\, B_{n,q}^{(\alpha,\beta)}(x, y) \right.$$

$$\left. - (\alpha x + \beta y)\, B_{n,k}^{(\alpha,\beta)}(x, y) \right\} \Omega_{\alpha,\beta}^k$$

$$+ \sum_{k=1}^{n+1} (-1)^{n-k+1} B_{n,k-1}^{(\alpha,\beta)}(x, y)\, \Omega_{\alpha,\beta}^k.$$

Therefore

$$(8.2)\quad B^{(\alpha,\beta)}_{n+1,k}(x, y) = [(\alpha + n)x + (\beta + n)y - xy(D_x + D_y)] \cdot B^{(\alpha,\beta)}_{n,k}(x, y) + B^{(\alpha,\beta)}_{n,k-1}(x, y).$$

On the other hand, if we multiply both sides of (8.1) on the right by $\Omega_{\alpha,\beta}$ we get

$$\sum_{k=0}^{n} (-1)^{n-k} B^{(\alpha,\beta)}_{0,k}(x, y)\,\Omega^{k+1}_{\alpha,\beta}$$

$$= (xy)^n (D_x + D_y)^n [\alpha x + \beta y + xy(D_x + D_y)]$$

$$= (xy)^n \{n(\alpha + \beta)(D_x + D_y)^{n-1} + (\alpha x + \beta y)(D_x + D_x)^n\}$$

$$+ (xy)^n \{n(n-1)(D_x + D_y)^{n-1} + n(x + y)(D_x + D_y)^n + xy(D_x + D_y)^{n+1}\}$$

$$= n(\alpha + \beta + n - 1)(xy)^n (D_x + D_y)^{n-1}$$

$$+ [(\alpha + n)x + (\beta + n)y](xy)^n (D_x + D_y)^n + (xy)^{n+1}(D_x + D_y)^{n+1}.$$

This implies

$$(8.3)\quad B^{(\alpha,\beta)}_{n+1,k}(x, y) = [(\alpha + n)x + (\beta + n)y]\,B^{(\alpha,\beta)}_{n,k}(x, y) - n(\alpha + \beta + n - 1)\,xy\,B^{(\alpha,\beta)}_{n-1,k} + B^{(\alpha,\beta)}_{n,k-1}(x, y)..$$

Comparing (8.3) with (8.2), we get

$$(8.4)\quad (D_x + D_y)\,B^{(\alpha,\beta)}_{n,k}(x, y) = n(\alpha + \beta + n - 1)\,B^{(\alpha,\beta)}_{n-1,k}(x, y).$$

In the next place, it follows at once from (4.1) and (8.1) that

$$(8.5)\quad \sum_{k=j}^{n} (-1)^{k-j} C^{(\alpha,\beta)}_{n,k}(x, y)\,B^{(\alpha,\beta)}_{n,j}(x, y) = \delta_{n,j}$$

and

$$(8.6)\quad \sum_{k=j}^{n} (-1)^{n-k} B^{(\alpha,\beta)}_{n,k}(x, y)\,C^{(\alpha,\beta)}_{k,j}(x, y) = \delta_{n,j}.$$

A formula of a different kind can be obtained by applying each side of (8.1) to $A_r(x, y \mid \alpha, \beta)$. Since

$$(D_x + D_y)^n A_r(x, y \mid \alpha, \beta) = 0 \qquad (o \le r < n),$$

$$\Omega^k_{\alpha,\beta} A_r(x, y \mid \alpha, b) = A_{r+k}(x, y \mid \alpha, \beta),$$

we get

(8.7) $$\sum_{k=0}^{n} (-1)^{n-k} B_{n,k}^{(\alpha,\beta)}(x,y)\, A_{r+k}(x,y \mid \alpha,\beta) = 0 \qquad (0 \le r < n).$$

In either (8.5) or (8.6) take $j = n$. Since $C_{n,n}^{(\alpha,\beta)}(x,y) = 1$, we have

(8.8) $$B_{n,n}^{(\alpha,\beta)}(x,y) = 1.$$

Also it is easily verified that

$$\Omega_{\alpha,\beta}(x^{-\beta} y^{-\alpha}) = 0,$$

so that

$$\Omega_{\alpha,\beta}^{k}(x^{-\beta} y^{-\beta}) = 0 \qquad (k = 1, 2, 3, \ldots).$$

Thus (8.1) implies

$$(-1)^n B_{n,0}^{(\alpha,\beta)}(x,y) = (xy)^n (D_x + D_y)^n x^{-\beta} y^{-\alpha}.$$

A little manipulation leads to

(8.9) $$B_{n,0}^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} (\alpha)_k (\beta)_{n-k} x^k y^{n-k}.$$

