FIBONACCI NOTES
4: q-FIBONACCI POLYNOMIALS

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1. We shall make use of the notation of [1]. In addition we define

$$\phi_n(a) = \phi_n(a,q) = \sum_{2k<n} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2} a^{n-2k-1} \qquad (n \geqslant 1). \tag{1.1}$$

Since

$$\begin{bmatrix} n-k \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} = q^{n-2k} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} ,$$

it is clear that

$$\phi_{n+1}(a) - a\phi_n(a) = \sum_{2k \leqslant n} \left(\begin{bmatrix} n-k \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} \right) q^{k^2} a^{n-2k} = \sum_{0<2k \leqslant n} q^{n-2k} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} q^{k^2} a^{n-2k}$$

$$= q^{n-1} \sum_{0<2k \leqslant n} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} q^{(k-1)^2} a^{n-2k} = q^{n-1} \sum_{2k<n-1} \begin{bmatrix} n-k-2 \\ k \end{bmatrix} q^{k^2} a^{n-2k-2} .$$

Hence

$$\phi_{n+1}(a) - a\phi_n(a) = q^{n-1} \phi_{n-1}(a) \qquad (n > 1). \tag{1.2}$$

The first few values of $\phi_n(a)$ are easily computed by means of (1.1) or (1.2).

$$\phi_1(a) = 1, \quad \phi_2(a) = a, \quad \phi_3(a) = a^2 + q, \quad \phi_4(a) = a^3 + q(1+q)a,$$
$$\phi_5(a) = a^4 + q(1+q+q^2)a^2 + q^4, \quad \phi_6(a) = a^5 + q(1+q+q^2+q^3)a^3 + q^4(1+q+q^2)a .$$
$$\phi_7(a) = a^6 + q(1+q+q^2+q^3+q^4)a^4 + q^4(1+q+q^2)(1+q^2)a^2 + q^9 .$$

If we put $\phi_0(a) = 0$ then (1.2) holds for all $n \geqslant 1$. By means of (1.2) we can define $\phi_n(a)$ for all integral n. It is convenient to put

$$\overline{\phi}_n(a) = \overline{\phi}_n(a,q) = (-1)^{n-1} \phi_{-n}(a). \tag{1.3}$$

Then (1.2) becomes

$$\overline{\Phi}_n(a) = q^n (a\overline{\Phi}_{n-1}(a) + \overline{\Phi}_{n-2}(a)) \qquad (n \geqslant 2), \tag{1.4}$$

where

$$\overline{\Phi}_0(a) = 0, \qquad \overline{\Phi}_1(a) = q . \tag{1.5}$$

The next few values of $\overline{\Phi}_n(a)$ are

$$\overline{\Phi}_2(a) = q^3 a, \quad \overline{\Phi}_3(a) = q^4(1+q^2a^2), \quad \overline{\Phi}_4(a) = q^7((1+q)a + q^3a^3),$$
$$\overline{\Phi}_5(a) = q^9(1 + (q^2+q^3+q^4)a^2 + q^6a^4),$$
$$\overline{\Phi}_6(a) = q^{13}((1+q+q^2)a + (q^3+q^4+q^5+q^6)a^3 + q^8a^5).$$

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Put

(1.6)
$$\Phi(a,x) = \sum_{n=0}^{\infty} \bar{\phi}_n(a)x^n .$$

Then by (1.4) and (1.5),

$$\Phi(a,x) = qx + \sum_{n=2}^{\infty} q^n(a\bar{\phi}_{n-1}(a) + \bar{\phi}_{n-2}(a))x^n ,$$

so that

(1.7)
$$\Phi(a,x) = qx + qx(a+qx)\Phi(a,qx).$$

Thus

$$\begin{aligned}\Phi(a,x) &= qx + qx(a+qx)\left\{q^2x + q^2x(a+q^2x)\Phi(a,q^2x)\right\}\\ &= qx + q^3x^2(a+qx) + q^3x^2(a+qx)(a+q^2x)\Phi(a,q^2x).\end{aligned}$$

