

SOME BINOMIAL SUMS

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1. Put

$$(1.1) \quad A(n) = \sum_{k=0}^{n+1} (-1)^k \left\{ \binom{n}{k} - \binom{n}{k-1} \right\}^3,$$

where it is understood that

$$\binom{n}{-1} = \binom{n}{n+1} = 0 \quad (n \geq 0).$$

Consideration of this sum was suggested by the following problem proposed by H. W. Gould [1]. Let

$$A_p(n) = \sum_{0 \leq 2k \leq n} (-1)^k \left\{ \binom{n}{k} - \binom{n}{k-1} \right\}^p.$$

Then

$$A_2(2m+1) = (2m+1)A_1(2m+1).$$

It is noted that this result does not hold for even n .

Since

$$A(n) = \sum_{k=0}^{n+1} (-1)^{n-k+1} \left\{ \binom{n}{n-k+1} - \binom{n}{n-k} \right\}^3 = \sum_{k=0}^{n+1} (-1)^{n-k+1} \left\{ \binom{n}{k-1} - \binom{n}{k} \right\}^3,$$

so that

$$(1.2) \quad A(n) = (-1)^n A(n),$$

therefore

$$(1.3) \quad A(2m+1) = 0.$$

However (1.2) gives no information about $A(2m)$. By (1.1) we have

$$\begin{aligned} A(n) &= \sum_{k=0}^n (-1)^k \binom{n}{k}^3 - 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n}{k-1} + 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2 \\ &- \sum_{k=1}^{n+1} (-1)^k \binom{n}{k-1}^3 = 2 \sum_{k=0}^n (-1)^k \binom{n}{k}^3 - 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n}{k-1} + 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2. \end{aligned}$$

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Thus if we put

$$S_0(n) = \sum_{k=0}^n (-1)^k \binom{n}{k}^3, \quad S_1(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n}{k-1},$$

$$S_2(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2.$$

it is clear that

$$(1.4) \quad A(n) = 2S_0(n) - 3S_1(n) + 3S_2(n).$$

In the next place, we have

$$S_2(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2 = \sum_{k=0}^{n+1} (-1)^{n-k+1} \binom{n}{n-k+1} \binom{n}{n-k}^2$$

$$= \sum_{k=0}^{n+1} (-1)^{n-k+1} \binom{n}{k-1} \binom{n}{k}^2,$$

so that

$$(1.5) \quad S_2(n) = (-1)^{n+1} S_1(n)$$

and (1.4) becomes

$$(1.6) \quad A(n) = 2S_0(n) - 3 \{1 + (-1)^n\} S_1(n).$$

In particular we have

$$(1.7) \quad \begin{cases} A(2m) = 2S_0(2m) - 6S_1(2m) \\ A(2m+1) = 2S_0(2m+1). \end{cases}$$

It is well known (see for example [2, p. 13], [3, p. 243]) that $S_0(2m+1) = 0$, while

$$(1.8) \quad S_0(2m) = (-1)^m \frac{(3m)!}{(m!)^3}.$$

However $S_1(n)$ does not seem to be known.

2. In order to evaluate $S_1(2m)$ we proceed as follows. We have

$$S_1(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \left\{ \binom{n+1}{k} - \binom{n}{k} \right\} = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n+1}{k} - S_0(n)$$

$$= \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n+1}{k} \left\{ \binom{n+1}{k} - \binom{n}{k-1} \right\} - S_0(n)$$

so that

$$(2.1) \quad S_1(n) = T_0(n) - T_1(n) - S_0(n),$$

where

$$T_0(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1}{k}^2, \quad T_1(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n+1}{k} \binom{n}{k-1}.$$

Now

$$\begin{aligned} T_1(n) &= \sum_{k=0}^{n+1} (-1)^{n-k+1} \binom{n}{n-k+1} \binom{n+1}{n-k+1} \binom{n}{n-k} \\ &= (-1)^{n+1} \sum_{k=0}^{n+1} (-1)^k \binom{n}{k-1} \binom{n+1}{k} \binom{n}{k}. \end{aligned}$$

that is,

$$(2.2) \quad T_1(n) = (-1)^{n+1} T_1(n).$$

Therefore $T_1(2m) = 0$ and (2.1) yields

$$(2.3) \quad S_1(2m) = T_0(2m) - S_0(2m).$$

In the next place

$$\begin{aligned} T_0(n) &= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1}{k}^2 = \sum_{k=0}^n (-1)^{n-k} \binom{n}{n-k} \binom{n+1}{n-k}^2 \\ &= (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1}{k+1}^2 = (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1}{k+1} \left\{ \binom{n+2}{k+1} - \binom{n+1}{k} \right\} \\ &= -(-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1}{k+1} \binom{n+1}{k} + (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+2}{k+1} \left\{ \binom{n+2}{k+1} - \binom{n+1}{k} \right\} \\ &= -(-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1}{k+1} \binom{n+1}{k} + (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+2}{k+1}^2 \\ &\quad - (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1}{k+1} \left\{ \binom{n+1}{k} + \binom{n+1}{k+1} \right\} \\ &= -2(-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1}{k+1} \binom{n+1}{k} - (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1}{k}^2 \\ &\quad + (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+2}{k+1}^2. \end{aligned}$$

so that

$$(2.4) \quad \begin{aligned} \{1 + (-1)^n\} T_0(n) &= -2(-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1}{k} \binom{n+1}{k+1} \\ &\quad + (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+2}{k+1}^2. \end{aligned}$$

For $n = 2m + 1$, (2.4) gives no information about $T_0(2m + 1)$; indeed each sum on the right vanishes. For $n = 2m$, however, (2.4) becomes

$$(2.5) \quad 2T_0(2m) = -2 \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{2m+1}{k} \binom{2m+1}{k+1} \\ + \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{2m+2}{k+1}^2.$$