This is equivalent to

(8.10) $$\sum_{n=0}^{\infty} B_{n,0}^{(\alpha,\beta)}(x,y) \frac{n!}{z^n} = (1 - xz)^{-\alpha} (1 - yz)^{-\beta}.$$

We remark that $B_{n,0}^{(\alpha,\beta)}(x,y)$ satisfies the following recurrence:

(8.11) $$B_{n+1,0}^{(\alpha,\beta)}(x,y) = (\alpha x + \beta y + x^2 D_x + y^2 D_y)\, B_{n,0}^{(\alpha,\beta)}(x,y),$$

Indeed, by (8.9),

$$(\alpha x + \beta y + x^2 D_x + y^2 D_v)\, B_{n,0}^{(\alpha,\beta)}(x,y)$$
$$= (\alpha x + \beta y) \sum_{k=0}^{n} \binom{n}{k} (\alpha)_k (\beta)_{n-k} x^n y^{n-k}$$
$$+ \sum_{k=0}^{n} k \binom{n}{k} (\alpha)_k (\beta)_{n-k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} (n-k) \binom{n}{k} (\alpha)_k (\beta)_{n-k} x^k y^{n-k+1}.$$

The coefficient of $x^k y^{n-k+1}$ on the right is equal to

$$\alpha \binom{n}{k-1} (\alpha)_{k-1} (\beta)_{n-k+1} + \beta \binom{n}{k} (\alpha) (\beta)_{n-k}$$

$$+ (k-1) \binom{n}{k-1} (\alpha)_{k-1} (\beta)_{n-k+1} + (n-k) \binom{n}{k} (\alpha)_k (\beta)_{n-k}$$

$$= \binom{n}{k-1} (\alpha)_k (\beta)_{n-k+1} + \binom{n}{k} (\alpha)_k (\beta)_{n-k+1} = \binom{n+1}{k} (\alpha)_k (\beta)_{n-k+1}.$$

It follows from (8.11) that

$$B_{n,o}^{(\alpha,\beta)}(x, y) = (\alpha x + \beta y + x^2 D_x + y^2 D_y)^n \cdot 1. \tag{8.12}$$

9. When $\alpha = \beta = 0$, (8.1) reduces to

$$(xy)^n (D_x + D_y)^n = \sum_{k=o}^{n} B_{n,k}^{(o,o)}(x, y) (xy (D_x + D_y))^k, \tag{9.1}$$

while (8.3) becomes

$$B_{n+1,k}^{(o,o)}(x, y) = n (x + y) B_{n,k}^{(o,o)}(x, y) - n (n - 1) xy B_{n-1,k}^{(o,o)}(x, y) + B_{n,k-1}^{(o,o)}(x, y). \tag{9.2}$$

It is evident from (9.1) that

$$B_{n,o}^{(o,o)}(x, y) = 0 \qquad (n > 0). \tag{9.3}$$

For brevity, put

$$b_{n,k} = \frac{1}{(n-1)!} B_{n,k}^{(o,o)}(x, y) \qquad (n \geq 1).$$

Then (9.2) becomes

$$b_{n+1,k} - (x + y) b_{n,k} + xy b_{n-1,k} = \frac{1}{n} b_{n,k-1} \qquad (k \geq 1). \tag{9.4}$$

For $k = 1$, (9.4) reduces to

$$b_{n+1,1} - (x + y) b_{n,1} + xy b_{n-1,1} = 0 \qquad (n > 1). \tag{9.5}$$

The recurrence (9.5) implies

$$b_{n,1} = c_1 x^n + c_2 y^n,$$

where c_1, c_2 are constant. Since

$$b_{1,1} = 1, \qquad b_{2,1} = x + y,$$

we get

$$b_{n,1} = \frac{x^n - y^n}{x - y} \equiv \sigma_n. \tag{9.6}$$

Next, for $k = 2$, we have

$$b_{n+1,2} - (x + y)\, b_{n,2} + xy\, b_{n-1,2} = \frac{n}{1}\sigma_n. \tag{9.7}$$

Since

$$b_{1,2} = b_{2,0} = 1,$$

(9.7) holds for $n \geq 1$. It follows that

$$(1 - (x + y) z + xyz^2) \sum_1^\infty b_{n+1,2}\, z^n = \sum_1^\infty \frac{n}{1} \sigma_n z^n.$$