Continuing in this way we get

(1.8)
$$\Phi(a,x) = \sum_{k=0}^{\infty} q^{\frac{1}{2}(k+1)(k+2)}x^{k+1}(a+qx)(a+q^2x)\cdots(a+q^kx).$$

Since

$$(a+qx)(a^2+qx)\cdots(a^2+q^kx) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{1}{2}j(j+1)}a^{k-j}x^j ,$$

(1.8) becomes

$$\begin{aligned}\Phi(a,x) &= \sum_{k=0}^{\infty} q^{\frac{1}{2}(k+1)(k+2)}x^{k+1} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{1}{2}j(j+1)}a^{k-j}x^j\\ &= \sum_{n=0}^{\infty} x^{n+1} \sum_{2j\le n} \begin{bmatrix} n-j \\ j \end{bmatrix} q^{\frac{1}{2}j(j+1)+\frac{1}{2}(n-j+1)(n-j+2)}a^{n-2j} .\end{aligned}$$

It follows that

(1.9)
$$\bar{\phi}_{n+1}(a) = \sum_{2j\le n} \begin{bmatrix} n-j \\ j \end{bmatrix} q^{\frac{1}{2}(n+1)(n+2)-nj+j(j-1)}a^{2n-j} .$$

Since

$$\phi_{n+1}(a) = \sum_{2j\le n} \begin{bmatrix} n-j \\ j \end{bmatrix} q^{j^2}a^{n-2j} ,$$

it is clear that

$$\bar{\phi}_{n+1}(a) = q^{n+1}\phi_n(q^{(n+1)/2}a),$$

that is,

(1.10)
$$\bar{\phi}_n(a) = q^n\phi_n(q^{n/2}a) .$$

2. It is evident that

(2.1)
$$F_n(q) = \phi_n(1,q).$$

Also it follows from

$$F'_{n+1}(q) = \sum_{2k\le n} q^{(k+1)^2} \begin{bmatrix} n-k \\ k \end{bmatrix}$$

that

(2.2)
$$F'_n(q) = q^n\phi_n(q^{-1},q).$$

We have defined [1] the q-Lucas number

(2.3) $$L_n(q) = F_{n+2}(q) - q^n F'_{n-2}(q).$$

Hence, by (2.1) and (2.2),

(2.4) $$L_n(q) = \phi_{n+2}(1,q) - q^2\phi_{n-2}(q^{-1},q).$$

In the next place put

(2.5) $$\phi_n^*(a) = \phi_n^*(a,q) = \phi_n(a,q^{-1}).$$

When q is replaced by q^{-1}, it is easily verified that

$$\begin{bmatrix} n-k \\ k \end{bmatrix} \to q^{k(2k-n)} \begin{bmatrix} n-k \\ k \end{bmatrix}.$$

Hence

$$\phi_{n+1}(a,q^{-1}) = \sum_{2k \leqslant n} \begin{bmatrix} n-k \\ k \end{bmatrix} q^{k^2-nk} a^{n-2k},$$

so that

(2.6) $$q^{n^2/2}\phi^*_{n+1}(a,q) = \phi_{n+1}(aq^{-n/2},q).$$

In particular we have

(2.7) $$q^{n^2/2}F_{n+1}(q^{-1}) = \phi_{n+1}(q^{-n/2},q)$$

and

(2.8) $$q^{\frac{1}{2}(n^2+1)}F'_n(q^{-1}) = \phi_n(q^{\frac{1}{2}(n+1)},q).$$

3. Returning to the recurrence (1.2), we have

(3.1) $$a\phi_n(a) = \phi_{n+1}(a) - q^{n-1}\phi_{n-1}(a).$$

Thus

$$a^2\phi_n(a) = \phi_{n+2}(a) - (1+q)q^{n-1}\phi_n(a) + q^{2n-3}\phi_{n-2}(a)$$

and

$$a^3\phi_n(a) = \phi_{n+3}(a) - (1+q+q^2)q^{n-1}\phi_{n+1}(a) + (1+q+q^2)q^{2n-3}\phi_{n-1}(a) - q^{3n-6}\phi_{n-3}(a).$$