It is known [3, p. 243] that

$$(2.6) \quad \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{2m+1}{k} \binom{2m+1}{k+1} = (-1)^m \frac{(3m+1)!}{m!m!(m+1)!}$$

and

$$(2.7) \quad \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{2m+2}{k+1}^2 = (-1)^m \frac{2(3m+2)!}{m!m!(m+1)!(2m+1)!}.$$

Substituting from (2.6) and (2.7) in (2.5), we get

$$(2.8) \quad T_0(2m) = (-1)^m \frac{(3m+1)!}{(m!)^3(2m+1)}.$$

Therefore by (2.3) and (1.8)

$$(2.9) \quad S_1(2m) = (-1)^m \frac{(3m)!}{m!m!(m-1)!(2m+1)}.$$

Finally, by (1.6) and (2.9),

$$(2.10) \quad A(2m) = -2(-1)^m \frac{(3m)!(m-1)}{(m!)^3(2m+1)}.$$

This completes the evaluation of the sum $A(2m)$. Note that we have not evaluated $S_1(2m+1)$.

3. For completeness we give a simple proof of (1.8), (2.6) and (2.7). We assume Saalschütz's theorem [2, p. 9]:

$$(3.1) \quad \sum_{k=0}^n \frac{(-n)_k (a)_k (b)_k}{k! (c)_k (d)_k} = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n},$$

where

$$(a)_k = a(a+1)\cdots(a+k-1), \quad (a)_0 = 1$$

and

$$(3.2) \quad c+d = -n+a+b+1.$$

We rewrite (3.1) in the following way:

$$(3.3) \quad \sum_{r=0}^j \frac{(-j)_r (a+j)_r (b+c-a+1)_r}{r! (b+1)_r (c+1)_r} = \frac{(a-b)_j (a-c)_j}{(b+1)_j (c+1)_j};$$

the condition (3.2) is automatically satisfied. Multiplying both sides of (3.3) by $(a)_j x^j / j!$ and summing over j , it follows that

$$\sum_{j=0}^{\infty} \frac{(a)_j (a-b)_j (a-c)_j}{j! (b+1)_j (c+1)_j} x^j = \sum_{j=0}^{\infty} \frac{(a)_j}{j!} x^j \sum_{r=0}^{\infty} \frac{(-j)_r (a+j)_r (b+c-a+1)_r}{r! (b+1)_r (c+1)_r} \\ = \sum_{r=0}^{\infty} (-1)^r \frac{(a)_{2r} (b+c-a+1)_r}{r! (b+1)_r (c+1)_r} x^r \sum_{j=0}^{\infty} \frac{(a+2r)_j}{j!} x^j = \sum_{r=0}^{\infty} (-1)^r \frac{(a)_{2r} (b+c-a+1)_r}{r! (b+1)_r (c+1)_r} x^r (1-x)^{-a-2r}.$$

Now take $a = -n$ and we get

$$(3.4) \quad \sum_{j=0}^{\infty} \frac{(-n)_j (-n-b)_j (-n-c)_j}{j!(b+1)_j (c+1)_j} x^j = \sum_{r=0}^{\infty} (-1)^r \frac{(-n)_{2r} (b+c+n-1)_r}{r!(b+1)_r (c+1)_r} x^r (1-x)^{n-2r}.$$

For $n = 2m$ and $x = 1$, (3.4) reduces to

$$(3.5) \quad \sum_{j=0}^{\infty} \frac{(-2m)_j (-2m-b)_j (-2m-c)_j}{j!(b+1)_j (c+1)_j} = (-1)^m \frac{(2m)!(b+c+2m+1)_m}{m!(b+1)_m (c+1)_m}.$$

Now let b, c be non-negative integers. Then (3.5) yields

$$(3.6) \quad \sum_{j=0}^{2m} (-1)^m \binom{2m}{j} \binom{2m+b+c}{j+b} \binom{2m+b+c}{j+c} \\ = (-1)^m \frac{(2m)!(3m+b+c)!(2m+b+c)!}{m!(m+b)(m+c)!(2m+b)!(2m+c)!}.$$

For $b = c = 0$ we get (1.8); for $b = 0, c = 1$ we get (2.6); for $b = c = 1$ we get (2.7).

REFERENCES

1. E 2395, *Amer. Math. Monthly*, 80 (1973), p. 75; solution, 80 (1973), p. 1146.
2. W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge, 1935.
3. L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge, 1966.

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$$\frac{1}{k} \log \frac{1+\sqrt{5}}{2}$$

as $n \rightarrow \infty$. Since this limiting value is an irrational number, the sequence (u_n) is u.d. mod 1.

REMARK. Let p and q be non-negative integers. Then the sequence

$$p, q, p+q, p+2q, 2p+3q, \dots$$

or (H_n) , $n = 1, 2, \dots$ with

$$H_n = qF_{n-1} + pF_{n-2} \quad (n \geq 3), \quad H_1 = p, \quad H_2 = q$$

possesses the property shown in Theorem 1. For if $v_n = \log H_n^{1/k}$, we have

$$v_{n+1} - v_n \rightarrow \frac{1}{k} \log \frac{1+\sqrt{5}}{2}$$

as $n \rightarrow \infty$.

Theorem 2. Let p, q, p^* and q^* be non-negative integers. Let (H_n) be the sequence

$$p, q, p+q, p+2q, 2p+3q, \dots$$

and (H_n^*) the sequence

$$p^*, q^*, p^*+q^*, p^*+2q^*, 2p^*+3q^*, \dots$$

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