Since

$$\frac{1}{1 - (x + y) z + xyz^2} = \sum_0^\infty \sigma_{n+1} z^n,$$

we get

$$b_{n+1,2} = \sum_{j=1}^{n} \frac{1}{j} \sigma_j \sigma_{n-j+1}.$$

Generally (9.4) implies

$$\sum_{n=k-1}^{\infty} (b_{n+1,k} - (x + y)\, b_{n,k} + xyb_{n-1,k})\, z^n = \sum_{n=k-1}^{\infty} \frac{n}{1} b_{n,k-1} z^n,$$

that is,

$$\sum_{n=k-1}^{\infty} b_{n+1,k} z^n = \frac{1}{1 - (x + y) z + xyz^2} \sum_{n=k-1}^{\infty} \frac{n}{1} b_{n,k-1} z^n. \tag{9.8}$$

Therefore, as above, we get

$$b_{n+1,k} = \sum_{j=k-1}^{n} \frac{1}{j} b_{j,k-1} \sigma_{n-j+1}. \tag{9.9}$$

We may rewrite (9.9) in the form

$$b_{n+k,k} = \sum_{j=0}^{n} \frac{1}{j+k-1} b_{j+k-1,k-1} \sigma_{n-j+1}. \tag{9.10}$$

Using this formula we get

$$b_{n+3,3} = \sum_{0 \le i \le j \le n} \frac{1}{(i+1)(j+2)} \sigma_{i+1} \sigma_{j-i+1} \sigma_{n-j+1},$$

$$b_{n+4,4} = \sum_{0 \le i \le j \le k \le n} \frac{1}{(i+1)(j+2)(k+3)} \sigma_{i+1} \sigma_{j-i-1} \sigma_{k-j+1} \sigma_{n-k+1}.$$

and so on

We may also mention an operational formula for $b_{n,k}$. Define the operator D_z^{-1} by means of

$$D_z^{-1} f(z) = \int_0^z f(t)\,dt.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n} \sigma_n z^n = D_z^{-1} \sum_{n=0}^{\infty} \sigma_{n+1} z^n = D_z^{-1} \frac{1}{(1-xz)(1-zy)}.$$

By (9.7)

$$\sum_{n=1}^{\infty} b_{n+1,2} z^n = \frac{1}{(1-xz)(1-yz)} D_z^{-1} \frac{1}{(1-xz)(1-yz)},$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{n+1} b_{n+1,2} z^n = [D_z^{-1} (1-xz)^{-1} (1-yz)^{-1}]^2 \cdot 1.$$

At the next stage we get

$$\sum_{n=2}^{\infty} b_{n+1,3} z^n = [(1-xz)^{-1} (1-yz)^{-1} D_z^{-1}]^2 \ (1-xz)^{-1} (1-yz)^{-1}$$

The general formula is

$$\sum_{n=k-1}^{\infty} b_{n+1,k} z^n \tag{9.11}$$

$$= [(1-xz)^{-1} (1-yz^{-1} D_z^{-1}]^{k-1} (1-xz)^{-1} (1-yz)^{-1} \ (k \ge 1).$$

10. A generating function for $B_{n,k}^{(\alpha,\beta)}(x, y)$ in the general case can be found in the following way. It follows from (8.5) that

$$(10.1) \qquad \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=j}^{n} (-1)^{k-j} C_{n,k}^{(\alpha,\beta)}(x, y)\, B_{k,j}^{(\alpha,\beta)}(x, y) = \frac{z^j}{j!}.$$

By (7.4) we have

$$\sum_{n=k}^{\infty} \frac{1}{n!} C_{n,k}^{(\alpha,\beta)}(x, y) = \frac{1}{k!} F^k(x, y)\, \Phi_{\alpha,\beta}(x, y).$$

Since $C_{n,k}^{(\alpha,\beta)}(x, y)$ is homogeneous of degree $n - k$, this implies

$$\sum_{n=k}^{\infty} \frac{z^n}{n!} C_{n,k}^{(\alpha,\beta)}(x, y) = \frac{z^k}{k!} F^k(xz, yz)\, \Phi_{\alpha,\beta}(xz, yz).$$

Thus (10.1) becomes

$$\frac{z^j}{j!} = \sum_{k=j}^{\infty} (-1)^{k-j} B_{k,j}^{(\alpha,\beta)}(x, y) \sum_{n=k}^{\infty} \frac{z^n}{n!} C_{n,k}^{(\alpha,\beta)}(x, y)$$
$$= \Phi_{\alpha,\beta}(xz, yz) \sum_{k=j}^{\infty} (-1)^{k-j} B_{k,j}^{(\alpha,\beta)}(x, y) \frac{z^k F^k(xz, yz)}{k!}.$$

Multiplying by v^j and summing over j, we get

$$(10.2) \qquad e^{zv} = \Phi_{\alpha,\beta}(xz, yz) \sum_{k=0}^{\infty} \sum_{j=0}^{n} (-1)^{k-j} B_{k,j}^{(\alpha,\beta)}(x, y) \frac{z^k F^k(xz, yz)}{k!} v^j.$$