This suggests the general formula

(3.2) $$a^k\phi_n(a) = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{jn-\frac{1}{2}j(j+1)}\phi_{n+k-2j}(a),$$

where $k \geqslant 0$ but n is an arbitrary integer.

Clearly (3.2) holds for $k = 0, 1, 2, 3.$ Assuming that it holds up to and including the value k, we have, by (3.1),

$$\begin{aligned} a^{k+1}\phi_n(a) &= \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{jn-\frac{1}{2}j(j+1)} \left\{ \phi_{n+k-2j+1}(a) - q^{n+k-2j-1}\phi_{n+k-2j-1}(a) \right\} \\ &= \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{jn-\frac{1}{2}j(j+1)}\phi_{n+k-2j+1}(a) \\ &\quad + \sum_{j=1}^{k+1} (-1)^j \begin{bmatrix} k \\ j-1 \end{bmatrix} q^{jn-\frac{1}{2}j(j+1)+k-j+1}\phi_{n+k-2j+1}(a) \\ &= \sum_{j=0}^{k+1} (-1)^j \left\{ \begin{bmatrix} k \\ j \end{bmatrix} + \begin{bmatrix} k \\ j-1 \end{bmatrix} q^{k-j+1} \right\} q^{jn-\frac{1}{2}j(j+1)}\phi_{n+k-2j+1}(a) \\ &= \sum_{j=0}^{k+1} (-1)^j \begin{bmatrix} k+1 \\ j \end{bmatrix} q^{jn-\frac{1}{2}j(j+1)}\phi_{n+k-2j+1}(a). \end{aligned}$$

This completes the proof of (3.2).

Special cases of interest are obtained by taking $n = k, -k, 0, 1$ in (3.2). We get

$$a^k\phi_k(a) = \sum_{j=0}^{k} (-1)^j \left[{k \atop j}\right] q^{kj-\frac{1}{2}j(j+1)}\phi_{2k-2j}(a), \tag{3.3}$$

$$a^k\phi_{-k}(a) = \sum_{j=0}^{k} (-1)^j \left[{k \atop j}\right] q^{-kj-\frac{1}{2}j(j+1)}\phi_{-2j}(a), \tag{3.4}$$

$$0 = \sum_{j=0}^{k} (-1)^j \left[{k \atop j}\right] q^{-\frac{1}{2}j(j+1)}\phi_{k-2j}(a), \tag{3.5}$$

$$a^k = \sum_{j=0}^{k} (-1)^j \left[{k \atop j}\right] q^{-\frac{1}{2}j(j-1)}\phi_{k-2j+1}(a). \tag{3.6}$$

Note that in approximately half the terms on the right of (3.6) the subscript $k-2j+1$ is positive but is negative in the remaining terms. Also, if we prefer, we may eliminate negative subscripts in (3.4), (3.5), and (3.6) by making use of (1.10).

It is clear from (1.1) that we may put

$$a^k = \sum_{2j\le k} (-1)^j q^j C_{k,j}\phi_{k-2j+1}(a), \tag{3.7}$$

where the coefficients $C_{k,j}$ are independent of a. This formula is of course not the same as (3.6). To determine $C_{k,j}$ we multiply both sides of (3.6) by a and then apply (3.1). We get

$$a^{k+1} = \sum_{2j\le k} (-1)^j q^j C_{k,j} \left\{ \phi_{k-2j+2}(a) - q^{k-2j}\phi_{k-2j}(a) \right\}$$

$$= \sum_{2j\le k} (-1)^j q^j C_{k,j}\phi_{k-2j+2}(a) + \sum_{2j\le k+1} (-1)^j q^{k-j+1} C_{k,j-1}\phi_{k-2j+2}(a).$$

It follows that

$$C_{k+1,j} = C_{k,j} + q^{k-2j+1}C_{k,j-1} \qquad (2j \le k). \tag{3.8}$$

The first few values of $C_{k,j}$ are easily computed by means of (3.8).

$n \backslash k$	0	1	2	3
0	1			
1	1			
2	1	1		
3	1	$1+q$		
4	1	$1+q+q^2$	$1+q$	
5	1	$1+q+q^2+q^3$	$1+2q+q^2+q^3$	
6	1	$1+q+q^2+q^3+q^4$	$1+2q+2q^2+2q^3+q^4+q^5$	$1+2q+q^2+q^3$
7	1	$1+q+q^2+q^3+q^4+q^5$	$1+2q+2q^2+3q^3+2q^4+2q^5+q^6+q^7$	$1+3q+3q^2+3q^3+2q^4+q^5+q^6$

It is evident from (3.8) that $C_{k,j}$ is a polynomial in q with nonnegative coefficients and that

(3.9) $$C_{k,0} = 1 \qquad (k = 0, 1, 2, \cdots),$$

(3.10) $$C_{k,j} = 0 \qquad (2j > k).$$

Also it is easily seen that

(3.11) $$C_{k,1} = \frac{1-q^{k-1}}{1-q} \qquad (k \geqslant 1).$$

To get $C_{k,2}$ we take $j = 2$ in (3.8). Thus

$$C_{k+1,2} - C_{k,2} = q^{k-3}C_{k,1} = q^{k-3}\,\frac{1-q^{k-1}}{1-q},$$

which holds for $k \geqslant 3$. Hence

$$C_{k+1,2} = \frac{1}{1-q}\sum_{j=3}^{k} q^{j-3}(1-q^{j-1}),$$

which reduces to

(3.12) $$C_{k+1,2} = \begin{bmatrix} k-2 \\ 1 \end{bmatrix} + q\begin{bmatrix} k-1 \\ 2 \end{bmatrix}.$$

In the next place, taking $j = 3$ in (3.8),

$$C_{k+1,3} = C_{k,3} = q^{k-5}C_{k,2} \qquad (k \geqslant 5).$$

We find that

(3.13) $$C_{k+1,3} = q^{-1}\begin{bmatrix} k-2 \\ 2 \end{bmatrix} + \begin{bmatrix} k-1 \\ 3 \end{bmatrix} - q^{-1} - 1.$$

By means of (3.8) it can be proved that

(3.14) $$\deg C_{k,j} = jk - \tfrac{1}{2}j(3j+1).$$

The proof is by induction on k. The second term on the right of (3.8) is of higher degree than the first term, so that

$$\deg C_{k+1,j} = k - 2j + 1 + \deg C_{k,j-1} = (k-2j+1) + (j-1)k - \tfrac{1}{2}(j-1)(3j-2) = j(k+1) - \tfrac{1}{2}j(3j+1).$$

It would be of interest to find a simple explicit formula for $C_{k,j}$. The problem is equivalent to inverting

(3.15) $$u_n = \sum_{2k \leqslant n} \begin{bmatrix} n-k \\ k \end{bmatrix} q^{k^2} v_{n-2k} \qquad (n = 0, 1, 2, \cdots).$$

In this connection the following two inversion theorems may be mentioned:

I. $$u_r = \sum_{2s \leqslant r} \begin{bmatrix} r \\ s \end{bmatrix} v_{r-2s} \qquad (r = 0, 1, 2, \cdots)$$

if and only if

$$v_r = \sum_{2s \leqslant r} (-1)^s q^{\frac{1}{2}s(s-1)}\,\frac{1-q^r}{1-q^{r-s}} \begin{bmatrix} r-s \\ s \end{bmatrix} v_{r-2s} \qquad (r = 0, 1, 2, \cdots).$$