Consider the equation

$$(10.3) \qquad zF(xz, yz) = u,$$

that is,

$$\frac{e^{xz} - e^{yz}}{xe^{yz} - ye^{xz}} = u.$$

This reduces to

$$e^{(x-y)z} = \frac{1 - xu}{1 - yu},$$

so that

$$(10.4) \qquad z = \frac{1}{x - y} \log \frac{1 + xu}{1 + yu} = \sum_{1}^{\infty} (-1)^{n-1} \frac{1}{n} \sigma_n u^n,$$

where as above

$$\sigma_n = (x^n - y^n)/(x - y).$$

Since

$$\Phi_{\alpha,\beta}(xz, yz) = (1 + xzF(xz, yz))^\alpha (1 + yzF(xz, yz))^\beta = (1 + xu)^\alpha (1 + yu)^\beta,$$

(10.2) becomes

$$e^{zv} = (1 + xu)^\alpha (1 + yu)^\beta \sum_{k=0}^{\infty} \frac{u^k}{k!} \sum_{j=0}^{k} (-1)^{k-j} B_{k,j}^{(\alpha,\beta)}(x, y)\, v^j. \tag{10.5}$$

But, by (10.4),

$$e^{zv} = \left(\frac{1 + xu}{1 + yu}\right)^{v/(x-y)},$$

so that (10.5) may be replaced by

$$\sum_{k=0}^{\infty} \frac{u^k}{k!} \sum_{j=0}^{k} B_{k,j}^{(\alpha,\beta)}(x, y)(x - y)^j v^j = \left(\frac{1 - xu}{1 - yu}\right)^{-v} (1 - xu)^{-\alpha} (1 - yu). \tag{10.6}$$

In particular, for $v = 0$, (10.6) reduces to

$$\sum_{k=0}^{\infty} \frac{k!}{u^k} B_{k,0}^{(\alpha,\beta)}(x, y) = (1 - xu)^{-\alpha} (1 - yu)^{-\beta}, \tag{10.7}$$

which is evidently in agreement with (8.10).

For $\alpha = \beta = 0$, (10.6) becomes

$$\sum_{k=0}^{\infty} \frac{u^k}{k!} \sum_{j=0}^{k} B_{k,j}^{(0,0)}(x, y)(x - y)^j v^j = \left(\frac{1 - xu}{1 - yu}\right)^{-v}. \tag{10.8}$$

Since

$$\left(\frac{1 - xu}{1 - yu}\right)^{-v} = \sum_{r=0}^{\infty} \frac{(v)_r}{r!} x^r u^r \sum_{s=0}^{\infty} \frac{(-v)_s}{s!} y^s v^s,$$

we get

$$\sum_{j=0}^{k} B_{k,j}^{(0,0)}(x, y)(x - y)^j v^j = \sum_{r=0}^{k} \binom{k}{r} (-v)_r (v)_{k-r}. \tag{10.9}$$

The general result is only slightly more complicated, namely

$$\sum_{j=0}^{k} B_{k,j}^{(\alpha,\beta)}(x, y)(x - y)^j v^j = \sum_{r=0}^{k} \binom{k}{r} (\alpha + v)_r (\beta - v)_{k-r}. \tag{10.10}$$

It follows from (10.6), (10.7) and (10.8) that

$$\sum_{j=0}^{k} B_{k,j}^{(\alpha,\beta)}(x, y)\, v^j = \sum_{r=0}^{k} \binom{k}{r} B_{k-r,0}^{(\alpha,\beta)}(x, y) \sum_{j=0}^{r} B_{r,j}^{(0,0)}(x, y)\, v^j$$

and therefore

$$B_{k,j}^{(\alpha,\beta)}(x, y) = \sum_{r=0}^{k} B_{r,j}^{(0,0)}(x, y)\, B_{k-r,0}^{(\alpha,\beta)}(x, y). \tag{10.11}$$

Comparing coefficients of v^j on both sides of (10.8), we get

$$\sum_{k=j}^{\infty} \frac{u^k}{k!} B_{k,j}^{(0,0)}(x, y) = \frac{(-1)^j}{j!}\left(\log \frac{1-xu}{1-yu}\right)^j.$$

Differentiation with respect to u gives

$$\sum_{k=j-1}^{\infty} \frac{u^k}{k!} B_{k+1,j}^{(0,0)}(x, y)$$
$$= \frac{(-1)^{j-1}}{(j-1)!}(x-y)^{-j+1}\left(\log \frac{1-xu}{1-yu}\right)^{j-1} \frac{1}{(1-xu)(1-yu)}$$

which is equivalent to (9.8).

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