II. $$u_r = \sum_{2s \leqslant r} \left\{ \begin{bmatrix} r \\ s \end{bmatrix} - \begin{bmatrix} r \\ s-1 \end{bmatrix} \right\} v_{r-2s} \qquad (r = 0, 1, 2, \cdots)$$

if and only if

$$v_r = \sum_{2s \leqslant r} (-1)^s q^{\frac{1}{2}s(s+1)} \begin{bmatrix} r-s \\ s \end{bmatrix} u_{r-2s} \qquad (r = 0, 1, 2, \cdots).$$

For proof of these and some related inversion theorems see [2].

4. Returning to the recurrence (1.2) we now construct a second solution $\psi_n(a) = \psi_n(a,q)$ such that

(4.1) $$\psi_0(a) = 1, \qquad \psi_1(a) = a$$

and of course

(4.2) $$\psi_{n+1}(a) = a\psi_n(a) + q^{n-1}\psi_{n-1}(a) \qquad (n \geqslant 1).$$

Put

(4.3) $$\Psi(a,x) = \sum_{n=0}^{\infty} \psi_n(a)x^n .$$

Then

$$\Psi(a,x) = 1 + ax + \sum_{n=2}^{\infty} (a\psi_{n-1}(a) + q^{n-2}\psi_{n-2}(a))x^n = 1 + ax\Psi(a,x) + x^2\Psi(a, qx),$$

so that

(4.4) $$\Psi(a,x) = \frac{1}{1-ax} + \frac{x^2}{1-ax}\Psi(a,qx).$$

Iteration of (4.4) yields

(4.5) $$\Psi(a,x) = \sum_{r=0}^{\infty} \frac{q^{r(r-1)}x^{2r}}{(ax)_{r+1}} .$$

Hence

$$\Psi(a,x) = \sum_{r=0}^{\infty} q^{r(r-1)}x^{2r} \sum_{s=0}^{\infty} \begin{bmatrix} r+s \\ s \end{bmatrix} a^s x^s = \sum_{n=0}^{\infty} x^n \sum_{2r \leqslant s} \begin{bmatrix} n-r \\ r \end{bmatrix} q^{r(r-1)} a^{n-2r} ,$$

which implies

(4.6) $$\psi_n(a) = \sum_{2r \leqslant n} \begin{bmatrix} n-r \\ r \end{bmatrix} q^{r(r-1)} a^{n-2r} .$$

We have therefore

(4.7) $$q^{\frac{n}{2}}\psi_n(a) = \phi_{n+1}(q^{\frac{1}{2}}a) .$$

Finally we mention the following continued fraction formula.

(4.8) $$a + \frac{q}{a+}\frac{q^2}{a+}\cdots\frac{q^n}{a} = \frac{\phi_{n+2}(a)}{q^{n/2}\phi_{n+1}(q^{-\frac{1}{2}}a)} = \sum_{2k \leqslant n+1} \begin{bmatrix} n-k+1 \\ k \end{bmatrix} q^{k^2} a^{n-2k+1} \Big/ \sum_{2k \leqslant n} \begin{bmatrix} n-k \\ k \end{bmatrix} q^{k(k+1)} a^{n-2k}.$$

An equivalent result has been obtained by Hirschhorn [3].

REFERENCES

1. L. Carlitz, "Fibonacci Notes—3: q-Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 12, No. 7 (December, 1974), pp. 317–322.
2. L. Carlitz, "Some Inversion Formulas," *Rendiconto del Circolo Mathematico di Palermo,* Series 2, Vol. 12 (1963) pp. 183–199.
3. M.D. Hirschhorn, "Partitions and Ramanujan's Continued Fraction," *Duke Mathematical Journal,* Vol. 39 (1972), pp. 789–791.

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